CHARACTERISTIC CLASSES FOR SYMPLECTIC FOLIATIONS

Chal Benson

1. Introduction. A transverse symplectic structure for a foliation is given by a closed 2-form that vanishes on the leaves and is non-degenerate in the transverse direction. That is, if $\mathfrak{F}$ is a codimension $2q$ foliation on M, then $\omega \in \Omega^2(M)$ is a transverse symplectic structure if

$$(1.1)\qquad \begin{aligned} &\text{(a)}\ d\omega = 0,\\ &\text{(b)}\ i_V\omega = 0 \text{ for all vector fields } V \text{ tangent to } \mathfrak{F},\\ &\text{(c)}\ \omega^q \neq 0 \text{ at all points of } M. \end{aligned}$$

Equivalently, $(\mathfrak{F}, \omega)$ is given by a Haefliger cocycle of submersions into a symplectic manifold N^{2q} with transition germs preserving the symplectic structure [7].

Haefliger has introduced a classifying space, $B\Gamma_{\mathrm{Sp}(q)}$, for codimension $2q$ symplectic foliations [7]. A related space is $B\bar{\Gamma}_{\mathrm{Sp}(q)}$, the homotopy fibre of the map $\nu : B\Gamma_{\mathrm{Sp}(q)} \to B\,\mathrm{Sp}(q) = BU(q)$ classifying the normal bundle to the universal $\Gamma_{\mathrm{Sp}(q)}$-structure. This space classifies (symplectically) framed symplectic foliations. The central goal of this paper is to obtain new results on the algebraic topology of these spaces. This is done in Sections 4 and 5 where it is shown that certain homology and homotopy groups for $B\Gamma_{\mathrm{Sp}(q)}$ and $B\bar{\Gamma}_{\mathrm{Sp}(q)}$ are enormous.

The theorems we obtain are similar in flavor to known results on the classifying spaces for real, Riemannian, and complex foliations. See for example [9]. Such theorems are biproducts of the study of characteristic classes for these types of foliations. The characteristic classes furnished by the usual construction in the symplectic context (outlined in Section 2) are difficult to analyze and there are no known non-trivial examples for them. These difficulties are circumvented here by the introduction of a new and more manageable family of invariants in Section 2.

The characteristic classes are constructed using differential geometric methods which do not involve properties of Haefliger's classifying space. The final results follow in part from explicit computations for certain examples described in Section 3.

A special class of symplectic foliations are the Kähler foliations. These are modeled on a Kähler manifold N^{2q} where the transition germs preserve the Kähler structure. Such foliations are studied by Matsuoka and Morita who consider characteristic classes related to the ones here [15].

In this paper, all manifolds and foliations are of type C^∞. Cohomology groups are taken with real coefficients unless otherwise specified. When M is a manifold, $H^\bullet(M)$ should be thought of as the (de Rahm) cohomology of the complex $(\Omega^\bullet(M), d)$ of smooth forms on M.

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2. Definition of the characteristic classes. There are several alternative approaches to the construction of characteristic classes for foliations. The classes considered here will be obtained by extending the Weil algebra construction as developed by Kamber and Tondeur. We begin by briefly reviewing this theory in the context of symplectic foliations. For details, we refer the reader to [13].

We write $\mathrm{Sp}(q)$ for the real symplectic group $\mathrm{Sp}(q,\mathbf{R})\subset \mathrm{GL}(2q,\mathbf{R})$ and $\mathrm{sp}(q)$ for its Lie algebra. The symplectic Weil algebra is the DG algebra given by

$$W(\mathrm{sp}(q))=\wedge(\mathrm{sp}(q)^*)\otimes S(\mathrm{sp}(q)^*),$$

a tensor product of exterior and symmetric algebras. The truncated algebra $W(\mathrm{sp}(q))_{2q}$ is the quotient of $W(\mathrm{sp}(q))$ by the ideal generated by

$$\bigoplus_{j>2q}(1\otimes S^j(\mathrm{sp}(q)^*)).$$

If $(\mathfrak{F},\omega,s)$ is a framed symplectic foliation of codimension $2q$ on M (here s denotes a framing of the symplectic normal bundle $\mathrm{Sp}(\nu\mathfrak{F})$) then we obtain a characteristic homomorphism

$$\Delta_*(\mathfrak{F},\omega,s)\colon H^{\boldsymbol{\cdot}}(W(\mathrm{sp}(q))_{2q})\to H^{\boldsymbol{\cdot}}(M).$$

The map $\Delta_*(\mathfrak{F},\omega,s)$ can be realized as a map $W(\mathrm{sp}(q))_{2q}\to\Omega^{\boldsymbol{\cdot}}(M)$ by using any Bott connection in $\mathrm{Sp}(\nu\mathfrak{F})$.

The map $\Delta_*(\mathfrak{F},\omega,s)$ and the cohomology of $W(\mathrm{sp}(q))_{2q}$ can be described simply in terms of a cohomologically equivalent subcomplex

$$\hat{A}=\wedge(y_1,y_2,\ldots,y_q)\otimes\mathbf{R}[e_1,e_2,\ldots,e_q]_{2q}.$$

Here e_i is a symbol of degree $4i$ with $de_i=0$, and y_i has degree $(4i-1)$ with $dy_i=e_i$. The subscript $2q$ denotes truncation by the elements of degree greater than $4q$.

Let β be any Bott connection in $\mathrm{Sp}(\nu\mathfrak{F})$. Chern–Weil theory yields canonical representing forms $e_i(\beta)\in\Omega^{4i}(M)$ for the symplectic Pontrjyagin classes and forms $y_i(\beta,s)\in\Omega^{4i-1}(M)$ with $dy_i(\beta,s)=e_i(\beta)$. The forms $e_i(\beta)$ are determined by applying the invariant polynomials $e_i\colon\mathrm{sp}(q)\to\mathbf{R}$, characterized by the identity

$$\det\left(I-\frac{t}{2\pi i}A\right)=1+\sum_{j=1}^{q}(-1)^j e_j(A)t^{2j}, \tag{2.1}$$

to the curvature tensor for β. This determines a map of DG algebras

$$\Delta(\beta)\colon\hat{A}\to\Omega^{\boldsymbol{\cdot}}(M),$$

which realizes $\Delta_*(\mathfrak{F},\omega,s)$ on cohomology.

A vector space basis for $H^{\boldsymbol{\cdot}}(W(\mathrm{sp}(q))_{2q})$ is given by the "Vey basis" of classes

$$y_Ie_J=y_{i_1}\wedge y_{i_2}\wedge\cdots\wedge y_{i_s}\otimes e_1^{j_1}e_2^{j_2}\cdots e_q^{j_q}$$

whose indices (I, J) satisfy certain "admissibility conditions". Adopting the notation

$$|J| = |(j_1, j_2, \ldots, j_q)| = \sum_{l=1}^{q} l j_l, \tag{2.2}$$

these can be written as

$$\begin{aligned} &\text{(a)}\ \ I \neq \phi,\ 1 \le i_1 < i_2 < \cdots < i_s \le q, \\ &\text{(b)}\ \ |J| \le q, \\ &\text{(c)}\ \ i_1 + |J| > q, \\ &\text{(d)}\ \ i_1 \le \text{the index of the first non-zero } j_l. \end{aligned} \tag{2.3}$$

It follows immediately from the Heitsch rigidity theorem [8] that all the classes $y_I e_J$ are invariant under deformations through framed foliations. Moreover, there are no known examples showing any of these classes to be non-trivial. These shortcomings will be circumvented here by considering an easy extension of the theory outlined above.

