SOME SEIFERT FIBER SPACES WHICH ARE BOUNDARIES

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Hamrick and Royster [6] proved that all compact flat manifolds are boundaries. Their argument uses a refined version of a theorem of R. Stong together with M. Gordon's result [5]. Recently this was generalized to some almost flat manifolds by Farrell and Zdravkovska [4].

In this article we shall prove certain Seifert fiber spaces are boundaries. This class of Seifert fiber spaces contains all flat manifolds, those almost flat manifolds covered by the first part of [4], and more importantly, all the manifolds of non-positive sectional curvature satisfying Assumption B below.

We formulate the generalization of Stong's theorem [6] in a more general setting so that we can apply it to Seifert fiber spaces. Then we use the same argument as in [6] and [4]. An important point is that the Seifert fiber structure gives rise to a free $(\mathbf{Z}_2)^n$ action on a finite covering of the manifold.

In this paper, a Seifert fibering $M \to B$ will mean a smooth closed manifold M with an injective Seifert fiber structure where the typical fiber is a flat torus T^n. More precisely, M is a smooth closed manifold such that

(i) $\pi_1 M$ contains a normal subgroup $\mathbf{Z}^n$ $(n>0)$,
(ii) there exists $\mathbf{R}^n \subset \mathrm{Diffeo}(\tilde{M})$ containing $\mathbf{Z}^n$ as a uniform lattice, and
(iii) $\mathbf{R}^n$ is normalized by $\pi_1 M$.

Of course, we are considering $\pi_1 M$ as a subgroup of $\mathrm{Diffeo}(\tilde{M})$ where $\tilde{M}$ is the universal covering of M. Generally, the $\mathbf{R}^n$-action on $\tilde{M}$ does not yield a torus action on M, but the universal covering $\tilde{M}$ splits as a direct product $\mathbf{R}^n \times W$, where W is a simply connected smooth manifold on which $Q = \pi_1 M/\mathbf{Z}^n$ acts properly discontinuously with compact quotient. Thus, $B = Q \backslash W$ and

$$M = \pi_1 M \backslash (\mathbf{R}^n \times W) = Q \backslash (T^n \times W).$$

Note that even though Q acts on $T^n \times W$ as a group of covering transformations, it does not act freely on the W-factor. In general, the base space B is an orbifold where the fibers over regular (= unbranched) points of B are called typical. Otherwise, they are called singular. Singular fibers are finitely covered by the typical fiber T^n and are flat Riemannian manifolds.

We shall denote $\pi_1 M$ simply by π. $C_\pi(\mathbf{Z}^n)$ denotes the centralizer of $\mathbf{Z}^n$ in π. We make two assumptions as follows.

ASSUMPTION A. $C_\pi(\mathbf{Z}^n)$ has finite index in π.

ASSUMPTION B. $C_\pi(\mathbf{Z}^n)/\mathbf{Z}^n$ has no 2-torsion.

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The first assumption means that $\mathbf{Z}^n$ is almost central in π. The finite covering $\hat{M}$ of M with $\pi_1\hat{M}=C_\pi(\mathbf{Z}^n)$ has a Seifert fiber structure nicer than that of M. Let $\hat{Q}=C_\pi(\mathbf{Z}^n)/\mathbf{Z}^n$. Then $\hat{M}=\hat{Q}\backslash(T^n\times W)$ has tori for all of its singular fibers. The isotropy groups $\hat{Q}_w$ are all finite and act on T^n as translations.

The first factor of $T^n\times W$ yields an action of T^n on $\hat{M}$ (notice that M itself does not admit a T^n action in general). Inside the T^n action, there is a $(\mathbf{Z}_2)^n$ action on $\hat{M}$. The second assumption is really the requirement that the $(\mathbf{Z}_2)^n$ action on $\hat{M}$ be free. We now state our results.

MAIN THEOREM. *Let M be a Seifert fibered manifold satisfying Assumptions* A *and* B *above. If $\pi_1 M$ has no 2-torsion, then M is a boundary.*

One can apply the Main Theorem to some nice Seifert fibered manifolds which satisfy Assumptions A and B. We state a few cases.

