THE CLASSIFICATION OF PL FIBRATIONS

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Here all PL spaces are locally finite and all PL maps are proper, hence triangulable [13]. By a PL fibration we mean a PL map which is a Hurewicz fibration. A concordance between PL fibrations over B is a PL fibration over $B \times I$ which restricts to the given fibrations over $B \times \partial I$.

By a PL fibration with fibres of the homotopy type of the PL space K and with a fibre homotopy trivialization we mean a PL fibration $p: E \to B$ together with a continuous fibrewise map $j: B \times K \to E$, which is a fibre homotopy equivalence. A concordance of same includes a fibre homotopy equivalence from $B \times I \times K$ to the total space (over $B \times I$) as part of the data. (A more traditional definition would be to additionally require the maps j in fibrations and concordances to be PL inclusions. We show in Corollary 4.6 below that these two definitions give rise to the same set of concordance classes. This would also follow much more simply from a sequel [17] in which we show that PL fibrations are fibrations in the PL category.)

PL fibrations were first studied by Hatcher [12], who showed that the fibres of a PL fibration over a connected base are naturally (via fibre transport) simple homotopy equivalent (see [5] for a generalization to compact ANR-fibrations). In addition, he asserted the following:

ASSERTION A [12, 3.1].

(i) PL fibrations are classified up to concordance by the classifying space of the category of compact PL spaces and CEPL maps (PL maps with contractible fibres).

(ii) PL fibrations with fibres of the homotopy type of K and with fibre homotopy trivializations are classified up to concordance by the category of compact PL pairs (L, K) with $K \subset L$ a homotopy equivalence, and CEPL maps which restrict to the identity map of K.

ASSERTION B [12, 9.1]. Let $C(M)$ denote the space of concordances of the PL manifold M and let $BC(M)$ be its classifying space. Then passage from concordance bundles to PL fibrations induces a homotopy equivalence from $\varinjlim_k BC(M \times I^k)$ to the basepoint component of the classifying space for PL fibrations with fibres of the homotopy type of M and with fibre homotopy trivializations.

ASSERTION C [12, 9.1]. The natural map $BC(M) \to \varinjlim_k BC(M \times I^k)$ is highly connected in a "stability range" depending on the connectivity and dimension of M.

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In [18], Waldhausen developed machinery to compute the rational homotopy groups of a space $\Omega\,\mathrm{Wh}^{\mathrm{PL}}(K)$, conjecturally equivalent to the classifying space for PL fibrations with fibres of the homotopy type of K and with fibre homotopy trivializations. Several papers have been published which carry out these computations and give deductions via Assertions B and C above regarding the rational homotopy groups of automorphism groups of manifolds (e.g., [2], [3], [8], [9]).

Unfortunately, although Hatcher's [12] is insightful and has been highly influential, the arguments given there for Assertions A, B and C are now well known to be insufficient. Arguments for Assertions B and C have been announced, although none have yet appeared. Here, we provide and prove a suitable reformulation of Assertion A and show that $\Omega\,\mathrm{Wh}^{\mathrm{PL}}(K)$ does classify PL fibrations with fibres of the homotopy type of K and with fibre homotopy trivializations.

Let $\mathcal{S}$ be the category of finite ordered simplicial complexes and ordered simplicial maps (i.e., the vertices of each simplex are ordered, compatibly with the orderings for its faces, and the maps respect this order) and let $c\mathcal{S}$ be the subcategory whose maps are CEPL. For an ordered simplicial complex K, let $c\mathcal{S}(K)$ be the category of finite ordered simplicial pairs (L, K); with $K \subset L$ a homotopy equivalence, and ordered simplicial CEPL maps which restrict to the identity map of K.

THEOREM 1. *The classifying space $Bc\mathcal{S}$ classifies* PL *fibrations up to concordance. Moreover,* PL *fibrations with fibres of the homotopy type of K and with fibre homotopy trivializations are classified up to concordance by $Bc\mathcal{S}(K)$.*

Let $\mathcal{C}$ be the category of finite simplicial sets (c.s.s. complexes) and let $c\mathcal{C}$ and $c\mathcal{C}(K)$ be the simplicial set analogues of the categories $c\mathcal{S}$ and $c\mathcal{S}(K)$ above. (The analogue of a CEPL map is a map whose geometric realization has contractible fibres. We call such maps contractible.) To an ordered simplicial complex K we may associate a simplicial set whose k-simplices are the ordered simplicial maps of the standard k-simplex Δ^k into K. This identifies $\mathcal{S}$ with a full subcategory of $\mathcal{C}$ and induces $\iota: c\mathcal{S} \to c\mathcal{C}$ and $\iota: c\mathcal{S}(K) \to c\mathcal{C}(K)$. Waldhausen defines $\Omega\,\mathrm{Wh}^{\mathrm{PL}}(K)$ to be classifying space $Bc\mathcal{C}(K)$ (e.g., [19, §3]), so that the connection between $\Omega\,\mathrm{Wh}^{\mathrm{PL}}(K)$ and PL fibrations is given by the following.

THEOREM 2. *The maps $\iota: c\mathcal{S} \to c\mathcal{C}$ and $\iota: c\mathcal{S}(K) \to c\mathcal{C}(K)$ are homotopy equivalences.*

Following Quillen [16] this means that the induced maps of classifying spaces are homotopy equivalences.

REMARK. The order preserving property of the maps in $c\mathcal{S}$ seems a bit unnatural geometrically. Our classification theorem actually proceeds by first classifying PL fibrations by a more geometrically natural category $c\mathcal{D}$ of convex cell complexes and then comparing $c\mathcal{D}$ with $c\mathcal{S}$ via first derived complexes (which are naturally ordered). As there is no natural comparison between an unordered simplicial complex and its first derived, we do not know if the analogous category of unordered simplicial complexes is equivalent to these.

We also consider the much simpler case of PL quasifibrations (PL maps which are quasifibrations), showing these to be classified up to concordance by the classifying spaces of the homotopy equivalent categories $h\mathcal{S}$ and $h\mathcal{C}$, where the prefix h denotes the subcategory of $\mathcal{S}$ or $\mathcal{C}$ whose maps are homotopy equivalences (after geometric realization). We also give the unpublished result of Waldhausen that if $h\mathcal{C}_K$ is the subcategory of $h\mathcal{C}$ whose objects have the homotopy type of K, then the natural map from $Bh\mathcal{C}_K$ to the classifying space for Hurewicz fibrations (without finiteness conditions) with fibres of the homotopy type of K is a homotopy equivalence.

We give an outline of our constructions and methods in §1, and state our primary classification theorem (Theorem 1.1) as well as the basic tool results needed later. We prove Theorem 1.1 in §2, and prove Theorem 2 and the absolute case of Theorem 1 in §3.

