PRIME IDEALS IN CLOSED SUBALGEBRAS OF L^∞

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Let $\mathbf{D}$ denote the open disc and let H^∞ denote the algebra of bounded analytic functions on $\mathbf{D}$. A prime ideal in a commutative algebra A is an ideal Q such that whenever $f, g \in A$ and $fg \in Q$, either $f \in Q$ or $g \in Q$. In [11, p. 396] the following question is asked: Let Q be a nonzero prime ideal in H^∞ such that $Q \neq H^\infty$, and suppose Q is finitely generated; do we then have $Q = \{f \in H^\infty : f(\zeta) = 0\}$, where $\zeta \in \mathbf{D}$? In the first section of this paper, we shall answer this question affirmatively. After this work was completed, I learned that R. Mortini also obtained this result ([14], [15]).

Let C denote the algebra of continuous functions on the unit circle, $\partial\mathbf{D}$. In §2, we shall show that $H^\infty + C$ has no nontrivial finitely generated prime ideals. However, there do exist proper closed subalgebras of L^∞ which have nontrivial finitely generated prime ideals.

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1. Finitely generated prime ideals in H^∞. Let B be a closed subalgebra of L^∞. The maximal ideal space of B is denoted $M(B)$. By maximal ideal we mean a proper ideal of B contained in no other proper ideal of B. Because each such ideal is the kernel of a nonzero complex homomorphism on B, we think of $M(B)$ as the space of nonzero complex homomorphisms on B. With the weak-*topology, $M(B)$ is a compact Hausdorff space. In the usual way, we think of $\mathbf{D}$ as a subset of $M(H^\infty)$. By the Corona theorem, $\mathbf{D}$ is a dense subset of $M(H^\infty)$. If B contains H^∞, then the space $M(B)$ can be identified with a closed subset of $M(H^\infty)$. If B properly contains H^∞, then B contains $H^\infty + C$. Thus $M(B) \subseteq M(H^\infty) - \mathbf{D}$ $(= M(H^\infty + C))$. We shall identify a function in B with its Gelfand transform.

In this section, our main tool is the analytic structure of the Gleason parts of H^∞. For $x \in M(H^\infty)$, the Gleason part containing x is denoted $P(x)$. If f denotes a function in H^∞ and $x \in M(H^\infty)$ is such that $f(x) = 0$, then the order of the zero of f at x is the supremum of the positive integers n such that f can be factored as $f = f_1 \cdots f_n$, $f_j \in H^\infty$ and $f_j(x) = 0$ for $j = 1, 2, \ldots, n$. The order of the zero of f at x will be denoted by $\operatorname{Ord} Z(f; x)$. The zero set f in $M(B)$ is denoted $Z_B(f)$. We shall also use the following lemma (usually referred to as Nakayama's lemma).

LEMMA 1.1 [12, p. 11]. *Let A be a commutative ring with identity, M a finitely generated A module and J an ideal of A. Suppose that $JM = M$. Then there exists an element $a \in A$ of the form $a = 1 + b$, $b \in J$, such that $aM = 0$.*

We shall apply Lemma 1.1 to the case in which $A = H^\infty$ and M is a finitely generated ideal of A. Since H^∞ has no zero divisors, if we produce a proper ideal J such that $JM = M$, our conclusion is that $M = 0$.

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The proof of the main result will require two lemmas.

LEMMA 1.2. *Let I be a finitely generated nontrivial prime ideal in H^∞. Let $f_1, \dots, f_n$ denote a set of generators for I. Then*

$$P(x) \not\subset \bigcap_{j=1}^{n} Z_{H^\infty}(f_j) \quad \textit{for any } x \in M(H^\infty + C).$$

Proof. Suppose there exists $x \in M(H^\infty + C)$ such that $P(x) \subseteq \bigcap_{j=1}^{n} Z_{H^\infty}(f_j)$. Let $h \in I$. Then $P(x) \subseteq Z_{H^\infty}(h)$. By [9], h has a zero of infinite order at x. Therefore, there exist h_1 and h_2 in H^∞ such that $h_1(x) = h_2(x) = 0$ and $h = h_1 h_2$. Since I is prime, either $h_1 \in I$ or $h_2 \in I$. In either case, $h \in I \cdot (\operatorname{Ker} x)$. Therefore $I \subseteq I \cdot (\operatorname{Ker} x) \subseteq I$. Thus $I(\operatorname{Ker} x) = I$. Taking $A = H^\infty$, $M = I$ and $J = \operatorname{Ker} x$ in Lemma 1.1, we see that I must be zero, a contradiction. □

An interpolating sequence is a Blaschke sequence $\{z_n\}$ such that for each bounded sequence of complex numbers $\{w_n\}$, there exists a bounded analytic function $f \in H^\infty$ with $f(z_n) = w_n$.