COROLLARY 1 [6]. *Every compact flat Riemannian manifold is a boundary.*

COROLLARY 2 [4]. *Let M be a compact infranilmanifold. If a 2-Sylow subgroup of the holonomy group acts effectively on the center of the nilradical of $\pi_1 M$, then M is a boundary.*

COROLLARY 3. *Let M be a closed manifold with non-positive sectional curvature. Suppose the maximal normal abelian subgroup of $\pi_1 M$ is non-trivial, say $\mathbf{Z}^n$, and $C_{\pi_1 M}(\mathbf{Z}^n)/\mathbf{Z}^n$ has no 2-torsion. Then M is a boundary.*

COROLLARY 4. *Let M be a compact complete affine manifold. Suppose $\pi_1 M$ has a non-trivial radical, and hence $\pi_1 M$ contains a normal $\mathbf{Z}^n$ $(n>0)$. If Assumptions* A *and* B *are satisfied, then M is a boundary.*

In order to prove the Main Theorem, the following facts are needed.

LEMMA 1. *Let G be a finite 2-group with a faithful representation in* $\mathrm{GL}(n,\mathbf{Z})$. *Consider the short exact sequence of G-modules $0\to\mathbf{Z}^n\to(\frac{1}{2}\mathbf{Z})^n\to(\mathbf{Z}_2)^n\to 0$. Let $\Sigma_G=H^0(G;(\mathbf{Z}_2)^n)$, and let $0\to\mathbf{Z}^n\to\tilde{\Sigma}_G\to\Sigma_G\to 0$ be the exact sequence above restricted to $\Sigma_G\subset(\mathbf{Z}_2)^n$. Let $\varphi:\Sigma_G\to G$ be any injective homomorphism. If*

$$(1+\varphi(\bar{s}))(s)\neq 0 \quad \textit{for every } s\in\tilde{\Sigma}_G-\mathbf{Z}^n$$

(where $\bar{s}\in\Sigma_G$ is the image of $s\in\tilde{\Sigma}_G$), then $\varphi(\Sigma_G)$ is not in $Z(G)$, the center of G.

LEMMA 2. *Let $\hat{M}\to M$ be a regular covering of a smooth closed manifold M with deck transformation group G a finite 2-group. Suppose $\hat{M}$ has a free $(\mathbf{Z}_2)^n$ action which is normalized by the G action in* $\mathrm{Diffeo}(\hat{M})$. *For any subgroup H of G, let*

$$\Sigma_H=\{\bar{s}\in(\mathbf{Z}_2)^n:\bar{s}g=g\bar{s}\ \textit{for all } g\in H\}.$$

For any injective homomorphism $\varphi:\Sigma_H\to H$, define

$$E_\varphi=\{\hat{x}\in\hat{M}:\bar{s}\hat{x}=\varphi(\bar{s})\hat{x}\ \textit{for all } \bar{s}\in\Sigma_H\}.$$

Suppose, for any subgroup $H<G$ and any injective homomorphism $\varphi:\Sigma_H\to H$, that E_φ is empty whenever $\varphi(\Sigma_H)\subset Z(H)$. Then M is a boundary.

Lemma 1, which is purely algebraic, is essentially due to M. Gordon [5]. Lemma 2 is a generalization of Theorem 10 in [6]. The proof of Lemma 9 and Theorem 10 of [6] work in this more general situation under the hypothesis: For any subgroup $H<G$ and any injective homomorphism $\varphi: \Sigma_H \to H$, E_φ is empty whenever $\varphi(\Sigma_H) \subset Z(H)$. One replaces T^n by our $\hat{M}$ in their argument.

Proof of the Main Theorem. Consider the commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc} & & & & 1 & & 1 & & \\ & & & & \downarrow & & \downarrow & & \\ 1 & \to & \mathbf{Z}^n & \to & \hat{\pi} & \to & \hat{Q} & \to & 1 \\ & & \downarrow = & & \downarrow & & \downarrow & & \\ 1 & \to & \mathbf{Z}^n & \to & \pi & \to & Q & \to & 1, \\ & & & & \downarrow & & \downarrow & & \\ & & & & G & \overset{=}{\to} & G & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 1 & & 1 & & \end{array}$$

where $\hat{\pi} = C_\pi(\mathbf{Z}^n)$. Let $\hat{M}$ be the regular covering of M with $\pi_1 \hat{M} = \hat{\pi}$. Then the deck transformation group G is finite by Assumption A.