Section 4 is devoted to the case of PL fibrations with fibre homotopy trivializations, and includes unpublished material of Waldhausen. Section 5 contains proofs of some tool lemmas stated in §1. This is followed by §6, which gives a categorical interpretation of the functor on PL spaces represented by the classifying space of a discrete category.

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1. Basic constructions. Let $\mathfrak{D}$ be the category of all finite convex cell complexes in Euclidean space and all maps $f: L_1 \to L_2$ such that if σ is a cell of L_1, then f restricts on σ to a linear map $f_1: \sigma \to \tau$, with τ a cell of L_2 such that

(1) f_1 admits a linear section carrying interior points to interior points, and

(2) if $\tau_1 \subset \tau$ is a face, then $f_1^{-1}\tau_1$ is a face of σ.

We say that $\mathcal{E} \subset \mathfrak{D}$ is special if

(1) Finite simplicial complexes are objects of $\mathcal{E}$.

(2) Restrictions of domain and codomain of maps in $\mathcal{E}$ remain in $\mathcal{E}$.

(3) Cartesian products of complexes (with cells the Cartesian products of the cells of the factors) and maps in $\mathcal{E}$ are in $\mathcal{E}$, as are the projection maps.

Note that $\mathfrak{D}$ itself is special and that there is a unique smallest special subcategory $\mathcal{P}$, whose objects are the subcomplexes of finite products of simplicial complexes, with polysimplices (products of simplices) as cells, and whose morphisms are restrictions of ambient projection maps.

For a subcategory $\mathcal{E}$ of $\mathfrak{D}$, let $c\mathcal{E}$ (resp. $h\mathcal{E}$) be the subcategory of $\mathcal{E}$ whose maps are CEPL (resp. homotopy equivalences). We think of the prefixes c or h as denoting subcategories of "weak equivalences" to be denoted generically by a prefix w as in [18]. We prove the following.

THEOREM 1.1. *Let $\mathcal{E}$ be a special subcategory of $\mathfrak{D}$. Then $Bc\mathcal{E}$ (resp. $Bh\mathcal{E}$) classifies* PL *fibrations (resp. quasifibrations) up to concordance.*

Thus, for $w = c$ or h, any two such $Bw\mathcal{E}$ are homotopy equivalent provided that their fundamental groups are countably generated. Clearly this is true of $Bw\mathcal{P}$. The following gives such homotopy information in many cases and implies the absolute case of Theorem 1.

THEOREM 1.2. *Let $\mathcal{E}$ be a special subcategory of $\mathfrak{D}$ containing the category of finite simplicial complexes and simplicial maps. Then the natural map $\chi: w\mathcal{S} \to w\mathcal{E}$ is a homotopy equivalence for $w=c$ or h.*

We prove this in §3.

First, we give a categorical interpretation of the set of homotopy classes of maps from a PL space to the classifying space of a category. Note first that if $\mathcal{L}$ is a partially ordered set (hence category) then the classifying space $B\mathcal{L}$ is naturally a simplicial complex. Let $\mathcal{C}$ be a category and B a PL space. Define a simplicial functor from B to $\mathcal{C}$ to be a triangulation $B\mathcal{L} \cong B$ ($\cong$ denotes a PL homeomorphism) together with a contravariant functor $F: \mathcal{L} \to \mathcal{C}$, where $\mathcal{L}$ is a partially ordered set. Two simplicial functors are concordant if there is a simplicial functor on $B \times I$ which restricts to the given functors on $B \times \partial I$. We write $\{B, \mathcal{C}\}$ for the set of concordance classes of simplicial functors from B to $\mathcal{C}$.

The following theorem ought to be folklore. We prove it in §6.

THEOREM 1.3. *The set of homotopy classes of maps from B to $B\mathcal{C}$ is naturally isomorphic to the set of concordance classes of simplicial functors from B to $\mathcal{C}$.*

Passage from simplicial functors on B to PL fibrations or quasifibrations over B will be given by a triangulated iterated mapping cylinder functor to be defined below. The idea of using iterated mapping cylinders to classify PL fibrations is originally due to Hatcher [12]. Modulo change of notation, he defines the iterated mapping cylinder, $M(f_1, \dots, f_k)$, of maps $f_i: L_i \to L_{i-1}$ by induction, with $M(f_k)$ the cylinder of f_k, and with $M(f_i, \dots, f_k)$ the mapping cylinder of the inductively defined composite $M(f_{i+1}, \dots, f_k) \to L_i \xrightarrow{f_i} L_{i-1}$. There is a natural, inductively defined projection map of $M(f_1, \dots, f_k)$ onto the k-simplex Δ^k, which may be identified with the iterated mapping cylinder of k copies of the identity map of a point. The construction as stated is defined unambiguously and functorially in the topological category; but is not well-defined in PL, as the simplicial mapping cylinder of [6] or [20] depends strongly on the triangulation of a map, with isomorphisms between the mapping cylinder of a simplicial map and a subdivision being noncanonical and not fibrewise. (One way of seeing the non-canonicity is that the natural embedding of the topological mapping cylinder of a degeneracy $\Delta^2 \to \Delta^1$ in the join $\Delta^2 * \Delta^1$ is onto a non-PL subspace. Another, and related way is that pushouts are not functorial in the PL category.) We generalize Cohen's simplicial mapping cylinder [6] to an iterated mapping cylinder $C(f_1, \dots, f_k)$ of maps of convex cell complexes, and generalize Whitehead's proof [20, §10] that this construction is fibrewise homeomorphic to the topological iterated mapping cylinder $M(f_1, \dots, f_k)$ (Lemma 1.4 below).

Note that if Δ^k is viewed as the classifying space of the partially ordered set $\underline{k} = v_0 < \cdots < v_k$ and $F: \underline{k} \to$ spaces is defined by $F(v_i < v_j) = f_{i+1} \circ \cdots \circ f_j: L_j \to L_i$, then $M(f_1, \dots, f_k) \to \Delta^k$ is precisely the 2-sided bar construction $B(\text{ob } \underline{k}, \underline{k}, F) \to B\underline{k} \cong \Delta^k$ of [15, §12], sometimes referred to as the homotopy colimit of F. For a general simplicial functor $F: \mathcal{L} \to$ spaces we write $\epsilon_F: M(F) \to B\mathcal{L} \cong B$ for the natural projection of the homotopy colimit of F onto $B\mathcal{L} \cong B$. Once again this is precisely the assembly procedure given by Hatcher [12, p. 109].

For L any convex cell complex (possibly infinite) let $\mathfrak{L}$ be its category of cells, the partially ordered set of the cells of $\mathfrak{L}$, ordered by inclusion. Write L' for $B\mathfrak{L}$, the abstract first derived complex of L, simplicially isomorphic to any first derived subdivision.