We start by adding an extra 2-dimensional generator ω to the Weil algebra to form $W(\mathrm{sp}(q), 2) = W(\mathrm{sp}(q)) \otimes S(\omega)$. The differential and $\mathrm{sp}(q)$-operations on $W(\mathrm{sp}(q))$ are all extended to $W(\mathrm{sp}(q), 2)$ by sending ω to 0. The truncated algebra $W(\mathrm{sp}(q), 2)_{2q}$ is obtained by dividing out the ideal generated by

$$\bigoplus_{j+2k>2q} (1 \otimes S^j(\mathrm{sp}(q)^*) \otimes \omega^k).$$

As before, we consider a framed symplectic foliation $(\mathfrak{F}, \omega, s)$ on M with β a Bott connection in $\mathrm{Sp}(\nu\mathfrak{F})$. The resulting map $W(\mathrm{sp}(q)) \to \Omega^{\bullet}(M)$ is extended to $W(\mathrm{sp}(q), 2)$ by sending the extra generator ω to the transverse symplectic form $\omega \in \Omega^2(M)$. This gives a map of DG algebras which factors through $W(\mathrm{sp}(q), 2)_{2q}$. One can see this by combining the usual proof of Bott's vanishing theorem with the fact that ω is $\mathfrak{F}$-transverse [condition (1.1)(b)]. We obtain a characteristic homomorphism $\Delta_*(\mathfrak{F}, \omega, s): H^{\bullet}(W(\mathrm{sp}(q), 2)_{2q}) \to H^{\bullet}(M)$, which is our extension of the usual construction.

The techniques described in [13] again allow one to replace $W(\mathrm{sp}(q), 2)_{2q}$ by a cohomologically equivalent subcomplex $\wedge(y_1, \ldots, y_q) \otimes \mathbf{R}[e_1, \ldots, e_q, \omega]_{2q}$. Here the y's and e's are as before and the $2q$ subscript denotes truncation by the monomials $e_J \omega^k$ with $|J| + k > q$. In terms of this complex, the map $\Delta_*(\mathfrak{F}, \omega, s)$ is given on the cochain level by using a Bott connection to construct forms $e_i(\beta)$ and $y_i(\beta, s)$ as before and by sending the generator ω to $\omega \in \Omega^2(M)$. The Vey basis for $H^{\bullet}(W(\mathrm{sp}(q), 2)_{2q})$ is given by the classes $\omega, \omega^2, \ldots, \omega^q$ together with the classes $y_I e_J \omega^k$ satisfying

$$\begin{aligned} &\text{(a)}\ \ I \neq \phi, \\ &\text{(b)}\ \ |J| + k \le q, \\ &\text{(c)}\ \ i_1 + |J| + k > q, \\ &\text{(d)}\ \ i_1 \le \text{the index of the first non-zero } j_l. \end{aligned} \tag{2.4}$$

The characteristic classes $y_I e_J \omega^k$ with $k > 0$ will be called *ω-classes*. These are new characteristic invariants for framed symplectic foliations which supplement the old invariants $y_I e_J$. It is clear that the classes $\Delta_*(\mathfrak{F}, \omega, s)(y_I e_J \omega^k) \in H^\cdot(M)$ are natural under pull-backs by transverse maps and are concordance invariants of $(\mathfrak{F}, \omega, s)$. It follows that $\Delta_*(\mathfrak{F}, \omega, s)$ factors through a universal homomorphism

$$\Delta_*: H^\cdot(W(\mathrm{sp}(q), 2)_{2q}) \to H^\cdot(B\bar{\Gamma}_{\mathrm{Sp}(q)}).$$

One might hope to obtain characteristic classes for (unframed) symplectic foliations by considering the complex $W(\mathrm{sp}(q), 2, U(q))_{2q}$ of $U(q)$-basic elements in $W(\mathrm{sp}(q), 2)_{2q}$. Here we are identifying the unitary group $U(q)$ with the maximal compact subgroup $O(2q) \cap \mathrm{Sp}(q)$ of $\mathrm{Sp}(q)$. However, $H^\cdot(W(\mathrm{sp}(q), 2, U(q))_{2q}$ contains only classes of the form $e_J \omega^k$ and no exotic elements. This results from the fact that the restriction map on invariant polynomials $I(\mathrm{Sp}(q)) \to I(U(q))$ is injective [13].

The construction here is closely related to the construction in [15] and [4] of characteristic classes for (unitarily) framed Kähler foliations and to the measure classes for $\mathrm{SL}(q)$-foliations studied in [10] and [11]. The classes for framed Kähler foliations are represented by closed forms $u_I \wedge \tilde{c}_J \wedge \omega^k$. Here ω denotes the transverse Kähler 2-form and $u_I, \tilde{c}_J$ arise from the Chern–Weil construction applied to the Chern polynomials, using the canonical Bott connection given locally by pulling back the Hermitian connection on the model manifold. If a framed Kähler foliation is viewed as a framed symplectic foliation, then the symplectic characteristic classes are determined by these Kähler classes using the identities

$$e_i = \sum_{j+l=2i} (-1)^l \tilde{c}_j \tilde{c}_l$$

and

$$y_i = \begin{cases} 2u_{2i} & \text{for } i \leq \lfloor q/2 \rfloor \\ 0 & \text{for } i > \lfloor q/2 \rfloor . \end{cases} \tag{2.5}$$

We conclude this section by remarking that the ω-classes arise naturally in the Gelfand–Fuks cohomology of the Lie algebra $\mathcal{S}_q$ of formal Hamiltonian vector fields. This is the Lie algebra of infinite jets at 0, $j_0^\infty(V)$ of vector fields V on $\mathbf{R}^{2q}$, with flows preserving the usual symplectic form $\alpha \in \Omega^2(\mathbf{R}^{2q})$. It is explained in [3] (for example) how a framed symplectic foliation $(\mathfrak{F}, \omega, s)$ on M gives rise to a characteristic map $H^\cdot(\mathcal{S}_q) \to H^\cdot(M)$. It is well known that there is a canonical map $W(\mathrm{sp}(q)) \to \wedge(\mathcal{S}_q^*)$ which factors through $W(\mathrm{sp}(q))_{2q}$ and relates the two constructions. One can extend this map to $W(\mathrm{sp}(q), 2)_{2q}$ by sending the generator ω to the 2-cocycle on $\mathcal{S}_q$ given by $(j_0^\infty(V), j_0^\infty(W)) \mapsto \alpha(V(0), W(0))$. It is shown in [2] that this map relates the extended Weil algebra construction to the formal vector fields construction. The computation of $H^\cdot(\mathcal{S}_q)$ is an unsolved problem. Partial results can be found in [5], [6], and [16].