From the Seifert fiber structure of M, we know that π sits in the fiber-preserving homeomorphism group $C^\infty(W, \mathbf{R}^n) \circ \{\mathrm{GL}(n, \mathbf{R}) \times \mathrm{Diffeo}(W)\}$ (see [9] for a proof of this fact), where $C^\infty(W, \mathbf{R}^n)$ is the space of all smooth maps of W into $\mathbf{R}^k$. The group law of the semi-direct product is

$$(\lambda, g, h)(\lambda', g', h') = (\lambda + g\lambda' h^{-1}, gg', hh')$$

and acts on $\mathbf{R}^n \times W$ by

$$(\lambda, g, h)(x, w) = (g(x) + \lambda h(w), hw)).$$

Elements of $\hat{\pi} = C_\pi(\mathbf{Z}^n)$ are of the form $(\lambda, 1, h)$.

Any $t \in \mathbf{R}^n$ can be viewed as a constant map $W \to \mathbf{R}^n$ via $t = (t, 1, 1)$ so that $(t, 1, 1)(\lambda, g, h)(-t, 1, 1) = (t - g(t), 1, 1)(\lambda, g, h)$. Therefore, $\mathbf{R}^n$ centralizes $\hat{\pi}$ (since elements of $\hat{\pi}$ have the second slot 1). Hence $\hat{M}$ admits a T^n-action. The $(\mathbf{Z}_2)^n$ action contained in T^n is free on $\hat{M}$ because $\hat{Q}$ has no 2-torsion by Assumption B.

Observe that the composite

$$\pi \to C^\infty(W, \mathbf{R}^n) \circ \{\mathrm{GL}(n, \mathbf{R}) \times \mathrm{Diffeo}(W)\} \to \mathrm{GL}(n, \mathbf{R})$$

comes from the automorphisms of $\mathbf{Z}^n$ induced by the conjugation by elements of π. Since $\hat{\pi} = C_\pi(\mathbf{Z}^n)$ is the kernel of this map, $\pi \to \mathrm{GL}(n, \mathbf{Z})$ factors through the injective homomorphism $G \to \mathrm{GL}(n, \mathbf{Z})$.

As $\hat{M}$ has a free $(\mathbf{Z}_2)^n$-action, it is a boundary [2]. Therefore, as far as the bounding problem is concerned, we may assume that G is a 2-group by the standard Stiefel–Whitney number argument. From now on, we assume that G is a 2-group.

Consider the G-module $\mathbf{Z}^n$, and form the short exact sequence of G-modules, $0 \to \mathbf{Z}^n \to \tilde{\Sigma}_G \to \Sigma_G \to 0$, as in Lemma 1. Then

$$\tilde{\Sigma}_G = \{s \in (\tfrac{1}{2}\mathbf{Z})^n : s - g(s) \in \mathbf{Z}^n \text{ for all } g \in G \subset \mathrm{GL}(n, \mathbf{Z})\},$$

where $\mathbf{Z}^n = \tilde{\Sigma}_G \subset \pi$ and $\tilde{\Sigma}_G \subset \mathbf{R}^n$ which acts on $\mathbf{R}^n \times W$ by translation along the first factor. Therefore, Σ_G is the maximal subgroup of $(\mathbf{Z}_2)^n$ centralizing the deck transformation group G, so it acts on M. Σ_G is effective since $\tilde{\Sigma}_G \subset$ normalizer of π.

We would like to show that the covering $\hat{M} \to M$ and the action of $(\mathbf{Z}_2)^n$ on $\hat{M}$ satisfy the conditions in Lemma 2. We prove the condition is true for G itself. For a subgroup H of G, the proof is exactly the same.