LEMMA 1.3. *Let I be a finitely generated nontrivial prime ideal in H^∞. Let $f_1, \dots, f_n$ denote a set of generators for I. Then*

$$x \notin \bigcap_{j=1}^{n} Z_{H^\infty}(f_j) \quad \textit{for any } x \in M(H^\infty + C).$$

Proof. Suppose there exists $x \in M(H^\infty + C)$ with $x \in \bigcap_{j=1}^{n} Z_{H^\infty}(f_j)$. By the Corona theorem, there exists a sequence $\{z_n\} \subseteq \mathbf{D}$ such that (i) $z_m \to x(z)$ and (ii) $f_j(z_m) \to 0$ as $m \to \infty$ $(j = 1, 2, \dots, n)$. Let $\{z_m^1\}$ denote an interpolating subsequence of $\{z_m\}$ and let $z_1, \dots, z_{n+1}$ denote distinct points of $\overline{\{z_m^1\}} \cap M(H^\infty + C)$. By Lemma 1.2, there must exist a generator f such that f does not vanish on Gleason parts corresponding to at least two points, say $P(x_1)$ and $P(x_2)$. Thus f has a zero of finite order n_k at x_k for $k = 1, 2$. Factor f as $f = g_1 \cdots g_{n_1}$ with $g_j(x_1) = 0$ for $j = 1, \dots, n_1$. Since I is a prime ideal, there exists j_0 with $g_{j_0} \in I$. The function g_{j_0} has the following properties:

(iii) $\operatorname{Ord} Z(g_{j_0}; x_1) = 1$,
(iv) $g_{j_0}(x_2) = 0$, and
(v) $\operatorname{Ord} Z(g_{j_0}; x_2) \leq \operatorname{Ord} Z(f; x_2) = n_2$.

Property (iv) holds because $x_2 \in \overline{\{z_n^1\}}$, $g_{j_0} \in I$, and the generators of I satisfy (ii) above. By factoring g_{j_0} as above, we may assume that $\operatorname{Ord} Z(g_{j_0}; x_2) = 1$. Let $g_{j_0} = b \cdot q$, where b is a Blaschke product and $q \in H^\infty$ has no zeroes on $\mathbf{D}$. By [9], if $q(x) = 0$ then $q = 0$ on $P(x)$. Since g_{j_0} does not vanish on $P(x_1)$ or $P(x_2)$, neither $q(x_1) = 0$ nor $q(x_2) = 0$. Therefore $q \notin I$. Since I is prime, we must have $b \in I$. Furthermore, $\operatorname{Ord} Z(b; x_1) = \operatorname{Ord} Z(b; x_2) = 1$. By [9], x_j lies in the closure of an interpolating subsequence $\{z_{n_j}\}$ of the zero sequence of b for $j = 1, 2$. Let O_{x_1} and O_{x_2} be disjoint neighborhoods of x_1 and x_2, respectively, in $M(H^\infty)$. Let b_{x_j} be the Blaschke product with zero sequence $O_{x_j} \cap Z_{\mathbf{D}}(b)$ for $j = 1, 2$. There exists a Blaschke product c such that $b = b_{x_1} \cdot b_{x_2} \cdot c$. Since I is prime and $b \in I$, either b_{x_1}, b_{x_2}, or c is in I. Since each generator vanishes on x_2, neither b_{x_1} nor c can be in I. Since $I \subset \operatorname{Ker} x_1$, b_{x_2} cannot be in I. Thus the proof of Lemma 1.3 is complete. □

THEOREM 1.1. *Let I be a finitely generated prime ideal of H^∞. Then there exists $\zeta \in \mathbf{D}$ such that $I = \{f \in H^\infty : f(\zeta) = 0\}$.*