Generalizing the work of Cohen [6], we define the iterated mapping cylinder $C(f_1, \dots, f_k)$ of maps $f_i: L_i \to L_{i-1}$ in $\mathfrak{D}$, for $1 \le i \le k$, to be the subcomplex of the iterated join $L_0' * \cdots * L_k'$ consisting of those simplices $\sigma_0 * \cdots * \sigma_k$ such that if $i<j$ and σ_i and σ_j are not the empty simplex, say $\sigma_i = \sigma_{i,0} < \cdots < \sigma_{in_i}$ and $\sigma_j = \sigma_{j0} < \cdots < \sigma_{jn_j}$, then $\sigma_{in_i} < f_{i+1} \circ \cdots \circ f_j(\sigma_{j0})$. Thus $C(f_1, \dots, f_k)$ is the classifying space of the partially ordered set obtained by adjoining the relations generated by $f_i(\sigma) < \sigma$ for $\sigma \subset L_i$ to the disjoint union of the categories $\mathfrak{L}_i$. The natural projection $\pi: C(f_1, \dots, f_k) \to \Delta^k$, the k-simplex, is the simplicial map taking L_i to the ith vertex of Δ^k, and is induced by the obvious map of partially ordered sets.

Let $F: \mathfrak{L} \to \mathfrak{D}$ be a simplicial functor on B. Let $\pi_F: C(F) \to B\mathfrak{L} \cong B$ be the simplicial map which restricts over each k-simplex, $x_0 < \cdots < x_k$ of $B\mathfrak{L}$ to $\pi: C(F(x_0 < x_1), \dots, F(x_{k-1} < x_k)) \to \Delta^k \cong x_0 < \cdots < x_k$, the last isomorphism preserving the order of the vertices. As above, π_F is induced by the obvious map of partially ordered sets.

We show the following in §5.

LEMMA 1.4. *Let $F: \mathfrak{L} \to \mathfrak{D}$ be a simplicial functor on B. Then $C(F)$ and $M(F)$ are fibrewise homeomorphic.*

This permits us to use topological information in deducing when $\pi_F: C(F) \to B\mathfrak{L}$ is a fibration or quasifibration.

The following is a straightforward generalization of an argument in [12, 2.1]. We give another argument in §5.

LEMMA 1.5. *Let F be a functor from a locally finite partially ordered set $\mathfrak{L}$ into the category of compact* ANR's. *Then $\epsilon_F: M(F) \to B\mathfrak{L}$ is a fibration if and only if each map $F(v<w)$ is cell-like* (CE) (*i.e., point inverses have the shape of a point*) *and is a quasifibration if and only if each map $F(v<w)$ is homotopy equivalence.*

Since CEPL maps are precisely those maps which are both PL and CE, we obtain the following.

COROLLARY 1.6. *Let $F: \mathfrak{L} \to \mathfrak{D}$ be a simplicial functor on B. Then*

$$\pi_F: C(F) \to B\mathfrak{L} \cong B$$

is a PL *fibration if and only if F takes value in $c\mathfrak{D}$ and is a* PL *quasifibration if and only if F takes value in $h\mathfrak{D}$.*

For simplicity, we give the following basic observation here. By a simplicial fibration (resp. quasifibration) over B we mean a simplicial map onto a triangulation of B which is a Hurewicz fibration (resp. quasifibration). A concordance between such is simplicial (quasi-) fibration over $B \times I$ which restricts to the given ones over $B \times \partial I$. Let $\Delta\,\mathrm{Fib}(B)$ (resp. $\Delta QF(B)$) be the set of concordance classes of same over B and let PL Fib(B) (resp. PL $QF(B)$) be the set of concordance classes of PL(quasi-) fibrations over B.

LEMMA 1.7. *The natural maps*

$$\Delta\,\mathrm{Fib}(B) \to \mathrm{PL\,Fib}(B) \quad \textit{and} \quad \Delta QF(B) \to \mathrm{PL}\; QF(B)$$

are isomorphisms.

Proof. By the Alexander trick of inductive coning, for any two triangulations of a given PL map $p: E \to B$ there is a triangulation of $p \times 1: E \times I \to B \times I$ which restricts to the given triangulations over $B \times \partial I$. □

The simplicial iterated mapping cylinder functor is seen to define a map $C: \{B, c\mathfrak{D}\} \to \Delta\,\mathrm{Fib}(B)$. Passage from simplicial fibrations to simplicial functors may be obtained from the following more highly structured version of a construction of [12] (see Remarks below).

Let $p: E \to L$ be a simplicial map with $L \cong B$. Define $\mu(p): \mathcal{L} \to \mathcal{P}$ as follows. (As above, $\mathcal{L}$ is the category of simplices of L.) For σ a simplex of L let $\mu(p)(\sigma)$ be $p^{-1}\hat{\sigma}$, where $\hat{\sigma}$ is the barycenter of σ, with cells $\sigma_1 \cap p^{-1}\hat{\sigma}$ for each simplex σ_1 with $p\sigma_1 = \sigma$. Note that if σ has vertices $v_0, \ldots, v_k$, then $\sigma_1 \cap p^{-1}\hat{\sigma}$ is naturally linearly isomorphic to $(\sigma_1 \cap p^{-1}v_0) \times \cdots \times (\sigma_1 \cap p^{-1}v_k)$, so that $\mu(p)(\sigma)$ is isomorphic to a subcomplex of $p^{-1}v_0 \times \cdots \times p^{-1}v_k$. For $\tau \subset \sigma$, let $\mu(p)(\tau < \sigma)$ be the restriction of the projection away from the inverse images of the vertices not in τ. Here the image of $\sigma_1 \cap p^{-1}\hat{\sigma}$ under $\mu(p)(\tau < \sigma)$ is $\sigma_1 \cap p^{-1}\hat{\tau}$.

The partially ordered set defining $C(\mu(p))$ has as objects the cells $\sigma_1 \cap p^{-1}\hat{\sigma}$ as above, mapping down onto the object σ of $\mathcal{L}$. Mapping $\sigma_1 \cap p^{-1}\hat{\sigma}$ to σ_1 induces an isomorphism over $\mathcal{L}$ from this category to the category of simplices of E. We obtain the following.

LEMMA 1.8. *$C(\mu(p))$ is fibrewise simplicially isomorphic to a first derived triangulation of p.*

Thus, any simplicial map has a well-defined combinatorial iterated mapping cylinder decomposition.

This, together with Corollary 1.6, gives us well-defined maps

$$\mu: \Delta\,\mathrm{Fib}(B) \to \{B, c\mathcal{P}\} \quad \text{and} \quad \mu: \Delta QF(B) \to \{B, h\mathcal{P}\}$$

such that $C \circ \mu = id$. Thus, Theorem 1.1 will follow if we display a concordance of simplicial functors from $\mu(C(F))$ to F. We do this in §2.