Suppose $\varphi : \Sigma_G \to G$ is an injective homomorphism with E_φ non-empty. We claim that $\varphi(\Sigma_G) \not\subset Z(G)$. According to Lemma 1, it will be sufficient to show that $(1 + \varphi(\bar{s}))(s) \neq 0$ for all $s \in \tilde{\Sigma}_G - \mathbf{Z}^n$.

Let Σ_G^* be the group of all liftings of Σ_G to $\tilde{M}$ so that $1 \to \pi \to \Sigma_G^* \to \Sigma_G \to 1$ is exact. Since $\tilde{\Sigma}_G \subset \mathbf{R}^n$ and $\mathbf{R}^n$ is normalized by π (see condition (iii) in the definition of the Seifert fibering), $\Sigma_G^* = \pi \cdot \tilde{\Sigma}_G$,

We choose a point $\hat{x} \in E_\varphi$ and a preimage $\tilde{x} \in \tilde{M}$. This choice induces a splitting of the short exact sequence $1 \to \pi \to \Sigma_G^* \to \Sigma_G \to 1$ as follows: For $\bar{s} \in \Sigma_G$, choose a preimage $s \in \tilde{\Sigma}_G$. Then there is a unique $\sigma \in \pi$ such that $s\tilde{x} = \sigma\tilde{x}$. We now define the desired splitting $\tilde{\psi} : \Sigma_G \to \Sigma_G^*$ by $\tilde{\psi}(\bar{s}) = \sigma^{-1}s$. It is not hard to see that $\tilde{\psi}$ is independent of the choices of s and $\tilde{x}$, and that it is an injective homomorphism.

There is a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \hat{\pi} & \xrightarrow{=} & \hat{\pi} & & & & \\
 & & \downarrow & & \downarrow & \overset{\tilde{\psi}}{\curvearrowleft} & & & \\
1 & \to & \pi & \to & \Sigma_G^* & \rightleftarrows & \Sigma_G & \to & 1 \\
 & & \downarrow & & \downarrow & \overset{\psi}{\curvearrowleft} & \downarrow = & & \\
1 & \to & G & \to & G \times \Sigma_G & \rightleftarrows & \Sigma_G & \to & 1. \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

Clearly $\Sigma_G^* / \hat{\pi} = G \times \Sigma_G$ is the group of liftings of the action (Σ_G, M) to $\hat{M}$. Let ψ be the composite $\Sigma_G \xrightarrow{\tilde{\psi}} \Sigma_G^* \to G \times \Sigma_G$. Then ψ is related to φ by $\psi(\bar{s}) = \varphi(\bar{s})^{-1}\bar{s}$. Therefore, $\psi : \Sigma_G \to G \times \Sigma_G$ is the splitting induced by the fixed point $x \in M$.

Recall that $\tilde{\Sigma}_G \to \Sigma_G$ maps s to $\bar{s}$ and has kernel $\mathbf{Z}^n \subset \hat{\pi}$. Since $\tilde{\psi}(\bar{s}) = \sigma^{-1}s$ and $\psi(\bar{s}) = \varphi(\bar{s})^{-1}\bar{s}$, σ maps to $\varphi(\bar{s})$ by the homomorphism $\pi \to G$. Therefore, if $\sigma = (\lambda, g, h)$, then $g = \varphi(\bar{s})$ (recall that G acts on $\mathbf{Z}^n$ faithfully).

Now $\tilde{\psi}(\bar{s})^{-1} = s^{-1}\sigma = (\lambda - s, \varphi(\bar{s}), h)$. Since $\tilde{\psi}(\bar{s})^2 = 1$, we have $\lambda + \varphi(\bar{s})\lambda h^{-1} = (1 + \varphi(\bar{s}))(s)$. Therefore

$$\sigma^2 = (\lambda + \varphi(\bar{s})\lambda h^{-1}, \varphi(\bar{s})^2, h^2) = ((1 + \varphi(\bar{s}))(s), 1, 1).$$

However, π does not have an element of order 2. Therefore, $(1 + \varphi(\bar{s}))(s) \neq 0$ for all $s \in \tilde{\Sigma}_G - \mathbf{Z}^n$. Now by Lemma 1, $\varphi(\Sigma_G) \not\subset Z(G)$. Thus the condition in Lemma 2 is satisfied. We conclude that M is a boundary. □

Proof of Corollaries 1 *and* 2. When M is flat, the nilradical of $\pi_1 M$ is, in fact, the maximal normal abelian subgroup so that the holonomy group acts faithfully. Therefore, Corollary 1 is a special case of Corollary 2.