3. Examples. One general type of example lives on the total space of the symplectic frame bundle to a symplectic manifold. Indeed, if (N^{2q}, α) is a symplectic manifold (with symplectic structure $\alpha \in \Omega^2(N)$) then $M = \mathrm{Sp}(TN)$ is foliated trivially by the fibres of $\mathrm{Sp}(q) \hookrightarrow M \xrightarrow{\pi} N$. This foliation $\mathfrak{F}$ has a transverse symplectic structure given by $\omega = \pi^*(\alpha)$. There is a canonical $\mathrm{Sp}(q)$-bundle isomorphism

$$\begin{aligned}\mathrm{Sp}(\nu\mathfrak{F}) &\cong \pi^*(\mathrm{Sp}(TN)) \\ &= \{(x,y)\in M\times M \mid \pi(x)=\pi(y)\}.\end{aligned}$$

From this, we obtain a framing $s: M\to \mathrm{Sp}(\nu\mathfrak{F})$ defined by $s(x)=(x,x)$.

We will consider the characteristic homomorphism

$$\Delta_*(\mathfrak{F},\omega,s): H^\bullet(W(\mathrm{sp}(q),2)_{2q})\to H^\bullet(M)$$

for this example. A Bott connection $\pi^*(\beta)$ in $\mathrm{Sp}(\nu\mathfrak{F})$ can be obtained by pulling back any connection $\beta: TM\to \mathrm{sp}(q)$ in $M=\mathrm{Sp}(TN)$. The resulting map $\Delta(\pi^*\beta): W(\mathrm{sp}(q), 2)\to \Omega^\bullet(M)$ given by the data $(\pi^*\beta, s)$ satisfies

$$\text{(3.1)}\qquad \begin{aligned}&\Delta(\pi^*\beta)=k(\beta), \text{ the Chern–Weil map for } \beta\\ &\quad\text{on } W(\mathrm{sp}(q)), \text{ and } \Delta(\pi^*\beta)(\omega)=\omega.\end{aligned}$$

Next consider the specific example given by $N=T^{2q}$, the $2q$-dimensional torus with its usual symplectic structure α. In this case $M=T^{2q}\times \mathrm{Sp}(q)$ and we can use for β, the flat connection

$$\beta: TM = TN\times T\mathrm{Sp}(q)\twoheadrightarrow T\mathrm{Sp}(q)\xrightarrow{\Theta}\mathrm{sp}(q).$$

Here Θ is given by left translation of vectors to the identity (the Cartan–Maurer form). Let ψ: $\wedge^\bullet(\mathrm{sp}(q)^*)\hookrightarrow \Omega^\bullet(\mathrm{Sp}(q))$ be the usual inclusion dual to Θ (as left invariant forms on the group $\mathrm{Sp}(q)$). The Chern–Weil map $k(\beta)$: $W(\mathrm{sp}(q))\to\Omega^\bullet(M)$ is characterized by $k(\beta)(e_i)=0$ for all i and $k(\beta)(y_i)=\psi(y_i)$. This shows that $\Delta_*(\mathfrak{F},\omega,s)$ is determined on admissible classes by

$$\Delta_*(\mathfrak{F},\omega,s)(y_I\omega^k)=[\alpha]^k\times[\psi(y_I)]$$

(cohomology cross product) and $\Delta_*(\mathfrak{F}, \omega, s)(y_I e_J\omega^k)=0$ for $J\neq(0,0,0,\dots,0)$.

To complete the computation of $\Delta_*(\mathfrak{F},\omega,s)$, it remains only to describe the Chevelley–Eilenberg homomorphism

$$\begin{aligned}H^\bullet(\mathrm{sp}(q))=\wedge(y_1,y_2,\dots,y_q)&\to H^\bullet(\mathrm{Sp}(q))\\ y_I&\mapsto[\psi(y_I)].\end{aligned}$$

Identifying $U(q)$ with a maximal compact subgroup of $\mathrm{Sp}(q)$, we have

$$\begin{aligned}H^\bullet(\mathrm{Sp}(q)) &= H^\bullet(U(q))\\ &= H^\bullet(u(q))=\wedge(u_1,u_2,\dots,u_q).\end{aligned}$$

Using identity (2.5), we see that

$$\psi(y_{i_1}\wedge\cdots\wedge y_{i_s})=2^s u_{2i_1}\wedge u_{2i_2}\wedge\cdots\wedge u_{2i_s},$$

where it is understood that $u_{2i_l}=0$ if $2i_l>q$.

This discussion yields a complete determination of $\Delta_*(\mathfrak{F},\omega,s)$ for this example. In particular we have proved the following result.

3.2. THEOREM. *Let* $(\mathfrak{F},\omega,s)$ *be the framed symplectic foliation on* $\mathrm{Sp}(T^{2q})$ *described above. Then*

$$\{\Delta_*(\mathfrak{F},\omega,s)(y_I\omega^k)\mid y_I\omega^k \text{ is admissible with } y_I\in\wedge(y_1,\ldots,y_{\lfloor q/2\rfloor})\}$$

is a linearly independent set in $H^{\boldsymbol{\cdot}}(\mathrm{Sp}(T^{2q}))$.

One can also take N to be a complex torus with its usual Kähler metric and obtain a foliation on $U(N)$, the bundle of unitary frames. The above analysis then shows that the Kähler characteristic classes of the form $\Delta_*(\mathfrak{F},\omega,s)(u_I\omega^k)$ are a linearly independent set.

Next we turn to an interesting class of Kähler foliations. If H is a closed subgroup of a Lie group G, then it is well known that the foliation of G by left H-cosets is canonically framed. There is an extensive literature on the characteristic classes for such homogeneous foliations. See for example [1], [12], and [17]. Here we consider the case where G/H is a Hermitian symmetric space. We refer the reader to [14] for background material on Hermitian symmetric spaces. In this context, the left H-cosets give a framed Kähler foliation $(\mathfrak{F},\omega,s)$ on G.

The homogeneous Kähler structure on G/H has two components: a left G-invariant complex structure $J\colon T(G/H)\to T(G/H)$ and a left G-invariant and J-invariant Riemannian metric $\langle\,,\rangle$. There is a splitting $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ of G resulting from the symmetric structure. Here $\mathfrak{h}$ is the Lie algebra of H and we can identify $\mathfrak{m}$ with $T_{eH}(G/H)$. We make the following assumptions which are satisfied by a variety of classical examples described in [14].

3.3. CONDITIONS. (a) G is semi-simple and $\langle\,,\rangle$ is obtained by restriction of the Cartan–Killing form to $\mathfrak{m}$.

(b) J is obtained by restriction of $\mathrm{ad}(Z_0)$ to $\mathfrak{m}$, where Z_0 is some element in the center of $\mathfrak{h}$.

There are general relationships between the characteristic classes for such Hermitian symmetric foliations. The key to this is contained in the following lemma.