Proof. Let J be a maximal ideal containing I [17, p. 18]. There exists $\zeta \in M(H^\infty)$ such that $J = \operatorname{Ker} \zeta$. By Lemma 1.3, $\zeta \in \mathbf{D}$. Let $f \in I$. Since $f(\zeta) = 0$, there exists a positive integer N and a function $g \in H^\infty$ such that

$$g(\zeta) \neq 0 \quad \text{and} \quad f(z) = (z-\zeta)^N g(z) \quad \text{for all } z \in \mathbf{D}.$$

Since I is prime, the function h defined by $h(z) = (z - \zeta)$ for $z \in \mathbf{D}$ is in the ideal I. Let $k \in \operatorname{Ker} \zeta$. Then $k = h \cdot g$ for some $g \in H^\infty$. Thus $k \in I$. Hence $I = \operatorname{Ker} \zeta$, as desired. □

G. Tomassini [19] showed that if J is a maximal finitely generated ideal in H^∞, then there exists $\zeta \in \mathbf{D}$ such that $J = \operatorname{Ker} \zeta$.

The nontrivial finitely generated prime ideals in other subalgebras of L^∞ depend very much upon the particular algebra. This will become evident in the next section.

2. Finitely generated prime ideals in subalgebras of L^∞ containing $H^\infty + C$. In this section we study prime ideals in $H^\infty + C$. Let QC denote the algebra of functions in $H^\infty + C$ such that $\bar{f}$ is also in $H^\infty + C$. Let $QA = QC \cap H^\infty$. It is well known that $M(QA) = M(QC) \cup \mathbf{D}$. We shall show that $H^\infty + C$ has no finitely generated prime ideals, but that there exist Douglas algebras that do have such ideals.

LEMMA 2.1. *Let $f_1, \ldots, f_n \in H^\infty + C$. Let $x \in M(H^\infty + C)$ be such that $f_j \in \operatorname{Ker} x$ for $j = 1, 2, \ldots, n$. Then there exist $x_1, x_2 \in M(H^\infty + C)$ such that $x_1(f_j) = x_2(f_j) = 0$ for $j = 1, 2, \ldots, n$ and $x_1(q) \neq x_2(q)$ for some $q \in QC$.*

Proof. Let $\{z_m\}$ be a sequence of points in $\mathbf{D}$ such that $|z_m| \to 1$ and

$$z_m \in \bigcap_{j=1}^{n} \left\{ y \in M(H^\infty) : |x(f_j) - y(f_j)| < \frac{1}{m} \right\}.$$

Note that $f_j(z_m) \to 0$ as $m \to \infty$ for $j = 1, 2, \ldots, n$. Choose a subsequence, also denoted $\{z_m\}$, which is an interpolating sequence and

$$\lim_{n \to \infty} \prod_{m \neq n} \left| \frac{z_m - z_n}{1 - \bar{z}_m z_n} \right| = 1.$$

By [18], there exists $q \in QA$ such that

$$q(z_m) = \begin{cases} 0 & m \text{ even}, \\ 1 & m \text{ odd}. \end{cases}$$

Choose x_1 and x_2 in $\overline{\{z_m\}}^{M(H^\infty)} \cap M(H^\infty + C)$ such that $x_1(q) = 0$ and $x_2(q) = 1$. Then $x_1(f_j) = x_2(f_j) = 0$ for $j = 1, 2, \ldots, n$, as desired. □

THEOREM 2.1. *There exist no nontrivial finitely generated prime ideals in* $H^\infty + C$.

Proof. Suppose that $\{f_1, \ldots, f_n\}$ is a set of generators for I. Let $x \in M(H^\infty + C)$ be such that $I \subseteq \operatorname{Ker} x$. Using Lemma 2.1 above, choose x_1 and x_2 in $M(H^\infty + C)$ such that $I \subseteq \operatorname{Ker} x_1 \cap \operatorname{Ker} x_2$ and $x_1(q) \neq x_2(q)$ for some $q \in QC$. Let $O_{x'_j}$ be a neighborhood in $M(QC)$ of $x'_j = x_j \mid QC$, $j = 1, 2$, such that $O_{x'_1} \cap O_{x'_2} = \varnothing$. Choose $g_j \in QC$ $(= C(M(QC)))$ such that $g_j(x'_j) = 1$ and $g_j(M(QC) - O_{x'_j}) = 0$ for $j = 1, 2$. Then $g_1 g_2 = 0$. Therefore either $g_1 \in I$ or $g_2 \in I$, contradicting the fact that $I \subseteq \operatorname{Ker} x_1 \cap \operatorname{Ker} x_2$. □

The proof above shows that if J is a prime ideal in $H^\infty + C$, then there exists exactly one point $t \in M(QC)$ such that whenever $J \subseteq \operatorname{Ker} x$ we have $x(q) = t(q)$ for all $q \in QC$. Furthermore, as shown in [14], if J contains an interpolating Blaschke product then there exists a unique $x \in M(H^\infty + C)$ such that $J \subseteq \operatorname{Ker} x$.

