THE COHOMOLOGY OF THE ALTERNATING GROUPS

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Introduction. Let A_n be the alternating group on n letters. In this note we compute the mod p cohomology of A_n for p an odd prime. As expected the cohomology of A_n is closely related to the cohomology of S_n, where S_n is the symmetric group on n letters. The symmetric groups have long played an important role in the study of cohomology of groups begun by Eilenberg and Mac Lane [8] and Hopf [9]. Steenrod's fundamental work on cohomology operations ([21] and [22]), as well as Adem's calculation of the Adem relations, depend on the structure of S_n. Furthermore, work of Morse [15], Smith and Richardson [20], Steenrod [22], Dold [7], Nakoaka [17], and Milgram [12] taken together computes $H^t(S_n, \mathbf{Z}/p)$ as $\mathbf{Z}/p$ vector spaces using the homology of symmetric products.

Let p be an odd prime. Cárdenas [1] then computed $H^*(S_{p^2}, \mathbf{Z}/p)$ as a graded ring. The key step in Cárdenas' calculation was the determination, in cohomology, of the inclusion $R_2: \mathbf{Z}/p \times \mathbf{Z}/p \to S_{p^2}$ where R_2 is the regular representation. The determination of $R_n^*: H^*(S_{p^n}, \mathbf{Z}/p) \to H^*(\times^n \mathbf{Z}_p, \mathbf{Z}/p)$ for $n > 2$, where again $R_n: \times^n \mathbf{Z}/p \to S_{p^n}$ is the regular representation, was given independently by Mùi [16], Cooper [4], and the author [10]. This calculation was then used to determine $H^*(S_n, \mathbf{Z}/p)$.

Since p is odd R_n factors through the alternating group. By modifying the techniques used above we compute $R_n^*: H^*(S_{p^n}, \mathbf{Z}/p) \to H^*(A_{p^n}, \mathbf{Z}/p) \to H^*(\times^n \mathbf{Z}/p, \mathbf{Z}/p)$ and then use this computation to determine $H^*(A_{p^n}, \mathbf{Z}/p)$. This note is arranged so that the reader unfamiliar with the cohomology of S_n may simultaneously learn the structure $H^*(A_n, \mathbf{Z}/p)$ and $H^*(S_n, \mathbf{Z}/p)$, as well as the inclusion map between them, without referring to [16], [4], or [10].

1. Statement of results. Throughout this paper p is an *odd* prime and all cohomology is understood to be with $\mathbf{Z}/p$ coefficients. We begin by recalling some well-known facts. A p-Sylow subgroup K_p of a finite group K contains all cohomological information; more precisely $H^*(K)$ injects into $H^*(K_p)$ under the natural inclusion map and is in fact isomorphic to a subring of $H^*(K_p)$ that is invariant under the action of certain automorphisms. It is also well-known, dating back to Cauchy [3], that any p-Sylow subgroup of S_{p^n} and hence A_{p^n} is isomorphic to $\int^n \mathbf{Z}/p$, the n-fold wreath product of $\mathbf{Z}/p$. There is a family of natural inclusions [see diagram (1.1)], and we will write

(a) $T_{i,n}$ for $\times^{p^{n-i}}(\times^i \mathbf{Z}/p)$

(b) $\vartheta_{i,n}: T_{i,n} \to A_{p^n}$ is the composition of inclusions

(c) $\psi_{i,n} = \Phi \circ \vartheta_{i,n}: T_{i,n} \to S_{p^n}$ is the composition of $\vartheta_{i,n}$ with the natural inclusion $\phi: A_{p^n} \to S_{p^n}$.

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(1.1)

$$\begin{array}{ccccccc}
 & & & & & & S_{p^n} \\
 & & & & & & \uparrow \phi \\
 & & & & & & A_{p^n} \\
 & & & & & & \uparrow \\
\times^{p^{n-1}}(A_p) & \cdots \rightarrow & \times^{p^{n-l}}(A_{p^l}) & \rightarrow \cdots \rightarrow & \times^{p} A_{p^{n-1}} & \rightarrow & A_{p^{n-1}} \wr \mathbf{Z}/p \\
\Big\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Big| & & \times^{p^{n-l}}(\wr^{l}\mathbf{Z}/p) & & \times^{p}(\wr^{n-1}\mathbf{Z}/p) & & \wr^{n}\mathbf{Z}/p \\
\Big| & & \uparrow & & \uparrow & & \uparrow \\
\times^{p^{n-1}}(\mathbf{Z}/p) & & \times^{p^{n-l}}(\times^{l}\mathbf{Z}/p) & & \times^{p}(\times^{n-1}\mathbf{Z}/p) & & \times^{n}\mathbf{Z}/p
\end{array}$$