3.4. LEMMA. *Let G/H be a Hermitian symmetric space satisfying conditions 3.3. Let $\beta\colon TU(G/H)\to u(q)$ be the Hermitian connection for G/H and*

$$h(\beta)\colon \mathbf{R}[\tilde c_1,\tilde c_2,\ldots,\tilde c_q]\to\Omega^{\boldsymbol{\cdot}}(G/H)=\Omega^{\boldsymbol{\cdot}}(G)_H$$

be the Chern–Weil map for β. Then

$$h(\beta)(\tilde c_1)=\frac{1}{8\pi}\omega,$$

where ω is the Kähler form on G/H.

Proof. The invariant polynomial $\tilde c_1\colon u(q)\to\mathbf{R}$ is given by

$$\tilde c_1(A)=-\frac{1}{2\pi i}\operatorname{tr}(A),$$

where A denotes a $q\times q$ skew Hermitian matrix. Hence

$$h(\beta)(\tilde c_1)=-\frac{1}{2\pi i}\operatorname{tr}(\Omega)\in\Omega^2(U(G/H))_{U(q)},$$

where Ω is the $u(q)$-valued curvature form for β and the $U(q)$-subscript denotes basic elements. Theorem 4.9 from Chapter 9 of [14] states that $-2i\,\mathrm{tr}(\Omega) = \pi^*(\rho)$, where $\pi: U(G/H) \twoheadrightarrow G/H$ is projection and $\rho \in \Omega^2(G/H)$ is given by $\rho(X, Y) = S(X, JY)$. Here S is the Ricci tensor for β. This shows that

$$h(\beta)(\tilde{c}_1) = -\frac{1}{4\pi}\pi^*(\rho). \tag{3.5}$$

Writing K for the Cartan–Killing form on $\mathfrak{g}$, Proposition 9.7 from Chapter 11 of [14] shows that $S(X, Y) = -\frac{1}{2}K(X, Y)$ for $X, Y \in \mathfrak{m} \cong \mathfrak{g}/\mathfrak{h}$. Condition 3.3(b) is used here. Finally, condition 3.3(a) shows that $\rho(X, Y) = -\frac{1}{2}\langle X, JY\rangle$. The Kähler form is defined by $\omega(X, Y) = \langle X, JY\rangle$ so that this identity together with equation (3.5) complete the proof. □

3.6. COROLLARY. *Let G/H be a Hermitian symmetric space satisfying conditions* (3.3), *and* $(\mathfrak{F}, \omega, s)$ *the associated framed Kähler foliation on G. Then ω is an exact form.*

Proof. We obtain a map $W(u(q))_q \xrightarrow{\Delta} \Omega^\cdot(G)$ by using the pull-back of the Hermitian connection as a Bott connection. Then $\omega = 8\pi\Delta(\tilde{c}_1) = d(8\pi\Delta(u_1))$. □

Using Lemma 3.4, we see that the characteristic classes for a Hermitian symmetric foliation satisfy

$$\Delta_*(u_I\tilde{c}_J\omega^k) = (8\pi)^k\Delta_*(u_I\tilde{c}_1^k\tilde{c}_J).$$

Considerably more can be said in this context.

3.7. THEOREM. *Let G/H be a Hermitian symmetric space satisfying conditions 3.3. Let $2q = \dim_{\mathbf{R}}(G/H)$ and $(\mathfrak{F}, \omega, s)$ be the associated framed Kähler foliation on G. We write Δ_* for $\Delta_*(\mathfrak{F}, \omega, s)$. Then for ω-classes $u_I\tilde{c}_J\omega^k$ (with $k > 0$) we have*

(a) $\Delta_*(u_I\tilde{c}_J\omega^k) = 0$ *if* $i_1 + |J| + k > (q+1)$,

(b) $\Delta_*(u_I\tilde{c}_J\omega^k) = a\Delta_*(u_1u_{i_2}u_{i_3}\cdots u_{i_s}\omega^q) = (8\pi)^q a\Delta_*(u_1u_{i_2}u_{i_3}\cdots u_{i_s}\tilde{c}_1^q)$ *for some $a \in \mathbf{R}$ if $i_1 + |J| + k = (q+1)$.*

If in addition G is connected and G/H compact, then (a) *and* (b) *also hold for the admissible classes with $k = 0$.*

Proof. This is a version of Pittie's theorem on parabolic foliations [17] in the Kähler setting. Pittie's proof carries through here so we give only a brief summary.

Let $\alpha = u_I \wedge \tilde{c}_J \wedge \omega^k \in \Omega^\cdot(G)$ be a form representing $\Delta_*(u_I\tilde{c}_J\omega^k)$ and computed using the pull-back of the Hermitian connection in G/H.

One first shows that there is a form $\gamma \in \wedge^{2|J|+2k-2}(\mathfrak{g}^*)_H$ with $\gamma \wedge \omega \sim \tilde{c}_J \wedge \omega^k$ ("cohomologous to"). Of course when $k > 0$, this is obvious. When $k = 0$, one must use the additional hypotheses of connectivity and compactness and apply the hard Lefschetz theorem. An easy computation shows that we also have $u_I \wedge \tilde{c}_J \wedge \omega^k \sim u_I \wedge \gamma \wedge \omega$.

When $i_1 + |J| + k > (q+1)$, we compute that $d(8\pi u_1 \wedge u_I \wedge \gamma) = u_I \wedge \gamma \wedge \omega$. It follows that $\Delta_*(u_I\tilde{c}_J\omega^k) = 0$ in this case.

Next suppose that $i_1+|J|+k=(q+1)$. Here there are two cases to consider: $i_1=1$ and $i_1>1$. If $i_1=1$, then $|J|+k=q$ and one must have $\tilde{c}_J\wedge\omega_k=a\omega^q$ for some $a\in\mathbf{R}$. In this case $u_I\wedge\tilde{c}_J\wedge\omega^k=au_1\wedge u_{i_2}\wedge\cdots\wedge u_{i_s}\wedge\omega^q$. When $i_1>1$, we compute that

$$d(u_1\wedge u_I\wedge\gamma)=u_I\wedge\gamma\wedge\tilde{c}_1-u_1\wedge u_{i_2}\wedge\cdots\wedge u_{i_s}\wedge\gamma\wedge\tilde{c}_{i_1}.$$

Hence

$$\begin{aligned}u_I\wedge\gamma\wedge\omega^k &\sim 8\pi u_I\wedge\gamma\wedge\tilde{c}_1\\ &\sim 8\pi u_1\wedge u_{i_2}\wedge\cdots\wedge u_{i_s}\wedge\gamma\wedge\tilde{c}_{i_1}.\end{aligned}$$

Here $\deg(\gamma\wedge\tilde{c}_{i_1})=2q$ so we must have $\gamma\wedge\tilde{c}_{i_1}=b\omega^q$ for some $b\in\mathbf{R}$. Letting $a=8\pi b$, this shows that

$$u_I\wedge\tilde{c}_J\wedge\omega^k\sim au_1\wedge u_{i_2}\wedge\cdots\wedge u_{i_s}\wedge\omega^q. \qquad \square$$

Theorem 3.7 is a negative result in the sense that it puts severe limitations on the non-triviality theorems that one can obtain by studying Hermitian symmetric foliations. If such a foliation is viewed as a framed symplectic foliation, then the symplectic characteristic classes are determined by using identities (2.5). In view of the admissibility conditions (2.4), Theorem 3.7 shows that all the symplectic classes vanish. Despite this fact, we will return to Hermitian symmetric foliations in Section 5 when we turn to the study of $\pi_*(B\Gamma_{\mathrm{Sp}(q)})$.