In spite of the close connection of prime ideals in $H^\infty + C$ to division problems in $H^\infty + C$ ([4], [6]), it seems that very little is known about nonmaximal prime ideals in $H^\infty + C$.

Theorem 2.1 cannot be extended to an arbitrary closed subalgebra B of L^∞ containing H^∞. An example of such an algebra depends on results in [18]. A sequence $\{z_n\} \subseteq \mathbf{D}$ is said to be *thin* if it is an interpolating sequence and

$$\lim_{n \to \infty} \prod_{m \neq n} \left| \frac{z_m - z_n}{1 - \bar{z}_m z_n} \right| = 1.$$

Let b be a Blaschke product such that the zero sequence of b is thin. Let $x \in \overline{\{z_n\}}^{M(H^\infty + C)}$ and let m_x denote the unique representing measure of x. The closed support of m_x is denoted $\operatorname{supp} m_x$. Let $B = \{f \in L^\infty : f \mid \operatorname{supp} m_x \in H^\infty \mid \operatorname{supp} m_x\}$. Then $H^\infty \subseteq B \subseteq L^\infty$ and B is a closed subalgebra of L^∞. Furthermore, $M(B) = M(L^\infty) \cup \{y \in M(H^\infty + C) : \operatorname{supp} m_y \subseteq \operatorname{supp} m_x\}$ [3, p. 39]. We shall show that bB is a prime ideal in B. We first show that $Z_B(b) = \{x\}$. Let $y \in M(B) \cap Z_B(b)$. If $y \neq x$, choose θ_x and θ_y, disjoint neighborhoods in $M(H^\infty)$ of x and y (respectively), and let $W_x = \theta_x \cap Z_{H^\infty}(b)$ and $W_y = \theta_y \cap Z_{H^\infty}(b)$. By [18], there exists $q \in QC$ such that $q(W_x) = 0$ and $q(W_y) = 1$. Since $\operatorname{supp} m_x$ is an antisymmetric set for $H^\infty + C$, we must have $q \mid \operatorname{supp} m_x = 0$. But $\operatorname{supp} m_y \subseteq \operatorname{supp} m_x$, so $y(q) = 0$, a contradiction. Thus $Z_B(b) = \{x\}$. Let $f \in \operatorname{Ker} x$. Then $Z_B(b) \subseteq Z_B(f)$. By [1], $f/b \in B$. Thus $\operatorname{Ker} x \subseteq bB$. Therefore, $\operatorname{Ker} x = bB$ and bB is prime.

3. Finitely generated prime ideals in *QA*. We now consider the algebra $QA = H^\infty \cap QC$. In many ways the relationship of QA to QC is similar to the role the disc algebra plays in C [20]. We shall use Nakayama's lemma to prove the following theorem.

THEOREM 3.1. *Let I be a finitely generated prime ideal in QA. Then there exists $\zeta \in \mathbf{D}$ such that* $I = \{f \in QA : f(\zeta) = 0\}$.