REMARKS. Let $p: E \to L$ be a simplicial map and let $\mu'(p)$ be the functor which assigns to each $\sigma \in L$ the first derived complex of $\mu(p)(\sigma)$. Hatcher [12, p. 105] studied $\mu'(p)$ as a functor from $\mathcal{L}$ into PL spaces and noted that $M(\mu'(p))$ is fibrewise topologically homeomorphic to p. This forms the intuitional starting point of our approach, and our emphasis on convex cell complexes arises from Lemma 1.8 together with the fact that without the machinery we give here we see no way to provide a concordance of PL fibrations between p and $C(\mu'(p))$ for a PL fibration p.

By a compact ANR-fibration over a PL space B we mean a proper map, with ANR fibres and total space, which is a Hurewicz fibration. Write ANR Fib(B) for the set of concordance classes of same. By [5, Theorem 1] and Lemma 5.2 below this is equivalent to the set of isomorphism classes of Hilbert cube manifold bundles over B. As noted in [12, p. 103], Assertion B above is thus equivalent to the statement that the natural map j: PL Fib$(B) \to$ ANR Fib(B) is an isomorphism. Let $c\mathcal{PL}$ be Hatcher's category of compact PL spaces and CEPL maps. The constructions of [12] provide us with a well-defined commutative diagram

$$\begin{tikzcd} \mathrm{PL\,Fib}(B) \simeq \Delta\,\mathrm{Fib}(B) \arrow[rr, "\mu'"] \arrow[dr, "j"'] & & \{B, c\mathcal{PL}\} \arrow[dl, "M"] \\ & \mathrm{ANR\,Fib}(B). & \end{tikzcd}$$

However, even given a proof of Assertion B, and hence an inverse, j^{-1}, of j, the existence of a concordance of simplicial functors between $\mu(j^{-1}M(F))$ and F is highly non-obvious. The author does not see any reason why $Bc\mathcal{PL}$ should classify PL fibrations.

The techniques presented here have yet to be applied to PL bundles.

CONJECTURE. PL bundles are classified up to fibrewise isomorphism by $Bt\mathfrak{D}$, where $t\mathfrak{D}$ is the subcategory of $\mathfrak{D}$ whose maps are transverse cellular [1].

Finally, we shall need some information regarding the fibrewise PL properties of the iterated mapping cylinder $C(F)$.

First, note that if F is the constant functor to a complex L_1, then π_F is the projection map from the usual ordered triangulation of $L_1' \times B\mathcal{L}$ onto $B\mathcal{L}$. In general, restricting to a totally ordered subset of $\mathcal{L}$, say $x_0 < \cdots < x_k$, let G be the constant functor to $F(x_k)$ and let $\eta: G \to F$ be the natural map. We show the following in §5.

LEMMA 1.9. *If $F(x_i)$ is a convex cell for $0 \leq i \leq k$, then the induced map $C(\eta)$ is collapsible and transverse cellular as is the restriction of $C(\eta)$ to the fibres over points not in the $(k-1)$-simplex $x_0 < \cdots < x_{k-1}$. For general F, the restriction of $C(\eta)$ to these same fibres is collapsible, and $C(\eta)$ is* CEPL *if and only if the maps of F are* CEPL.

2. Classification of PL fibrations and quasifibrations. We prove Theorem 1.1. As noted above, for $w = c$ or h and for a simplicial functor $F: \mathcal{L} \to w\mathcal{E}$ on B it suffices to provide a concordance of simplicial functors between $\mu(\pi_F)$ (i.e., $\mu \circ C(F)$) and F, taking value in $w\mathcal{E}$. Preservation of the category $\mathcal{E} \subset \mathfrak{D}$ in the constructions below is a consequence of the definition of special subcategory.

Write $\mathcal{L}_1$ for the category of simplices of $B\mathcal{L}$ and define $\epsilon: \mathcal{L}_1 \to \mathcal{L}$ by $\epsilon(x_0 < \cdots < x_k) = x_k$. Then $F_1 = F \circ \epsilon$ is concordant to F by the proof of Theorem 1.3 below. We define a subfunctor $\rho_F: \mathcal{L}_1 \to \mathcal{E}$ of the cartesian product functor $\mu(\pi_F) \times F_1$ as follows. The cells of $\rho_F(x_0 < \cdots < x_k)$ are those product cells $A \times \sigma$ such that the carrier of A is contained in the closed star of σ. Here, the carrier of A is the smallest cell τ of $F(x_k)$ such that A is contained in the iterated mapping cylinder of the restriction of the maps to τ.

Let $p_1: \rho_F \to \mu(\pi_F)$ and $p_2: \rho_F \to F_1$ be induced by the projection maps. Then if x is an interior point of A, $p_1^{-1}(x)$ is the closed star of carrier A, and hence collapsible. For x an interior point of σ, $p_2^{-1}(x)$ is the fibre of π_H over an interior point of $x_0 < \cdots < x_k$, where H is the restriction of the maps of F to the closed star of σ and its iterated images. This is collapsible by Lemma 1.9. Thus, there are natural transformations

$$F \leftarrow \rho_F \to \mu(\pi_F)$$

through CEPL maps, giving rise to the desired concordance of simplicial functors (see §6) provided that the maps of ρ_F are in $w\mathcal{E}$. This is immediate for $w = h$, so let $w = c$.

If F is a constant functor, the point inverses of an interior point (x, y) of $A \times \sigma$ are isomorphic to those of x in $\mu(\pi_H)$ with H as above, which are contractible, as π_H is a fibration. If we restrict to a totally ordered subset of $\mathcal{L}$, we obtain a natural transformation $\eta: G \to F$, where G is a constant functor. Since f is CEPL whenever g and $f \circ g$ are CEPL, the following lemma suffices.

LEMMA 2.1. *Let $\eta: G \to F$ be a natural transformation between functors G, $F: \mathcal{L} \to c\mathfrak{D}$. Then the induced map $\rho_\eta: \rho_G \to \rho_F$ is* CEPL.

Proof. First note by inspection of the first derived functor that if $f: L_1 \to L_2$ is a map in $\mathfrak{D}$ and if x is an interior point of the cell σ of L_2, then $f^{-1}x$ is isomorphic to the category of cells τ of L_1 for which $f(\tau) = \sigma$.

Let $A \times \sigma$ be a cell of $\rho_F(x_0 < \cdots < x_k)$. If $\mu(\pi_\eta)(\bar{A}) = A$, then $\eta(\text{carrier } \bar{A}) = \text{carrier } A$. Thus, if τ is the smallest cell of $F(x_k)$ containing both σ and carrier A and if $\rho_\eta(\bar{A} \times \bar{\sigma}) = A \times \sigma$, and $\alpha(\bar{A} \times \bar{\sigma})$ is the smallest cell of $G(x_k)$ containing $\bar{\sigma}$ and carrier $\bar{A}$, then $\eta(\alpha(\bar{A} \times \bar{\sigma})) = \tau$. Thus, α defines a functor from the category of cells of ρ_G mapping onto $A \times \sigma$ to the category, $\mathcal{K}$, of cells of $G(x_k)$ mapping onto τ. Since the maps of η are CEPL, $\mathcal{K}$ is contractible, and it suffices to show that α is a homotopy equivalence.