Suppose M is infranil. Let $\mathbf{Z}^n$ be the center of the nilradical N of $\pi_1 M$. Then $n > 0$ and M has a Seifert fiber structure with typical fiber T^n (see [8] for details). It is known [1] that the holonomy $G = \pi_1 M/N$ is finite. We can assume without loss of generality that all elements of G have order a power of 2. Because G acts effectively on $\mathbf{Z}^n$, the centralizer of $\mathbf{Z}^n$ in $\pi_1 M$, $C_{\pi_1 M}(\mathbf{Z}^n)$, is precisely N. Note that $N/Z(N)$ is torsion-free. Therefore, M satisfies the assumptions of the theorem, and hence it is a boundary. □

Proof of Corollary 3. It is known [12] that every solvable subgroup of $\pi = \pi_1 M$ is a Bieberbach group and hence is finitely generated. Using this fact, one can show that π has a unique maximal normal abelian subgroup, say $\mathbf{Z}^n$. To verify that Assumption A is true, one can proceed as follows. Let $G = \pi/C_\pi(\mathbf{Z}^n)$. We claim that each element of G has a finite order. Suppose $\alpha \in \pi$ is such that its image $\bar{\alpha}$ in G has infinite order. Then the subgroup π' of π generated by $\mathbf{Z}^n$ and α is solvable and hence is a Bieberbach group of rank $n+1$. Therefore π' has a normal abelian subgroup of rank $n+1$. This implies that α^r commutes with $\mathbf{Z}^n$ for some $r > 0$, which in turn implies that $\bar{\alpha}$ has finite order. This contradiction shows that every element of G has finite order. To conclude that G is finite, note that the homomorphism $\pi \to \mathrm{GL}(n, \mathbf{Z})$, induced by conjugation, factors through G. In fact, the induced homomorphism $G \to \mathrm{GL}(n, \mathbf{Z})$ is injective. Thus, G is a finitely generated subgroup of $\mathrm{GL}(n, \mathbf{Z})$ whose elements have finite order. By Selberg's lemma, we conclude that G is finite.

By the flat torus theorem in [10], M has a Seifert fiber structure with typical fiber T^n. Now we apply the main theorem. □

Proof of Corollary 4. By L. Auslander [1, Theorem 3], such a manifold M has a Seifert fiber structure.

EXAMPLES. All manifolds mentioned in the corollaries are aspherical. However, our main theorem does not require the asphericity of the space. Following are examples which are *not* aspherical.

Take any manifold X from the corollaries. Let Y be a closed simply connected manifold on which the finite group $G = \pi/C_\pi(\mathbf{Z}^n)$ acts smoothly. (This action may not be effective or free.) Let G act on $\hat{X} \times Y$ diagonally. Then $M = (\hat{X} \times Y)/G$ is a Y-bundle over X, and is not aspherical. By the Main Theorem, M is a boundary.

Yau's solution to the Calabi conjecture and work of Cheegar–Gromoll and Fischer–Wolf yield a structure theorem for compact Kähler manifolds with nonvanishing first Betti number and vanishing first Chern class. (See, e.g., [7, p. xiii].) That these Kähler manifolds are Seifert manifolds satisfying Assumption A is a direct consequence of the structure theorems. Therefore, if $\pi_1 M$ also has no 2-torsion and satisfies Assumption B, then M is a boundary.

REMARKS. While Assumptions A and B guarantee bounding, they are by no means necessary. Let us examine a few examples.

1. The classical closed 3-dimensional Seifert manifolds are all boundaries. If $\pi_1 M$ is infinite, then Assumption A always holds. However, B may fail to hold in many cases including all the types mentioned in the last three corollaries.