Proof. Let $t_0 \in M(QA)$ be such that $I \subseteq \operatorname{Ker} t_0$. We shall show that $t_0 \notin M(QA) - \mathbf{D}$. Suppose $t_0 \in M(QA) - \mathbf{D}$. Let $f \in I$ and factor $f = qb$, where b is an inner function and $q=f\bar{b}$ is an outer function in QA. Then $f=q^{1/2}(q^{1/2}b)$. By [6], $q^{1/2}\in QA$. To see that $q^{1/2}b\in QA$, for each $t\in M(QC)$ let $E_t=\{s\in M(L^\infty): s(q)=t(q)$ for all $q\in QC\}$. If $b|E_t$ is constant, then $q^{1/2}b|E_t$ is constant. If $b|E_t$ is nonconstant, then $q|E_t$ must be identically zero. Thus $q^{1/2}b|E_t$ is constant. By Shilov's theorem [16], $q^{1/2}b\in QC$. Thus f is the product of two QA functions. We know that $f\in I\subseteq \operatorname{Ker} t_0$. If $t_0(q^{1/2}b)=0$ then $q^{1/2}|E_{t_0}=0$, so $t_0(q^{1/2})=0$. It is easy to see from this that $t_0(q^{1/2}b)=t_0(q^{1/2})=0$. Furthermore, since I is prime either $q^{1/2}\in I$ or $q^{1/2}b\in I$. Thus $I=I\cdot\operatorname{Ker} t_0$, contradicting Lemma 1.1. Therefore, we may assume that there exists $\zeta\in\mathbf{D}$ such that $f(\zeta)=0$ for all $f\in I$. Let $f\in I$ and let N be a positive integer such that $f(z)=(z-\zeta)^N g(z)$, where $g\in QA$ and $g(\zeta)\neq 0$. Then $(z-\zeta)\in I$ and hence $I=\{f: f(\zeta)=0\}$. □

4. Examples of prime ideals in subalgebras of H^∞. In [2] Dietrich has shown, using the continuum hypothesis, that any point $x\in M(H^\infty)-\mathbf{D}$ such that $P(x)$ is nontrivial has the property that Ker x contains a chain of prime ideals of infinite length. In this section we construct an infinite chain of prime ideals in Ker x without the use of the continuum hypothesis. Using this example it is possible to extend Dietrich's result to all points of $M(H^\infty+C)-M(L^\infty)$. We begin this section with a theorem about closed prime ideals in H^∞. This theorem seems to be known but has not yet appeared in the literature. In what follows, $Z(f)=Z_{\mathbf{D}}(f)$ and $\overline{Z(f)}$ is the closure of $Z(f)$ in $M(H^\infty)$.

THEOREM 4.1. *Let I be a closed prime ideal in H^∞ containing an interpolating Blaschke product. Then I is maximal.*

Recall that a prime ideal containing an interpolating Blaschke product is contained in a unique maximal ideal (see the remarks immediately following the proof of Theorem 2.1). The proof of Theorem 4.1 requires this fact and the following theorem.

THEOREM 4.2 [8, p. 208]. *Let $\{z_n\}$ be a sequence of points in $\mathbf{D}$. If $\{z_n\}$ is an interpolating sequence, then disjoint subsets of $\{z_n\}$ have disjoint closures in $M(H^\infty)$.*

LEMMA 4.3. *Let I be a prime ideal containing an interpolating Blaschke product b. Let x denote the unique element of $M(H^\infty+C)$ containing I in its kernel. If θ is an open subset of $M(H^\infty)$ containing $\{x\}$, then $x\in\overline{\theta\cap Z(b)}$. Furthermore, if b_1 denotes the Blaschke product with zero set $\theta\cap Z(b)$, then $b_1\in I$.*

Proof. Let x,θ, and b_1 be as above. Write $b=b_1b_2$. We claim that $x(b_2)\neq 0$, and hence $b_2\notin I$. Let U be an open subset of $M(H^\infty)$ containing x. Then $U\cap\theta$ is an open subset of $M(H^\infty)$ containing x. Since $x(b)=0$, by [9] $x\in\overline{Z(b)}$. Therefore $U\cap\theta\cap Z(b)\neq\varnothing$. Hence $x\in\overline{\theta\cap Z(b)}$. From this we see that $x(b_1)=0$. Since $Z(b_1)\cap Z(b_2)=\varnothing$, by Theorem 4.2 $x\notin\overline{Z(b_2)}$. Again by [9], $x(b_2)\neq 0$. Since I is prime and $b_2\notin I$, we must have $b_1\in I$. □

The techniques used to prove Theorem 4.1 are the same as those used to prove Theorem 1 of [1].