We use Quillen's Theorem A [16]. Let $\eta(\epsilon) = \tau$. Then $\alpha(\bar{A} \times \bar{\sigma}) \subset \epsilon$ if and only if carrier $\bar{A} \subset \epsilon$ and $\bar{\sigma} \subset \epsilon$. Thus, the comma category $\epsilon \backslash \alpha$ is the product of the category of all such $\bar{A}$ and the category of all such $\bar{\sigma}$. The latter has a terminal object $\epsilon \cap \eta^{-1}\sigma$, hence is contractible. The former is isomorphic to a point inverse in the fibre over an interior point of Δ^k of the natural map from $C(g_1, \ldots, g_k)$ to $C(f_1, \ldots, f_k)$, where g_i and f_i are the restrictions of the maps of G and F to the iterated images of ϵ and τ respectively. The result now follows from Lemma 1.9 and the fact that $\eta: \epsilon \to \tau$ is CEPL. □

3. Proof of Theorem 1 in the absolute case and of Theorem 2. The absolute case of Theorem 1 follows from Theorem 1.2 which we prove here. Let $\beta: w\mathcal{E} \to w\mathcal{S}$ be the abstract first derived functor, $\beta(L) = B\mathcal{L} = L'$ in our notations above. For L an ordered simplicial complex there is an ordered simplicial map $\epsilon: L' \to L$, natural with respect to ordered simplicial maps, which takes a vertex $\sigma_0 < \cdots < \sigma_k$ of L' to the last vertex of σ_k. We show that ϵ is CEPL, and hence induces a natural transformation from $\beta\chi$ to $1_{w\mathcal{S}}$, and hence a homotopy from

$B(\beta\chi)$ to $1_{Bw\mathcal{S}}$. Since ϵ preserves the simplices of L and pushouts of CE maps are CE it suffices to show that $\epsilon\colon \sigma' \to \sigma$ is CEPL for a simplex σ. This factors as the composite

$$\sigma' = \operatorname{cone}((\partial\sigma)') \xrightarrow{\operatorname{cone}\epsilon} \operatorname{cone}(\partial\sigma) \xrightarrow{\nu} \sigma,$$

where ν is the identity on $\partial\sigma$ and takes the cone point to the last vertex. Since ν is collapsible the result follows by induction on dimension.

To show that $B(\chi\beta)$ is homotopic to the identity, we construct a functor $\rho\colon w\mathcal{E} \to w\mathcal{E}$, and natural transformations $p_1\colon \rho \to \chi\beta$ and $p_2\colon \rho \to 1_{w\mathcal{E}}$. As in the proof of Theorem 1.1, $\rho(L)$ is the subcomplex of $L' \times L$ consisting of cells $A \times \sigma$ such that carrier $A \subset \operatorname{star} \sigma$, and p_1 and p_2 are induced by the projections. Here carrier has the usual meaning. Again p_1 and p_2 are trivially CEPL, and the proof that the maps of ρ are in $w\mathcal{E}$ is analogous to that of Lemma 2.1 but simpler.

REMARK. The argument above does not apply to the category of unordered simplicial complexes in place of $\mathcal{E}$ as the product of unordered simplicial complexes is not functorially triangulated.

The following version of Theorem 2 includes the case of quasifibrations.

THEOREM 3.1. *For $w = c$ or h, the natural map $\iota\colon w\mathcal{S} \to w\mathcal{C}$ is a homotopy equivalence. Moreover for any object K of $\mathcal{S}$, $\iota\colon c\mathcal{S}(K) \to c\mathcal{C}(K)$ is a homotopy equivalence.*

Proof. We give the proof in the absolute case, with the argument being identical for the case of pairs.

We apply Quillen's Theorem A. Since the geometric realization of $\iota(L)$ is naturally isomorphic to L, ι is an embedding onto the full subcategory of $w\mathcal{C}$ consisting of objects for which the characteristic maps of nondegenerate simplices are embeddings determined by their vertices. We call such objects simplicial complexes, identify $w\mathcal{S}$ with its image under ι and identify ι with the inclusion of this subcategory. Since $w\mathcal{S}$ is full, the comma categories $\iota(L)\backslash\iota$ are contractible. Let $f\colon \iota(L) \to L_1$ be a map in $w\mathcal{C}$. Since products and subcomplexes of simplicial complexes are simplicial complexes, the pullback over f induces a functor $f^*\colon L_1\backslash\iota \to \iota(L)\backslash\iota$ and a natural transformation $\eta\colon f_* f^* \to 1_{(L_1\backslash\iota)}$. Thus, $L_1\backslash\iota$ is contractible and the hypothesis of Theorem A is satisfied, provided each $L_1\backslash\iota$ is nonempty (i.e., that any finite simplicial set is the contractible simplicial image of a finite simplicial complex). We prove this by induction on the number of nondegenerate simplices. Let $L_1 = X \cup_h \Delta^k$ with $h\colon \partial\Delta^k \to X$, and let $f\colon Y \to X$ be contractible with Y a simplicial complex. Let Z be the pullback over h of Y and let $\bar{h}\colon Z \to Y$ be the induced map. Then $L = C(\bar{h}) \cup \operatorname{cone} Z$ is a simplicial complex and each map below is contractible.

$$L \xrightarrow{f_1} C(h) \cup \operatorname{cone} \partial\Delta^n \xrightarrow{f_2} X' \cup_{h'} \operatorname{cone} \partial\Delta^n = L_1' \xrightarrow{\epsilon} L_1.$$

Here $C(h)$ is the analogue of our mapping cylinder for simplicial sets, f_1 is the natural map, f_2 is induced by the natural collapse of $C(h)$ onto X' and ϵ is the analogue for simplicial sets of the map ϵ in the proof of Theorem 1.2 above. □

4. PL fibrations with fibre homotopy trivializations. We first specify our preferred classifying space for ordinary Hurewicz fibrations (without finiteness conditions) with fibres of the homotopy type of K. Let $h\mathcal{J}_K$ be the topological category whose objects are the compact ANR's of the homotopy type of K (embedded in the Hilbert cube) and whose space of morphisms from X to Y is the space of homotopy equivalences from X to Y. Then $Bh\mathcal{J}_K$ classifies fibrations as above by [15, §12].