Next we describe a specific Hermitian symmetric foliation. The group $U(q)$ can be considered as a closed subgroup of $SU(q+1)$ by letting

$$U(q)=\left\{\left(\begin{array}{c|c} a & \text{---}\,0\,\text{---} \\ \hline | & \\ 0 & B \\ | & \end{array}\right)\right\}$$

where $B\in U(q)$, $a\in S^1\subset\mathbf{C}$ and $a\det(B)=1$. Then $SU(q+1)/U(q)=\mathbf{C}P^q$, which is a famous Kähler manifold. In Chapter 11 of [14] it is shown that this is a Hermitian symmetric space satisfying conditions 3.3. The resulting framed Kähler foliation on $SU(q+1)$ has been studied by Kamber and Tondeur in [12] and by Matsuoka and Morita in [15]. It is known that

$$\Delta_*(u_1u_{i_2}u_{i_3}\cdots u_{i_s}\tilde{c}_1^q)=k_{(i_2,\ldots,i_s)}u_{i_2}\wedge\cdots\wedge u_{i_s}\wedge u_{q+1}$$

where $k_{(i_2,\ldots,i_s)}$ is a non-zero real number and we identify $H^\bullet(SU(q+1))$ with $\wedge(u_2,u_3,\ldots,u_{q+1})$. This fact together with Theorem 3.7 can be used to prove the following theorem.

3.8. THEOREM. *Let Δ_* be the characteristic homomorphism for the framed Kähler foliation of $SU(q+1)$ by cosets of $U(q)$. If $u_I\tilde{c}_J\omega^k$ is admissible and $i_1+|J|+k=q+1$, then $\Delta_*(u_I\tilde{c}_J\omega^k)$ is a non-zero multiple of*

$$u_{i_2}\wedge u_{i_3}\wedge\cdots\wedge u_{i_s}\wedge u_{q+1}\in H^\bullet(SU(q+1)).$$

Proof. In view of the above remarks, one need only show that the constants a in Theorem 3.7 are all non-zero. This can be done by using the identity

$$c_i(\mathbf{C}P^q) = \binom{q}{i} c_1(\mathbf{C}P^q) = \left(\frac{1}{8\pi}\right)^i \binom{q}{i} \omega$$

for the Chern classes of $\mathbf{C}P^q = SU(q+1)/U(q)$, together with the proof of Theorem 3.7. $\square$

Theorem 3.8 shows that if we fix (i_1, J, k) with $i_1 + |J| + k = q+1$, then

$$\{\Delta_*(u_{i_1} u_I \tilde{c}_J \omega^k) \mid u_I \in \Lambda(u_{i_1+1}, \ldots, u_q)\}$$

is a linearly independent set in $H^\bullet(SU(q+1))$. In view of Theorem 3.7, this is the best possible non-triviality example from among the Hermitian symmetric foliations.

4. Results on $H^*(B\bar{\Gamma}_{\mathrm{Sp}(q)})$. The existence of a framed symplectic foliation $(\mathfrak{F}, \omega, s)$ for which certain characteristic classes $\Delta_*(\mathfrak{F}, \omega, s)(y_I e_J \omega^k)$ form a linearly independent set implies that the corresponding universal classes

$$\Delta_*(y_I e_J \omega^k) \in H^\bullet(B\bar{\Gamma}_{\mathrm{Sp}(q)})$$

are also linearly independent. In addition to such independence considerations, one also asks about the variability of these classes.

4.1. DEFINITION. A finite set

$$\mathcal{S} = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset H^l(B\bar{\Gamma}_{\mathrm{Sp}(q)})$$

is independently variable if evaluation of $\mathcal{S}$ gives an epimorphism

$$H_l(B\bar{\Gamma}_{\mathrm{Sp}(q)}; \mathbf{Z}) \twoheadrightarrow \mathbf{R}^n.$$

A set of homogeneous classes $\mathcal{S} \subset H^\bullet(B\bar{\Gamma}_{\mathrm{Sp}(q)})$, finite in each degree, is independently variable when the classes of degree l are independently variable for each $l = 1, 2, 3, \ldots$.

One does not expect any classes of the form $\Delta_*(y_I e_J)$ to belong to independently variable sets in $H^\bullet(B\bar{\Gamma}_{\mathrm{Sp}(q)})$ since, as was noted in Section 2, all the $y_I e_J$ classes are rigid. In contrast, one has the following result concerning the ω-classes.

4.2. THEOREM. *Let $\mathcal{S} \subset H^\bullet(W(\mathrm{sp}(q), 2)_{2q})$ be a set of ω-classes. Then $\Delta_*(\mathcal{S}) \subset H^\bullet(B\bar{\Gamma}_{\mathrm{Sp}(q)})$ is independently variable if and only if it is an independent set.*

Proof. We begin by noting that if $(\mathfrak{F}, \omega)$ is a symplectic foliation on M, then so is $(\mathfrak{F}, t\omega)$ for all $t \neq 0$. In the following discussion, we will assume that $t > 0$.

The $\mathrm{Sp}(q)$-bundle $\mathrm{Sp}(\nu(\mathfrak{F}, \omega)) \subset \mathrm{GL}(\nu\mathfrak{F})$ is the set of ω-symplectic frames.

This means that for $x \in M$, the fibre $\mathrm{Sp}(\nu(\mathfrak{F}, \omega))_x$ is the set of all frames $(\vec{a}, \vec{b}) = (a_1, a_2, \ldots, a_q, b_1, b_2, \ldots, b_q)$ in $(\nu\mathfrak{F})_x$ where $\omega(a_i, b_j) = \delta_{ij}$ and $\omega(a_i, a_j) = 0 = \omega(b_i, b_j)$.

Let $1/\sqrt{t}: \mathrm{Sp}(\nu(\mathfrak{F}, \omega)) \to \mathrm{Sp}(\nu(\mathfrak{F}, t\omega))$ be given by $(\vec{a}, \vec{b}) \mapsto 1/\sqrt{t}(\vec{a}, \vec{b})$ on each fibre $\mathrm{Sp}(\nu(\mathfrak{F}, \omega))_x$. This is an isomorphism of $\mathrm{Sp}(q)$-bundles with inverse $\sqrt{t}: \mathrm{Sp}(\nu(\mathfrak{F}, t\omega)) \to \mathrm{Sp}(\nu(\mathfrak{F}, \omega))$. These isomorphisms are obtained by restriction of the automorphisms of $\mathrm{GL}(\nu\mathfrak{F})$ given by $1/\sqrt{t}I, \sqrt{t}I$ and the usual right action $\mathrm{GL}(\nu\mathfrak{F}) \times \mathrm{GL}(2q) \to \mathrm{GL}(\nu\mathfrak{F})$.