Proof of Theorem 4.1. Let b denote an interpolating Blaschke product contained in I. Let x denote the unique point of $M(H^\infty + C)$ with $I \subseteq \operatorname{Ker} x$. Let $g \in \operatorname{Ker} x$. We shall show that $g \in I$. Let $\{z_m\} = Z(b)$ and $\theta_n = \{z \in M(H^\infty): |g(z)| < 1/n\}$. Let $W_n = \theta_n \cap Z(b)$. Let b_n denote the factor of b with zero set W_n. Since $\{z_n\}$ is interpolating, the map $T: H^\infty/bH^\infty \to \ell^\infty$ defined by $T(f + bh^\infty) = (f(z_1), f(z_2), \dots)$ is a one-to-one map of H^∞/bH^∞ onto ℓ^∞. By a corollary to the open mapping theorem, there exists a constant K such that $\operatorname{dist}(f, bH^\infty) \le K \sup_m |f(z_m)|$. Choose $f_n \in H^\infty$ such that

$$f_n(z_m) = \begin{cases} g(z_m) & \text{if } z_m \in W_n, \\ 0 & \text{if } z_m \notin W_n. \end{cases}$$

Then $g - f_n \in b_n H^\infty$. By Lemma 4.3, $b_n \in I$. Therefore $g - f_n \in I$. Hence

$$\begin{aligned} \operatorname{dist}(g, I) &= \operatorname{dist}(f_n, I) \\ &\le \operatorname{dist}(f_n, bH^\infty) \\ &\le K \sup |f_n(z_m)| \\ &\le K/n. \end{aligned}$$

Hence $g \in I$, as desired. □

We shall see that the conclusion of Theorem 4.1 may not hold if I is not closed.

In what follows, B denotes a closed subalgebra of L^∞ containing H^∞. As usual, QA_B denotes the algebra of bounded analytic functions whose complex conjugates lie in B; that is, $QA_B = H^\infty \cap \bar{B}$. In §3 we considered the case $B = H^\infty + C$. In what follows, we shall construct an example of a nonmaximal prime ideal in QA_B. In particular, when $B = L^\infty$ we construct a chain of prime ideals in H^∞. Let $x \in M(H^\infty) - \mathbf{D}$ be such that $x(b_0) = 0$ for some interpolating Blaschke product b_0. Let $I_0 = \{f \in QA_B: f = bg, \text{ where } g \in QA_B,\ b_0/b \in H^\infty, \text{ and } x(b) = 0\}$. It is shown in [20] that for any Blaschke product b such a g can be found.

THEOREM 4.4. *The set I_0 is a nonmaximal prime ideal in QA_B.*

Proof. Let f_1 and f_2 be elements of I_0. We need to show that $f_1 + f_2 \in I_0$. For $j = 1, 2$ let $f_j = b_j g_j$, where $b_0/b_j \in H^\infty$, $g_j \in QA_B$, and $x(b_j) = 0$. Let $A = Z(b_1) \cap Z(b_2)$. By our assumptions on b_j we have $x \in \overline{Z(b_1)} \cap \overline{Z(b_2)}$. By Theorem 4.2, the set $A = Z(b_1) \cap Z(b_2)$ is nonempty. Since $(Z(b_1) - A) \cap (Z(b_2) - A) = \varnothing$, by Theorem 4.2, $x \notin \overline{(Z(b_1) - A)} \cap \overline{(Z(b_2) - A)}$. It is now easy to show that $x \in \bar{A}$. Let b be the interpolating Blaschke product with zero set A. Then $b_j/b \in H^\infty$ for $j = 1, 2$. Thus $f_1 + f_2 = bh$ for some $h \in H^\infty$. Since $h = \bar{b} \cdot (f_1 + f_2)$, we see that $\bar{h} = b \cdot (\bar{f}_1 + \bar{f}_2) \in H^\infty \cdot B \subseteq B$. Thus $h \in H^\infty \cap \bar{B} = QA_B$. We must now show that I_0 is prime.

Let $f, g \in QA_B$ be such that $fg \in I_0$. Then $fg = bh$ for some $h \in QA_B$. Since $x \in \overline{Z(b)}$, we see that $x \in \overline{(Z(f) \cap Z(b))} \cup \overline{(Z(g) \cap Z(b))}$. We may assume that $x \in \overline{Z(f) \cap Z(b)}$. Let b_1 denote the Blaschke product with zero set $Z(f) \cap Z(b)$. Then $b/b_1 \in H^\infty$ and $x \in \overline{Z(b_1)}$, so $x(b_1) = 0$ and $f = b_1 h$ for some $h \in QA_B$. Thus $f \in I_0$. Since I_0 does not contain the outer function $x(z) - z$, I_0 is not maximal. □

To construct a chain of prime ideals in H^∞ contained in Ker x, we use a well-known construction [13]. Let b_0 be as above. Let b_n be the Blaschke product with zero set equal to $Z(b_0)$. The order of the zeros will be chosen as follows.