For any of the categories $w\mathcal{A}$ defined above, let $w\mathcal{A}_K$ be the full subcategory whose objects have the homotopy type of K. Then there is an induced functor $\nu: w\mathcal{A}_K \to h\mathcal{J}_K$, compatible with all of the inclusion maps between the categories $w\mathcal{A}$, which corresponds, on passage to classifying spaces, to forgetting the PL and finiteness conditions on a PL fibration, and to passage from a quasifibration $p: E \to B$ to the induced Hurewicz fibration $\operatorname{Hur} p: E \times_B B^I \to B$.

Suppose given a PL fibration $p: E \to B$ with classifying map $B \cong B\mathcal{L} \xrightarrow{BF} Bc\mathcal{S}_K$. By the covering homotopy property, a nullhomotopy of $B\nu \circ BF$ induces a fibre homotopy trivialization of p. Moreover, the induced equivalence between the space of nullhomotopies of $B\nu \circ BF$ and the space of maps from B to $\Omega Bh\mathcal{J}_K$ shows that homotopy classes of nullhomotopies are in 1–1 correspondence with homotopy classes of fibre homotopy trivializations. Since homotopic trivializations are concordant, and since concordant PL fibrations are homotopy-canonically fibre homotopy equivalent, we obtain the following.

PROPOSITION 4.1. *Concordance classes of* PL *fibrations with fibres of the homotopy type of K and with fibre homotopy trivializations are classified by the homotopy fibre of* $B\nu: Bc\mathcal{S}_K \to Bh\mathcal{J}_K$.

The following is an unpublished result of Waldhausen, analogous to [12, 3.2].

PROPOSITION 4.2. *For any object K of* $\mathcal{C}$, *the natural maps*

$$Bc\mathcal{C}(K) \to Bc\mathcal{C}_K \to Bh\mathcal{C}_K$$

form a fibre sequence.

Proof. Let $\eta: c\mathcal{C}_K \to h\mathcal{C}_K$ be the natural inclusion. We show that for any object L of $\mathcal{C}_K$, the natural inclusion of $c\mathcal{C}(L)$ in the comma category L/η is a homotopy equivalence and that the hypotheses of Quillen's Theorem B hold for η.

For the first statement, note that $c\mathcal{C}(L)$ is the full subcategory of L/η whose objects are (weak) homotopy equivalences $L \xrightarrow{f} L_1$ which are inclusions. Denote this by $i_L: c\mathcal{C}(L) \subset L/\eta$. Let $\rho_L: L/\eta \to c\mathcal{C}(L)$ be given by $\rho_L(L \xrightarrow{f} L_1) = L \subset Z(f)$, where $Z(f)$ is the standard mapping cylinder functor in the simplicial category. Then the projection of $Z(f)$ onto L_1 induces natural transformations displaying ρ_L as a homotopy inverse to i_L.

For the second statement, it suffices to show that a contractible map $f: L_0 \to L_1$ induces a homotopy equivalence $f^*: L_1/\eta \to L_0/\eta$, so then the natural maps $Sd^n L \to L$ and $L \times I \to L$ induce homotopy equivalences of comma categories, and hence so does any weak homotopy equivalence (cf. §6).

Thus, let $f: L_0 \to L_1$ be contractible, and let $f_*: c\mathcal{C}(L_0) \to c\mathcal{C}(L_1)$ be obtained from pushing out by f. Because f is contractible, the pushout diagram

$$\begin{array}{ccc} L_2 & \longrightarrow & f_*L_2 \\ \cup & & \cup \\ L_0 & \xrightarrow{f} & L_1 \end{array}$$

induces a natural transformation from $i_{L_0}: c\mathcal{C}(L_0) \subset L_0/\eta$ to f^*f_*. Moreover, since f is contractible, the natural map from the double mapping cylinder $Z(f) \cup_{L_0} Z(f)$ onto L_1 is contractible, inducing a natural transformation from $f_*\rho_{L_0} f^* i_{L_1}$ to i_{L_1}, with ρ_{L_0} as above. □

Theorem 3.1 now gives the following.

COROLLARY 4.3. *The natural maps*

$$Bc\mathcal{S}(K) \to Bc\mathcal{S}_K \to Bh\mathcal{S}_K$$

form a fibre sequence.

Theorem 1 now follows from Proposition 4.1 and the following unpublished argument of Waldhausen (cf. [10, 4.6]).

THEOREM 4.4. *The natural map $Bh\mathcal{C}_K \to Bh\mathcal{J}_K$ is a homotopy equivalence.*

Proof. Define a simplicial category $\mathcal{C}_*$ as follows. The objects of $\mathcal{C}_n$ are the same as those of $h\mathcal{C}_K$ for all n. The morphisms of $\mathcal{C}_n$ from L_0 to L_1 are the weak homotopy equivalences of simplicial sets $L_0 \times \Delta^n \to L_1 \times \Delta^n$ which fiber over the projections to the canonical n-simplex Δ^n. The faces and degeneracies in the simplicial structure of $\mathcal{C}_*$ are obtained from pulling back along the simplex coordinate.

Let B be the geometric realization of the simplicial space $(B\mathcal{C})_*$. Considering the classifying space functor as the realization of the simplicial nerve and reversing the order of realization in the resulting bisimplicial set, we see that B is the classifying space of the topological category whose objects are those of $h\mathcal{C}_K$ and whose space of morphisms from L_0 to L_1 is the geometric realization of the simplicial mapping space of weak homotopy equivalences from L_0 to L_1. Thus, there is an induced homotopy equivalence from B to $Bh\mathcal{J}_K$ by [15, §12]. Moreover, the natural map of $Bh\mathcal{C}_K$ to $Bh\mathcal{J}_K$ factors through this equivalence via the identification of $h\mathcal{C}_K$ with $\mathcal{C}_0$ and the natural map $B\mathcal{C}_0 \to B = |B\mathcal{C}_*|$. Thus, it suffices to show that the total degeneracy map $s: \mathcal{C}_0 \to \mathcal{C}_n$ is a homotopy equivalence for all n.

Note that morphisms in $\mathcal{C}_n$ from L_0 to L_1 are in 1–1 correspondence with weak homotopy equivalences of simplicial sets from $L_0 \times \Delta^n$ to L_1, via the projection of $L_1 \times \Delta^n$ onto L_1. Note also that a morphism is in the image of the total degeneracy map s if and only if this map $L_0 \times \Delta^n \to L_1$ factors through the projection of $L_0 \times \Delta^n$ onto L_0. This induces an identification of L_0/s with the category whose objects are weak homotopy equivalences $L_0 \times \Delta^n \to L_1$ and whose morphisms are commutative diagrams

$$\begin{tikzcd} & L_1 \arrow[d] \\ L_0 \times \Delta^n \arrow[ur] \arrow[dr] & \\ & L_2 \end{tikzcd}$$

of weak homotopy equivalences. This category has an initial object given by the identity map of $L_0 \times \Delta^k$, and hence is contractible. The theorem follows from Quillen's Theorem A. $\square$

Note that a similar argument, using the existence of homeomorphisms from $M \times \Delta^k$ to M for M a Hilbert cube manifold, shows the following.