If $s: M \to \mathrm{Sp}(\nu(\mathfrak{F}, \omega))$ is a framing, then $1/\sqrt{t}s = 1/\sqrt{t} \circ s$ is a section in $\mathrm{Sp}(\nu(\mathfrak{F}, t\omega))$. In this way, a framed symplectic foliation $(\mathfrak{F}, \omega, s)$ yields a one parameter family $\{(\mathfrak{F}, t\omega, 1/\sqrt{t}s) \mid t > 0\}$ of framed symplectic foliations.

Now suppose that $\beta: T\mathrm{Sp}(\nu(\mathfrak{F}, \omega)) \to \mathrm{sp}(q)$ is a Bott connection for $(\mathfrak{F}, \omega)$. This means that β comes by restriction of a Bott connection γ in $\mathrm{GL}(\nu\mathfrak{F})$. It is not hard to show that the connection

$$(\sqrt{t})^*(\beta) = \beta \circ D(\sqrt{t}): T\mathrm{Sp}(\nu(\mathfrak{F}, t\omega)) \to \mathrm{sp}(q)$$

is also obtained by restriction of γ. Hence $(\sqrt{t})^*(\beta)$ is a Bott connection for $(\mathfrak{F}, t\omega)$.

Using β and $(\sqrt{t})^*(\beta)$ to compute representing forms e_i and y_i for $(\mathfrak{F}, \omega, s)$ and $(\mathfrak{F}, t\omega, 1/\sqrt{t}s)$, we obtain the same results in $\Omega^\bullet(M)$. This shows that

$$\Delta_*(\mathfrak{F}, t\omega, 1/\sqrt{t}s)(y_I e_J \omega^k) = t^k \Delta_*(\mathfrak{F}, \omega, s)(y_I e_J \omega^k) \tag{4.3}$$

for all admissible classes $y_I e_J \omega^k$.

Now suppose that $\mathcal{S} = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset H^\bullet(W(\mathrm{sp}(q), 2)_{2q})$ is a set of ω-classes with $\Delta_*(\mathcal{S}) \subset H^\bullet(B\bar{\Gamma}_{\mathrm{Sp}(q)})$ linearly independent. Standard methods of Haefliger imply that there is some framed symplectic foliation, say $(\mathfrak{F}, \omega, s)$ on M, with $\Delta_*(\mathfrak{F}, \omega, s)(\mathcal{S}) \subset H^\bullet(M)$ linearly independent. Without loss of generality, we assume all classes in $\mathcal{S}$ have degree l. The universal coefficient theorem shows that there are homology classes $\{z_1, z_2, \ldots, z_n\} \subset H_l(M, \mathbf{Z})$ with $A = [A_{ij}] = [\Delta_*(\mathfrak{F}, \omega, s)(\alpha_i)(z_j)]$ a nonsingular matrix.

Let $f_t: M \to B\bar{\Gamma}_{\mathrm{Sp}(q)}$ be the classifying map for $(\mathfrak{F}, t\omega, 1/\sqrt{t}s)$, where $t > 0$. Then $\Delta_*(\alpha_i)(f_{1*}(z_j)) = A_{ij}$ and using (4.3) we see that $\Delta_*(\alpha_i)(f_{t*}(z_j)) = A_{ij} t^{k_i}$, where $\alpha_i = y_I e_J \omega^{k_i}$, say. Now let $\vec{t} = (t_1, t_2, \ldots, t_n) \in (\mathbf{R}^+)^n$ and define $f_i(\vec{t}) \in \mathbf{R}$ by

$$f_i(\vec{t}) = \Delta_*(\alpha_i)\left(\sum_{j=1}^n f_{t_j*}(z_j)\right) = \sum_{j=1}^n A_{ij} t_j^{k_i}.$$

Here $\sum_{j=1}^n f_{t_j*}(z_j) \in H_l(B\bar{\Gamma}_{\mathrm{Sp}(q)}; \mathbf{Z})$, so that

$$\mathrm{Im}(f_i: (\mathbf{R}^+)^n \to \mathbf{R}) \subset \mathrm{Im}(\Delta(\alpha_i): H_l(B\bar{\Gamma}_{\mathrm{Sp}(q)}; \mathbf{Z}) \to \mathbf{R}).$$

Letting $F: (\mathbf{R}^+)^n \to \mathbf{R}^n$ be given by $F(\vec{t}) = (f_1(\vec{t}), \ldots, f_n(\vec{t}))$, we have

$$\mathrm{Im}(F) \subset \mathrm{Im}(H_l(B\bar{\Gamma}_{\mathrm{Sp}(q)}; \mathbf{Z}) \xrightarrow{\Delta_*(\mathcal{S})} \mathbf{R}^n).$$

One computes that the differential of F at the point $(1,1,\dots,1)$ is given by the Jacobian matrix

$$\begin{bmatrix} k_1 & & & 0 \\ & k_2 & & \\ & & \ddots & \\ 0 & & & k_n \end{bmatrix} A,$$

which is nonsingular since A is nonsingular and each k_i is positive. It follows that $\operatorname{Im}(F)$ contains an open set in $\mathbf{R}^n$. Since $\Delta_*(\mathcal{S})$ is an abelian group homomorphism, it follows that $\Delta_*(\mathcal{S})$ is surjective. □

By combining Theorems 3.2 and 4.2, we obtain a specific independently variable set.

4.4. THEOREM. *Let* $\mathcal{S} \subset H^{\bullet}(W(\mathrm{sp}(q),2)_{2q})$ *be defined by*

$$\mathcal{S} = \{y_I \omega^k \text{ admissible} \mid y_I \in \Lambda(y_1,\dots,y_{\lfloor q/2 \rfloor})\}.$$

Then $\Delta_*(\mathcal{S}) \subset H^{\bullet}(B\bar{\Gamma}_{\mathrm{Sp}(q)})$ *is an independently variable set. In particular, there are epimorphisms* $H_l(B\bar{\Gamma}_{\mathrm{Sp}(q)};\mathbf{Z}) \twoheadrightarrow \mathbf{R}^{d_l}$ *where* $d_l =$ *the number of classes of degree* l *in* $\mathcal{S}$.

5. Results on $\pi_*(B\Gamma_{\mathrm{Sp}(q)})$. There is a general method used to transform results on the cohomology of a classifying space $B\bar{\Gamma}$, for some type of framed foliations, into results on $\pi_*(B\Gamma)$. The main step employs the rational Hurewicz theorem to conclude that $H_*(B\bar{\Gamma};\mathbf{Q}) \cong \pi_*(B\bar{\Gamma}) \otimes \mathbf{Q}$ in a certain range of degrees $*$ depending on the connectivity of $B\bar{\Gamma}$. This idea is used by Hurder in a number of papers including [9] and [11].

For this method to work, one needs $B\bar{\Gamma}$ to be highly connected. We have seen that the class $\omega = \Delta_*(\omega) \in H^2(B\bar{\Gamma}_{\mathrm{Sp}(q)})$ is non-zero, so that $B\bar{\Gamma}_{\mathrm{Sp}(q)}$ cannot be even 2-connected. This complicates the application of Hurder's methods in the symplectic case.