Choose $\{a_{m,1}\}$ so that $\sum_m a_{m,1}(1-|z_m|)<\infty$ and $a_{m,1}\to\infty$ as $m\to\infty$. Suppose that $\{a_{m,n-1}\}$ has been chosen. Let $\{a_{m,n}\}$ be a sequence such that $a_{m,n}\to\infty$ and $\sum_m(\prod_{k=1}^n a_{m,k})(1-|z_m|)<\infty$. The Blaschke product b_n is defined by

$$b_n(z)=\prod_{m=1}^{\infty}\left(\frac{z-z_m}{1-\bar z_m z}\right)^{N_{m,n}},$$

where $N_{m,n}=\prod_{k=1}^n a_{m,k}$.

Let $b_{n,k}$ denote a factor of b_n. Let $b_{0,k}$ denote the factor of b_0 with the same zero set as $b_{n,k}$. Define the ideals I_n, $n\geq 1$, as follows:

$$\begin{aligned}I_n=\{f\in H^\infty : f=b_{n,k}h,\ h\in H^\infty,\ x\in\overline{Z(b_{0,k})},\ b_n/b_{n,k}\in H^\infty, \text{ and}\\ \text{Ord } Z(b_{n,k};z_{m,k})=d_m \text{ Ord } Z(b_{n-1};z_{m,k}), \text{ where } d_m\to\infty \text{ as}\\ m\to\infty \text{ and } \{z_{m,k}\}_{m=1}^\infty=Z(b_{0,k})\}.\end{aligned}$$

THEOREM 4.5. *Each ideal I_n is a prime ideal in H^∞.*

Proof. Let $f_1,f_2\in I_n$. Let b_{n,k_1} and b_{n,k_2} be such that $f_j=b_{n,k_j}h$ for $j=1,2$. Since $x\in\overline{Z(b_{0,k_1})}\cap\overline{Z(b_{0,k_2})}$, by Theorem 4.2 the set $A=Z(b_{0,k_1})\cap Z(b_{0,k_2})$ is nonempty. As before, $x\in\bar A$. Let b_{n,k_0} be the Blaschke product with zero set A and Ord $Z(b_{n,k_0};z_{m,k})=\min_{j=1,2}$ Ord$(b_{n,k_j};z_{m,k})$. Then $f_1+f_2=b_{n,k_0}h$ for some $h\in H^\infty$, $b_n/b_{n,k_0}\in H^\infty$, and $x\in\overline{Z(b_{0,k_0})}$. Finally,

$$\text{Ord } Z(b_{n,k_0};z_{m,k})=\min_{j=1,2}\text{Ord } Z(b_{n,k_j};z_{m,k})=\min_{j=1,2} d_{m,j}\text{ Ord } Z(b_{n-1};z_{m,k}).$$

Letting $f_m=\min_{j=1,2}d_{m,j}$, we have Ord $Z(b_{n,k_0};z_{m,k})=f_m$ Ord $Z(b_{n-1};z_{m,k})$ and $f_m\to\infty$, as desired. Thus I_n is an ideal.

To see that I_n is prime, choose $f_1,f_2\in H^\infty$ such that $f_1f_2\in I_n$. Write $f_1f_2=b_{n,k}h$ with $b_{n,k}$ and h satisfying the required conditions. Write $b_{n,k}=b_{n,k_1}\cdot b_{n,k_2}$, where $f_j/b_{n,k_j}\in H^\infty$. By our assumptions, for each m with $b_{n,k}(z_m)=0$,

$$\text{Ord } Z(b_{n,k};z_{m,k})=d_m\text{ Ord } Z(b_{n-1};z_{m,k})\quad\text{with } d_m\to\infty.$$