2. The Dold manifold in dimension 5, $M = S^1 \times_{\mathbf{Z}_2} \mathbf{C}P^2$. The free involution acts diagonally on the product space $S^1 \times \mathbf{C}P^2$ by translating in the S^1 factor and by sending $[z_1, z_2, z_3]$ to $[\bar{z}_1, \bar{z}_2, \bar{z}_3]$ on the $\mathbf{C}P^2$ factor. There is a (homologically injective) S^1-action on M induced by the first factor. It has $\mathbf{Z}_2$ isotropy with fixed point set homeomorphic to $\mathbf{R}P^2$, the fixed point set of the $\mathbf{Z}_2$ action on $\mathbf{C}P^2$. The Seifert structure is induced from the S^1-action. Therefore $\pi_1 S^1 = \mathbf{Z}$ is a subgroup of $\pi_1 M \cong \mathbf{Z}$ of index 2. Assumption A holds, but $C_\pi(\mathbf{Z})/\mathbf{Z} \cong \mathbf{Z}_2$. It is well known that M does not bound [3].

3. If one takes $\mathbf{R}P_3 \# \mathbf{R}P_3$, this is a Seifert fibering over $\mathbf{R}P_2$. Its fundamental group is the semi-direct product of $\mathbf{Z}$ with $\mathbf{Z}/2\mathbf{Z}$. $C_\pi(\mathbf{Z})/\mathbf{Z} = 1$, but π_1 contains 2-torsion. Of course, $\mathbf{R}P_3 \# \mathbf{R}P_3$ is a boundary.

4. Closely associated with each Seifert manifold M there is a (usually infinite) family of other distinct Seifert manifolds $\{M'\}$ called *Seifert relatives of M*. If M satisfies Assumptions A and B of the theorem, so will all the relatives M' of M satisfy Assumptions A and B. Thus, if $\pi_1(M')$ has no 2-torsion we can be assured that M' will bound. In particular, *if M is aspherical satisfying Assumptions* A *and* B *then all Seifert relatives M' of M are aspherical and bound.*

Let us recall that $\tilde{M} = \mathbf{R}^n \times W$ and $\pi = \pi_1(M)$ acts as covering transformations with the normal $\mathbf{Z}^n$ acting by translations on the first factor and the quotient $\pi/\mathbf{Z}^n = Q$ inducing a properly discontinuous (but not necessarily free) action on W with compact quotient the orbifold $B = Q\backslash W$. M Seifert-fibers over B with generic (or typical) fiber T^n. The Seifert construction is a reversal of this process (see [9]). It begins with an arbitrary group extension,

$$1 \to \mathbf{Z}^n \to \pi' \to Q \to 1 : a \in H^2_\phi(Q; \mathbf{Z}^n),$$

and embeds π' into $C^\infty(W, \mathbf{R}^n) \circ \{\mathrm{GL}(n, \mathbf{R}) \times \mathrm{Diffeo}(W)\}$, the group of fiber preserving diffeomorphisms of $\mathbf{R}^n \times W$. For the extension, $a : 1 \to \mathbf{Z}^n \to \pi_1 M \to Q \to 1$, the construction will recreate M, and for each of the other extensions a' it constructs the Seifert fibering $M' = \pi' \backslash (\mathbf{R}^n \times W) \to B$. It is easy to check that π' will satisfy Assumptions A and B if (and only if) $\pi = \pi_1 M$ does. But M' may be only an orbifold. However, if π' (which does act properly discontinuously) acts freely on $\mathbf{R}^n \times W$, then M' will be a Seifert *manifold.* Such a manifold M' is called a *Seifert relative of the Seifert manifold M.* An infinite number of distinct Seifert relatives of M appear if (and only if) $H^2_\phi(Q; \mathbf{Z}^n)$ is itself an infinite group. (This is a non-trivial consequence of an examination of a certain spectral sequence associated with the Seifert construction.)

As an illustration, if M is a closed manifold with non-positive sectional curvature, as in Corollary 3, then only a finite number of Seifert relatives of M will have non-positive sectional curvature. On the other hand, as the group $H^2_\phi(Q; \mathbf{Z}^n)$ will usually be infinite, all the other Seifert relatives will *not* have non-positive sectional curvature but, by our remarks, will still be boundaries.

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