A theorem of Quillen [19] ensures that the maps

$$\vartheta_n^* = \prod_{i=1}^{n} \vartheta_{i,n} : H^*(A_{p^n}) \to \prod_{i=1}^{n} H^*(T_{i,n})$$

and

$$\psi_n^* = \Phi^* \circ \vartheta_n^* : H^*(S_{p^n}) \to \prod_{i=1}^{n} H^*(T_{i,n})$$

are both injective on sets of multiplicative generators for $H^*(A_{p^n})$ and $H^*(S_{p^n})$, respectively. We shall see later that the relations on these generators are trivial to compute. Thus our major task is to compute ϑ_n^*. (See [10] for ψ_n^*.)

We also recall that $H^*(T_{n,n}) \cong E(e_1, \ldots, e_n) \otimes P(b_1, \ldots, b_n)$ where degree$(e_i) = 1$ and degree$(b_i) = 2$ for all i, and that $\beta_p(e_i) = b_i$ where β_p is the Bockstein operator associated with the exact sequence $0 \to \mathbf{Z}/p \to \mathbf{Z}/p^2 \to \mathbf{Z}/p \to 0$.

We now consider specific formal determinants with entries in $H^*(T_{n,n})$ which date back to Dickson's work in modular invariant theory ([5], [6]). Let

$$(1.2) \qquad L_n = \det \begin{vmatrix} b_1^{p^{n-1}} & \ldots & b_n^{p^{n-1}} \\ b_1^{p^r} & \ldots & b_n^{p^r} \\ b_1 & \ldots & b_n \end{vmatrix} \quad \text{for } 0 \le r \le n-1.$$

That is, the j, lth entry of L_n is $b_l^{p^j}$.

For $1 \le i \le n-1$,

$$(1.3) \qquad P_{i,n} = \det \begin{vmatrix} b_1^{p^n} & \ldots & b_n^{p^n} \\ \widehat{b_1^{p^i}} & \ldots & \widehat{b_n^{p^i}} \\ b_1 & \ldots & b_n \end{vmatrix}$$

That is, the $b_m^{p^i}$ row is omitted.

$$(1.4) \qquad Q_{i,n} = \frac{P_{i,n}}{L_n}.$$

We note that Moore [13] and Dickson [6] proved $Q_{i,r}$ is an integral polynomial mod p.

For $0 \le i \le n-1$,

$$(1.5) \qquad M_{i,n} = \det \begin{vmatrix} b_1^{p^{n-1}} & \cdots & b_n^{p^{n-1}} \\ \widehat{b_1^{p^i}} & \cdots & \widehat{b_n^{p^i}} \\ b_1 & \cdots & b_n \\ e_1 & \cdots & e_n \end{vmatrix}.$$

That is, the $b_m^{p^i}$ row is omitted. We remark that $M_{0,n} = \underline{L}_n$ in the notation of [10].

We now explain why these formal determinants appear in the computation of $H^*(A_{p^n})$. Let $\omega: \mathrm{GL}(n, \mathbf{Z}/p) \to \mathbf{Z}/2$ be the homomorphism that sends a matrix to its determinant raised to the $(p-1)/2$ power. Let $\mathrm{GL}^+(n, \mathbf{Z}/p) = \ker \omega$. Then, if $\bar{T}_{n,n} = \vartheta_{n,n}(T_{n,n})$ and $\bar{\bar{T}}_{n,n} = \phi(\bar{T}_{n,n})$ there are natural isomorphisms

$$(1.6) \qquad \frac{N_{\bar{\bar{T}}_{n,n}, S_{p^n}}}{\bar{\bar{T}}_{n,n}} \to \mathrm{GL}(n, \mathbf{Z}/p)$$

and

$$(1.7) \qquad \frac{N_{\bar{T}_{n,n}, A_{p^n}}}{\bar{T}_{n,n}} \to \mathrm{GL}^+(n, \mathbf{Z}/p),$$

where $N_{H,G}$ is the normalizer of H in G. Under these isomorphisms the action of the normalizers on $H^*(T_{n,n})$ are given as follows: If U_x in $\mathrm{GL}(n, \mathbf{Z}/p)$ represents the coset $x\bar{\bar{T}}_{n,n}$ in $N_{\bar{\bar{T}}_{n,n}, S_{p^n}}$ then $ad_x: H^*(T_{n,n}) \to H^*(T_{n,n})$ is the unique ring homomorphism determined by $ad_x(e_m) = U_x e_m$ and $ad_x(b_m) = U_x b_m$, where e_m, b_m are treated as the vectors $(0, \ldots, e, \ldots, 0)$ and $(0, \ldots, b_m, \ldots, 0)$ in $H^*(T_{n,n})$ with non-zero entries in the mth place.