THEOREM 4.5. *Let M be a Hilbert cube manifold and let $HE(M)$ be its space of self-homotopy equivalences. Let $HE^\delta(M)$ be the same monoid, but with the discrete topology. Then the natural map $BHE^\delta(M) \to BHE(M)$ is a homotopy equivalence.*

Let $p: E \to B$ be a PL fibration. By a strong fiber homotopy trivialization we mean a fibre homotopy trivialization $j: B \times K \to E$ which is a PL inclusion. Such inclusions also give rise to a notion of strong concordance. Theorem 1, together with passage to iterated mapping cylinder projections from simplicial functors $\mathcal{L} \to c\mathcal{S}(K)$, gives the following.

COROLLARY 4.6. *Concordance classes of* PL *fibrations with fibres of the homotopy type K and with fibre homotopy trivializations are in* 1–1 *correspondence with strong concordance classes of* PL *fibrations with fibres of the homotopy type of K and with strong fibre homotopy trivializations.*

5. Fine structure of iterated mapping cylinders. First we prove Lemma 1.9. If each $F(x_i) = \sigma_i$, a convex cell, then $C(G) = \sigma_k' \times \Delta^k$. Note that the boundary of $C(G)$ is the full inverse image of its image under $C(\eta)$ and that the analogous statement holds for the restriction of $C(\eta)$ to fibres of points not in $x_0 < \cdots < x_{k-1}$. Thus, by [6], it suffices to show that $C(\eta)$ is collapsible.

In the general case, let $F(x_i) = L_i$ for $0 \le i \le k$ and write $f_i: L_i \to L_{i-1}$ for $F(x_{i-1} < x_i)$, $1 \le i \le k$. Let $A * \sigma$ be a simplex of $C(F)$ with $A \subset C(f_1, \ldots, f_{k-1})$ and $\sigma \subset L_k'$. If σ is nonempty, say $\sigma = \sigma_0 < \cdots < \sigma_n$, then $A * \sigma$ is in the iterated mapping cylinder of the restriction of the maps f_i to the iterated images of σ_n. Simplices of $C(G)$ which map onto $A * \sigma$ are of the form $\bar{A} * \sigma$, where $\bar{A} \subset \sigma_0' \times \Delta^{n-1}$ maps onto A under the composite

$$\sigma_0' \times \Delta^{n-1} \xrightarrow{f_k \times 1} (f_k \sigma_0)' \times \Delta^{n-1} \xrightarrow{\eta} C(f_1, \ldots, f_{k-1}).$$

Thus, we are back in the special case. By induction, the map η in the above composite is collapsible onto its image, and $f_k \times 1$ is a collapsible retraction so the composite is collapsible onto its image by [6, Theorem 8.1]. Thus, the point inverse under $C(\eta)$ of an interior point of $A \times \sigma$ is collapsible. If A is empty, then point inverses are trivial. If σ is empty, then a similar argument to the one above works in the special case, while in the general case, the composite

$$L'_k \times \Delta^{k-1} \xrightarrow{f_k \times 1} L'_{k-1} \times \Delta^{k-1} \xrightarrow{\eta} C(f_1, \ldots, f_{k-1})$$

is CEPL by induction.

To prove Lemma 1.4, note that if $f_i: \sigma_i \to \sigma_{i-1}$ are maps of convex cells for $1 \leq i \leq k$, then $C(f_1, \ldots, f_k)$ is the cone on $C(f_1, \ldots, f_{k-1}, f_k \mid \partial\sigma_k)$ with the cone point at the centroid of σ_k. Since π_F is simplicial, this coning is fibrewise linear along the cone lines. If we show that the analogous statement holds for $M(f_1, \ldots, f_k)$, then Lemma 1.6 follows by an inductively defined cone by cone fibrewise homeomorphism.

Choose a linear section s_i for f_i, with $s_i(\text{interior } \sigma_{i-1}) \subset \text{interior } \sigma_i$. Let $g_i: \sigma_k \to \sigma_k$ be the composite $s_k \circ \cdots \circ s_i \circ f_i \circ \cdots \circ f_k$ for $1 \leq i \leq k$ and let g_{k+1} be the identity map of σ_k. Thus, $g_i \circ g_j = g_{\min(i,j)}$. Linearly embed σ_k as a neighborhood of the origin in $\mathbf{R}^n$, with 0 in the interior of $g_1(\sigma_k)$, and define $\theta: \sigma_k \times \Delta^k \to \sigma_k \times \Delta^k$ by

$$\theta(x, u) = \left(\sum_{i=0}^{k} \left(\frac{1}{2}\right)^{k-i} u_i g_{i+1}(x), u \right) = (\theta_u(x), u),$$

where

$$\begin{aligned} u &= (u_0, \ldots, u_k) \in \Delta^k \\ &= \left\{ (u_0, \ldots, u_k) \in \mathbf{R}^{k+1} \mid u_i \geq 0 \text{ for } 0 \leq i \leq k, \ \sum_{i=0}^{k} u_i = 1 \right\}. \end{aligned}$$

This induces $\bar{\theta}: M(f_1, \ldots, f_k) \to \sigma_k \times \Delta^k$. We shall show

(1) $\bar{\theta}$ is an embedding.

(2) The linear coning of $\bar{\theta}(M(f_1, \ldots, f_{k-1}, f_k \mid \partial\sigma_k))$ to the cone point $0 \times v_k$ is a homeomorphism onto $\bar{\theta}(M(f_1, \ldots, f_k))$. Here v_k is the last vertex of Δ^k.

Note that (1) is true by induction on k provided that θ_u is an embedding when $u_k \neq 0$. Identify Δ^{k+1} with the join of Δ^k and the origin in $\mathbf{R}^{k+1}$ and define $\phi_u: \sigma_k \to \sigma_k$ by

$$\phi_u(x) = \sum_{i=0}^{k} u_i g_{i+1}(x)$$

for $u \in \Delta^{k+1}$. Then $\theta_u = \phi_{\lambda u}$, where $\lambda u = ((1/2^k)u_0, \ldots, u_k)$. For $u \in \Delta^{k+1}$ with $u_k \neq 0$, define $\Psi_u: \sigma_k \to \mathbf{R}^n$ by

$$\Psi_u(x) = \frac{1}{u_k} x - \sum_{i=0}^{k-1} \frac{u_i}{w_i w_{i+1}} g_{i+1}(x),$$

where

$$w_i = \sum_{j=i}^{k} u_j.$$

Then (1) follows from the following lemma.

LEMMA 5.1. *ϕ_u and Ψ_u extend to inverse isomorphisms of* $\mathbf{R}^n$.