Therefore, for each m with $b_{n,k}(z_m)=0$ we must have

$$(*)\qquad \text{Ord } Z(b_{n,k_j};z_{m,k})\geq\frac{d_m}{2}\text{ Ord } Z(b_{n-1};z_{m,k})$$

for $j=1$ or $j=2$. Let $\{z'_{m,k_j}\}$ denote the distinct zeroes of b_{n,k_j} satisfying $(*)$. Then $x\in\overline{Z(b_{0,k})}=\overline{\{z'_{m,k_1}\}}\cup\overline{\{z'_{m,k_2}\}}$. Thus, we may assume that $x\in\overline{\{z'_{m,k_1}\}}$. If b_{n,k_0} denotes the Blaschke product

$$\prod_{m=1}^{\infty}\left(\frac{z-z'_{m,k_1}}{1-\bar z'_{m,k_1}z}\right)^{N_m},$$

where $N_m=$ Ord $Z(b_{n,k_1};z_{m,k_1})$, then

$$\text{Ord } Z(b_{n,k_0}; z_{m,k_1}) \geq \frac{d_m}{2} \text{ Ord } Z(b_{n-1}; z_{m,k_1}).$$

Thus, the conditions on b_{n,k_0} are satisfied and therefore $f_1 = b_{n,k_0} h \in H^\infty$. □

In order to conclude that $\operatorname{Ker} x$ contains an infinite chain of prime ideals, we must still show that $I_n \subsetneqq I_{n-1}$. If $f \in I_1$ then $f = b_{1,k} h$ for some $h \in H^\infty$. Let $b_{0,k}$ denote the Blaschke product with the same zero set as $b_{1,k}$ and $\text{Ord } Z(b_{0,k}; z_m) = 1$. Then $x \in \overline{Z(b_{0,k})}$ and $b_0/b_{0,k} \in H^\infty$ and hence $f \in I_0$. Now for each n, $b_{n,k} \in I_n$. Let $b_{n-1,k}$ denote the Blaschke product with the same zero set as $b_{n,k}$ and $\text{Ord } Z(b_{n-1,k}; z_{m,k}) = \text{Ord } Z(b_{n-1}; z_{m,k})$. Thus $b_{n-1,k}$ satisfies $b_{n,k}/b_{n-1,k} \in H^\infty$, $b_{n-1}/b_{n-1,k} \in H^\infty$, and (by construction)

$$\text{Ord } Z(b_{n-1,k}; z_{m,k}) = \text{Ord } Z(b_{n-1}; z_{m,k}) = d_m \text{ Ord } Z(b_{n-2}; z_{m,k}),$$

where $d_m \to \infty$ as $m \to \infty$. Therefore $b_{n-1,k} \in I_{n-1}$ and hence $b_{n,k} \in I_{n-1}$. Thus $I_n \subseteq I_{n-1}$ for all n.

Note that for any $f \in I_n$, we have $\text{Ord } Z(f; z_m) > \text{Ord } Z(b_{n-1}; z_m)$ for some m. Therefore $b_{n-1} \notin I_n$. Thus $I_n \subsetneqq I_{n-1}$.

COROLLARY 4.6. *Let $y \in M(H^\infty + C) - M(L^\infty)$. Then* $\operatorname{Ker} y$ *contains a chain of prime ideals of infinite length.*

Proof. Choose an interpolating Blaschke product b such that $|y(b)| < 1$ ([8, p. 177], [21]). Thus $b \mid \operatorname{supp} m_y$ is nonconstant. Since $\operatorname{supp} m_y$ is an antisymmetric set for $H^\infty + C$ and $(1/b) \mid \operatorname{supp} m_y = \bar{b} \mid \operatorname{supp} m_y$, b is not invertible in $M(H^\infty \mid \operatorname{supp} m_y)$. Therefore there exists $x \in M(H^\infty \mid \operatorname{supp} m_y)$ such that $x(b) = 0$. Let I_0 be the ideal of Theorem 4.4. Choose $b_{1,k} \in I_1$. For every positive integer N, there exists a finite Blaschke product c such that $cb_{1,k}/b_{0,k}^N \in H^\infty$, where $b_{0,k}$ denotes the Blaschke product with the same zeros as $b_{1,k}$ all of order 1. Thus $cb_{1,k} = b_{0,k}^N d$ for some Blaschke product d. Since $x \in \overline{Z(b_{0,k})}$ and $\operatorname{supp} m_x \subseteq \operatorname{supp} m_y$ we must have $|y(b_{0,k})| < 1$. Thus

$$|y(b_{1,k})| = |y(cb_{1,k})| = |y(b_{0,k}^N d)| \leq |y(b_{0,k})|^N.$$

Therefore $y(b_{1,k}) = 0$. Hence $I_1 \subset \operatorname{Ker} y$. Thus $\operatorname{Ker} y$ contains a chain of prime ideals of infinite length. □

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