We circumvent this difficulty by considering a related space introduced by Haefliger in [7]. The class $\omega \in H^2(B\bar{\Gamma}_{\mathrm{Sp}(q)})$ corresponds to a homotopy class of maps $\omega: B\bar{\Gamma}_{\mathrm{Sp}(q)} \to K(\mathbf{R},2)$. The homotopy fibre of ω will be written $B\bar{\bar{\Gamma}}_{\mathrm{Sp}(q)}$. It is shown in [7] that this space is $(2q-1)$-connected. Since $\pi_i(B\bar{\Gamma}_{\mathrm{Sp}(q)}) = \pi_i(B\bar{\bar{\Gamma}}_{\mathrm{Sp}(q)})$ for $i \neq 2$, the space $B\bar{\bar{\Gamma}}_{\mathrm{Sp}(q)}$ can be used to gain information on the higher homotopy groups of $B\bar{\Gamma}_{\mathrm{Sp}(q)}$.

Elementary obstruction theory shows that $B\bar{\bar{\Gamma}}_{\mathrm{Sp}(q)}$ is a classifying space for $\bar{\bar{\Gamma}}_{\mathrm{Sp}(q)}$-foliations as defined below.

5.1. DEFINITION. *A* $\bar{\bar{\Gamma}}_{\mathrm{Sp}(q)}$*-foliation* $(\mathfrak{F},\omega,\eta,s)$ *is a framed symplectic foliation* $(\mathfrak{F},\omega,s)$ *together with a* 1*-form* η *satisfying* $d\eta = \omega$.

Additional cohomology invariants for $\bar{\bar{\Gamma}}_{\mathrm{Sp}(q)}$-foliations will be introduced. The new invariants are given by closed forms divisible by η. We begin by adding a one-dimensional generator η to the $\hat{A}$-complex for $W(\mathrm{sp}(q),2)_{2q}$, described in Section 2, yielding

$$\mathfrak{A} = \Lambda(y_1, y_2, \ldots, y_q, \eta) \otimes \mathbf{R}[e_1, e_2, \ldots, e_q, \omega]_{2q}.$$

Here of course $d\eta = \omega$ in $\mathfrak{A}$. If $(\mathfrak{F}, \omega, \eta, s)$ is a $\bar{\bar{\Gamma}}_{\mathrm{Sp}(q)}$-foliation on M, then we obtain a DG algebra map $\Delta(\beta): \mathfrak{A} \mapsto \Omega^{\bullet}(M)$ by choosing a symplectic Bott connection β. We obtain a characteristic map $\Delta_*(\mathfrak{F}, \omega, \eta, s): H^{\bullet}(\mathfrak{A}) \to H^{\bullet}(M)$ which is independent of the choice of β.

The Vey basis for $H^{\bullet}(\mathfrak{A})$ is given by admissible classes of three types:

(a) $y_I e_J$ as in (2.4);

(b) $y_I e_J \omega^k$ as in (2.4), with $k \geq 1$ and $i_1 + |J| + k = (q+1)$;

(c) $y_I e_J \omega^k \eta$, where $I = \phi$ or $1 \leq i_1 < i_2 < \cdots < i_s \leq q$, $|J| + k = q$ and $i_1 \leq$ the index of the first non-zero j_l.

The classes of types (a) and (b) are the classes arising from $H^{\bullet}(W(\mathrm{sp}(q), 2)_{2q})$ which are potentially non-zero for framed symplectic foliations with an exact transverse symplectic form. The classes of type (c) are new invariants for $\bar{\bar{\Gamma}}_{\mathrm{Sp}(q)}$-foliations which will be called η-classes.

Suppose that G/H is a Hermitian symmetric space satisfying conditions (3.3). The proof of Corollary 3.6 shows that the resulting framed Kähler foliation $(\mathfrak{F}, \omega, s)$ on G becomes a $\bar{\bar{\Gamma}}_{\mathrm{Sp}(q)}$-foliation by specifying $\eta = 8\pi\Delta(u_1)$. Using this identity, it is easy to relate the η-classes to the Kähler characteristic classes for $(\mathfrak{F}, \omega, s)$. One has:

5.2. THEOREM. *Let G/H be a Hermitian symmetric space satisfying conditions* (3.3) *and $(\mathfrak{F}, \omega, \eta, s)$ the associated $\bar{\bar{\Gamma}}_{\mathrm{Sp}(q)}$-foliation on G.*

(a) *If $y_I \omega^q \eta \in H^{\bullet}(\mathfrak{A})$ is an η-class with $y_I \in \Lambda(y_1, y_2, \ldots, y_{\lfloor q/2 \rfloor})$, then*

$$\Delta_*(y_I \omega^q \eta) = \pm 8\pi\Delta_*(u_1 u_{2I} \omega^q).$$

Here $u_{2I} = u_{2i_1} \wedge u_{2i_2} \wedge \cdots \wedge u_{2i_s}$ if $y_I = y_{i_1} \wedge \cdots \wedge y_{i_s}$.

(b) $\Delta_*(y_I e_J \omega^k \eta) = 0$ *for all other η-classes.*

This result combined with Theorem 3.8 shows:

5.3. THEOREM. *The η-classes $\{\Delta_*(y_I \omega^q \eta) \mid y_I \in \Lambda(y_1, \ldots, y_{\lfloor q/2 \rfloor})\}$ for the $\bar{\bar{\Gamma}}_{\mathrm{Sp}(q)}$-foliation of $SU(q+1)$ by $U(q)$-cosets form a linearly independent set in $H^{\bullet}(SU(q+1))$.*

One can define independent variability for a set $\mathcal{S} \subset H^{\bullet}(B\bar{\bar{\Gamma}}_{\mathrm{Sp}(q)})$ as in Definition 4.1. The methods used in proving Theorem 4.2 also show that if $\mathcal{S} \subset H^{\bullet}(\mathfrak{A})$ is any set of η-classes, then $\Delta_*(\mathcal{S}) \subset H^{\bullet}(B\bar{\bar{\Gamma}}_{\mathrm{Sp}(q)})$ is independently variable if and only if it is linearly independent. In particular, if $\mathcal{S} = \{y_I \omega^q \eta \mid y_I \in \Lambda(y_1, \ldots, y_{\lfloor q/2 \rfloor})\}$ then $\Delta_*(\mathcal{S}) \subset H^{\bullet}(B\bar{\bar{\Gamma}}_{\mathrm{Sp}(q)})$ is independently variable.

In view of the high connectivity of $B\bar{\bar{\Gamma}}_{\mathrm{Sp}(q)}$, this can be converted into a result on the homotopy theory of $B\bar{\Gamma}_{\mathrm{Sp}(q)}$ and $B\Gamma_{\mathrm{Sp}(q)}$.

5.4. THEOREM. *Let $2q+1 \leq n \leq 4q-1$ and let*

$$b_n = \dim(\Lambda^{n-2q-1}(y_1, y_2, \ldots, y_{\lfloor q/2 \rfloor})).$$

Then there are epimorphisms $\pi_n(B\bar{\Gamma}_{\mathrm{Sp}(q)}) \twoheadrightarrow \mathbf{R}^{b_n}$ and $\pi_n(B\Gamma_{\mathrm{Sp}(q)}) \twoheadrightarrow \mathbf{R}^{b_n}$.