Proof. Let $\psi_u(x) = y$. One shows by induction on j that

$$\sum_{i=j}^{k} u_i g_{i+1}(x) = y - \sum_{i=0}^{j-1} \frac{u_i w_j}{w_i w_{i+1}} g_{i+1}(y)$$

for $0 \leq j \leq k$. The induction follows by applying g_{j+1} to both sides of the above. When $j = k$, the result follows.

To prove (2), we show that the coning map, α, restricts to a homeomorphism on the preimage of each $\theta_u(\sigma_k)$. If $u_k = 1$, $\alpha^{-1}\theta_u(\sigma_k) = \text{cone } \partial\sigma_k$, mapped homeomorphically onto σ_k. If $u_k < 1$, let

$$\bar{u} = \left(\frac{u_0}{1-u_k}, \ldots, \frac{u_{k-1}}{1-u_k}, 0\right).$$

Then $\alpha^{-1}\theta_u(\sigma_k)$ is homeomorphic, by a homeomorphism h, to the union of $\theta_{\bar{u}}(\sigma_k)$ and $\theta_{(1-t)u+t\bar{u}}(\partial\sigma_k)$ for $0 \leq t \leq 1$, which is in turn homeomorphic to the mapping cylinder of the restriction to $\partial\sigma_k$ of $f_i \circ \cdots \circ f_k$ for some i. Since this is contractible and since the inclusion of $\theta_u(\partial\sigma_k)$ factors through it, α must be onto. An easy calculation shows that the appropriate restriction of $\alpha \circ h \circ \theta_{(1-t)u+t\bar{u}}$ is the inclusion in the fibre over u of $\phi_{\lambda(u-sv_k)}$, where $s = tu_k/(1-u_k+tu_k)$. So it suffices to show that if $u \in \Delta^{k+1}$ with $u_k \neq 0$ and for $0 < t \leq 1$, the image of $\phi_{u-tu_kv_k}$ lies in the interior of the image of ϕ_u. Since $\phi_{u-tu_kv_k} = (1-t)\phi_u + t\phi_{u-u_kv_k}$, it suffices to assume $t = 1$. But $\Psi_u \circ \phi_{u-u_kv_k}(x)$ can be easily seen to be a sum $\sum_{i=0}^{k-1} r_i g_{i+1}(x)$ with $r_i \geq 0$ and $\sum_{i=0}^{k-1} r_i < 1$.

As for Lemma 1.5, the statement about quasifibrations is well-known (cf. [15, 7.6 and §12]). That the maps of F are CE whenever $\epsilon_F: M(F) \to B$ is a fibration follows as in [12], essentially by noting that a map $f: X \to Y$ between compact ANR's is cell-like if and only if Y is a Z-set in $M(f)$. To show that $\epsilon_F: M(F) \to B$ is a fibration when the maps of F are CE, note that the composite

$$M(F) \times Q \xrightarrow{\text{proj}} M(F) \xrightarrow{\epsilon_F} B\mathcal{L}$$

is fibrewise homeomorphic to $\epsilon_{F\times 1_Q}: M(F \times 1_Q) \to B\mathcal{L}$, where Q is the Hilbert cube. The maps of $F \times 1_Q$, being cell-like mappings of Hilbert cube manifolds, are near homeomorphisms [4, 43.2]. Thus, $\epsilon_{F\times 1_Q}$ is easily seen to be completely regular, and hence a bundle map [11]. Thus, ϵ_F is a fibrewise retract of a bundle, and hence a fibration.

Chapman and Ferry [5, Theorem 1] have shown that if E and B are locally compact metric spaces, with B locally finite dimensional and locally path connected, and if $p: E \to B$ is proper and a Hurewicz fibration with Hilbert cube manifold fibres, then p is a locally trivial bundle. Given this, the above argument produces the following.

LEMMA 5.2. *Suppose a commutative diagram of proper maps of locally compact metric spaces, with B as above:*

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "q"] \\ & B & \end{tikzcd}.$$

Suppose that p and q have ANR *fibres, that p is a Hurewicz fibration, and that f is* CE. *Then*

(1) *q is a Hurewicz fibration.*
(2) *The natural map $M(f) \to B \times I$ is a Hurewicz fibration, providing a concordance of fibrations between p and q.*

This would also follow from the fibrewise Edwards theorem given in [5] if B is an ANR.

6. Functors of PL spaces classified by discrete categories. We prove Theorem 1.3. The basic ingredient is the following result of Kan [14].

LEMMA 6.1. *Let X and Y be simplicial sets. Then the homotopy classes of maps between the geometric realizations of X and Y are naturally isomorphic to the direct limit over k of the homotopy classes of simplicial maps of the kth barycentric subdivision of X into Y.*

Here, the direct limit system is induced by the natural map $\epsilon: X' \to X$ described in §3.

Theorem 1.3 is stated for contravariant functors. However, the analogous statement for covariant functors is equivalent, as the classifying space of a category is naturally homeomorphic (by a map of simplicial complexes if the category is a partially ordered set) to the classifying space of its opposite category. We prove the covariant version.

A simplicial functor, of course, will correspond to the induced map of classifying spaces in the isomorphism of Theorem 1.3. Concordant functors clearly induce homotopic maps. Since simplicial maps and homotopies between nerves of categories are induced by functors and natural transformations, respectively, the lemma shows that any homotopy class from B to $B\mathcal{C}$ is induced by a simplicial functor and that any two simplicial functors with the same domain category are connected by a chain of natural transformations after sufficient subdivision. Since a natural transformation between two functors from $\mathcal{L}$ to $\mathcal{C}$ is induced by a functor from $\mathcal{L} \times \mathcal{I}$ to $\mathcal{C}$, where $\mathcal{I}$ is the partially ordered set $\{0, 1\}$ with $0 < 1$, natural transformations induce concordances of simplicial functors. Thus, it suffices to show that if $\mathcal{L}_0$ and $\mathcal{L}_1$ are partially ordered sets triangulating B and $B\mathcal{L}_0$ is a subdivision of $B\mathcal{L}_1$, then any simplicial functor defined on $\mathcal{L}_1$ is concordant to a simplicial functor defined on $\mathcal{L}_0$, particularly to the induced functor in the direct limit system if $\mathcal{L}_0$ is the category of simplices of $B\mathcal{L}_1$.

Subdivide $B\mathcal{L}_1 \times I$ inductively by the Alexander trick, to obtain $B\mathcal{L}_0$ along $B\mathcal{L}_1 \times 0$, $B\mathcal{L}_1$ along $B\mathcal{L}_1 \times 1$, and coning each $\sigma \times I$ in the interior, σ a simplex of $B\mathcal{L}_1$. Since coning is the geometric consequence of adjoining a terminal object to a category, this triangulation is induced by a partially ordered set $\mathcal{L}$. Now precompose the given functor with the functor $\mathcal{L} \to \mathcal{L}_1$ which takes a given vertex to the last vertex of the carrier of its projection into $B\mathcal{L}_1$.

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