Proof. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_{b_n}\}$ be a basis for $\Lambda^{n-2q-1}(y_1, y_2, \ldots, y_{\lfloor q/2 \rfloor})$. It will be shown that the composition

$$\pi_n(B\bar{\bar{\Gamma}}_{\mathrm{Sp}(q)}) \xrightarrow{\mathcal{H}} H_n(B\bar{\bar{\Gamma}}_{\mathrm{Sp}(q)}; \mathbf{Z}) \xrightarrow{G} \mathbf{R}^{b_n}$$

is surjective where $\mathcal{H}$ is the Hurewicz map and G is given by evaluation of $(\Delta_*(\alpha_1\omega^q\eta), \Delta_*(\alpha_2\omega^q\eta), \ldots, \Delta_*(\alpha_{b_n}\omega^q\eta))$. This will prove the theorem since for $n>2$, $\pi_n(B\bar{\Gamma}_{\mathrm{Sp}(q)}) \cong \pi_n(B\bar{\bar{\Gamma}}_{\mathrm{Sp}(q)})$ and for $n \geq 2q+1$, $\pi_n(B\bar{\Gamma}_{\mathrm{Sp}(q)}) \otimes \mathbf{Q} \cong \pi_n(B\Gamma_{\mathrm{Sp}(q)}) \otimes \mathbf{Q}$. The latter fact follows by tensoring the homotopy sequence for the fibration $B\bar{\Gamma}_{\mathrm{Sp}(q)} \hookrightarrow B\Gamma_{\mathrm{Sp}(q)} \twoheadrightarrow BU(q)$ with $\mathbf{Q}$.

Let $(\mathfrak{F}, \omega, \eta, s)$ be the $\bar{\bar{\Gamma}}_{\mathrm{Sp}(q)}$-foliation on $M = SU(q+1)$ considered previously. Let $f_t: M \to B\bar{\bar{\Gamma}}_{\mathrm{Sp}(q)}$ classify $(\mathfrak{F}, t\omega, t\eta, 1/\sqrt{t}s)$ for $t>0$. Since $B\bar{\bar{\Gamma}}_{\mathrm{Sp}(q)}$ is $(2q-1)$-connected, we can adjoin cells of dimension $\leq 2q$ to M and obtain a $(2q-1)$-connected CW complex $W \supset M$ and extended maps $F_t: W \to B\bar{\bar{\Gamma}}_{\mathrm{Sp}(q)}$. We know that $G: H_n(B\bar{\bar{\Gamma}}_{\mathrm{Sp}(q)}; \mathbf{Z}) \to \mathbf{R}^{b_n}$ is surjective even when restricted to the subgroup $\mathcal{G}$ generated by $\bigcup_{t>0} \mathrm{Im}(f_{t*})$. This is the same as the subgroup generated by $\bigcup_{t>0} \mathrm{Im}(F_{t*})$, since $n \geq 2q+1$ and only the $2q$ skeleton of M was altered to obtain W.

The rational Hurewicz theorem shows that

$$\pi_n(W) \otimes \mathbf{Q} \xrightarrow{\mathcal{H} \otimes \mathrm{id}} H_n(W; \mathbf{Z}) \otimes \mathbf{Q}$$

is an isomorphism when $n \leq 2(2q-1) = 4q-2$, and is onto when $n = 4q-1$. Since $H_n(W; \mathbf{Z})$ is a finitely generated abelian group, we conclude that $\mathcal{H}: \pi_n(W) \to H_n(W; \mathbf{Z})$ hits all non-torsion elements for $n \leq 4q-1$.

G vanishes on torsion elements in $H_n(B\bar{\bar{\Gamma}}_{\mathrm{Sp}(q)}; \mathbf{Z})$ since $\mathbf{R}^{b_n}$ is torsion free. Hence G is onto even when restricted to the subgroup $\mathcal{G}'$ generated by the images of all the non-torsion elements in $H_n(W; \mathbf{Z})$ under all the maps F_{t*}. We have also shown that when $(2q+1) \leq n \leq (4q-1)$, $\mathcal{G}'$ is contained in the group generated by the images of all the maps $F_{t*} \circ \mathcal{H} = \mathcal{H} \circ F_{t\#}$. □

One can obtain extensive improvements of Theorem 5.4 by using additional methods of Hurder. As before, we let $b_n = \dim \wedge^{n-2q-1}(y_1, \ldots, y_{\lfloor q/2 \rfloor})$ for $n = 2q+1, \ldots$. Form the free graded Lie algebra $\mathcal{L}$ with b_n generators of degree $n-1$ and let $a_n = \dim(\mathcal{L}^{n-1})$. If $q \geq 2$, then $\mathcal{L}$ has at least 2 generators ($\omega^q\eta$ and $y_1\omega^q\eta$) and it follows that $\{a_n\}$ has a subsequence tending to infinity.

5.5. THEOREM. *There are epimorphisms*

$$\pi_n(B\bar{\Gamma}_{\mathrm{Sp}(q)}) \twoheadrightarrow \mathbf{R}^{a_n}$$

and

$$\pi_n(B\Gamma_{\mathrm{Sp}(q)}) \twoheadrightarrow \mathbf{R}^{a_n}$$

for all $n \geq (2q+1)$.

This result can be proved by introducing a family of dual homotopy invariants for symplectic foliations and working by analogy with [9]. These invariants arise from $\pi^*(I_{2q})$, the dual homotopy of the algebra $I_{2q} = \mathbf{R}[e_1, \ldots, e_q, \omega]_{2q}$. If $(\mathfrak{F}, \omega, \eta, s)$ is a $\bar{\bar{\Gamma}}_{\mathrm{Sp}(q)}$-foliation then some of the lower-dimensional invariants are obtained by using the characteristic classes (arising from $H^\bullet(\mathfrak{A})$) and the Hurewicz map as in the proof of Theorem 5.4. In some sense, the higher dual homotopy invariants are iterated (dual) Whitehead products of the lower dimensional ones. Theorem 5.5 is proved by evaluating such higher dimensional invariants on iterated Whitehead products of non-zero classes detected in Theorem 5.4. An important observation is that all products of the cocycles used to obtain the Vey basis for $H^{\bullet}(\mathfrak{A})$ are zero. This makes the situation here formally identical to those considered in [9].

In conclusion, we note that Theorem 5.4 shows that $\pi_{2q+1}(B\bar{\bar{\Gamma}}_{\mathrm{Sp}(q)}) \neq 0$ and hence that $B\bar{\bar{\Gamma}}_{\mathrm{Sp}(q)}$ is at most $2q$-connected. It has been conjectured by McDuff that $B\bar{\bar{\Gamma}}_{\mathrm{Sp}(q)}$ is indeed $2q$-connected.

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Department of Mathematical Sciences
University of Missouri, St. Louis
St. Louis, Missouri 63121