As ad_x is a ring homomorphism it is apparent that ad_x acts on the formal determinant classes $L_n, Q_{i,n}, M_{i,n}$ via the determinant function; that is, $ad_x(L_n) = \det(U_x)L_n$. As a classical result in this theory implies that image $\theta^*_{n,n} \subset H^*(T_{n,n})^{\mathrm{GL}^+(n,\mathbf{Z}/p)}$, the $\mathrm{GL}^+(n, \mathbf{Z}/p)$ invariants of $H^*(T_{n,n})$; it should be expected that the formal determinants mentioned above enter into our computations.

We may now describe the maps $\theta^*_{n,n}$ and $\psi^*_{n,n}$.

DEFINITION 1.8.

(a) Let

$$U_1 = E(e_1 b_1^{(p-3)/2}) \otimes P(b_1^{(p-1)/2}) = E(M_{0,1} L_1^{(p-3)/2}) \otimes P(L_1^{(p-1)/2}).$$

(b) Let $W_1 = E(e_1 b_1^{p-2}) \otimes P(b_1^{p-1}) = E(M_{0,1} L_1^{p-2}) \otimes P(L_1^{p-1})$.

(c) For $n > 1$ let U_n be the subalgebra of $H^*(T_{n,n})$ generated by:

$$1, L_n^{(p-1)/2}, Q_{i,n}, M_{j,n} L_n^{(p-3)/2}$$

with $1 \le i \le n-1$, $0 \le j \le n-1$.

(d) For $n > 1$ let W_n be the subalgebra of $H^*(T_{n,n})$ generated by:

$$1, L_n^{p-1}, Q_{i,n}, M_{j,n} L_n^{p-2}, M_{j,n} M_{l,n} L_n^{p-3}$$

with $1 \le i \le n-1$, $0 \le j < l \le n-1$.

Note that U_n is a polynomial algebra tensored with an exterior algebra, whereas W_n is not (W_n has zero products; see [10]). Clearly W_n is contained in U_n. Furthermore if $x \in U_n$ then $x^2 \in W_n$, and if x and y are exterior classes in U_n then xy is in W_n.

The main result of this paper which permits all further computations is:

THEOREM 1.9. Image $\vartheta^*_{n,n} \cong U_n$.

THEOREM 1.10 ([16], [4], [10]). Image $\psi^*_{n,n} \cong W_n$.

More generally, for $1 \leq i \leq n$, $A_{p^{n-i}}$ operates on $T_{i,n}$ and on the algebra $\otimes^{p^{n-i}}(U_i)$ contained in $H^*(T_{i,n})$ by permuting the p^{n-1} copies of $\times^i \mathbf{Z}/p$, and we have

THEOREM 1.11. Image $\vartheta^*_{i,n} \cong (\otimes^{p^{n-i}}(U_i))^{A_{p^{n-i}}}$.

THEOREM 1.12 ([16], [4], [10]). Image $\psi^*_{i,n} \cong (\otimes^{p^{n-i}}(W_i))^{S_{p^{n-i}}}$.

COROLLARY 1.13. *$\phi^*: H^*(S_{p^n}) \to H^*(A_{p^n})$ restricted to the classes in the image $\psi^*_{i,n}$ is given by the natural inclusion*

$$\left(\bigotimes^{p^{n-i}}(W_i)\right)^{S_{p^{n-i}}} \to \left(\bigotimes^{p^{n-i}}(U_i)\right)^{A_{p^{n-i}}}.$$

Clearly in order to completely determine $H^*(A_{p^n})$ we must know when a class $x \in H^*(A_{p^n})$ has a non-trivial image under $\vartheta^*_{i,n}$ for more than one value of i.

DEFINITION 1.14. $x \in H^*(A_{p^n})$ is a multiple image class if $\vartheta^*_{i,n}(x) \neq 0$ for at least two different values of i.

Notice that if $x_1, x_2 \in H^*(A_{p^n})$ with x_1 detected only by $\vartheta^*_{i_1,n}$ and x_2 detected only by $\vartheta^*_{i_2,n}$, $i_1 \neq i_2$, then $x_1 + x_2$ is trivially a multiple image class.

DEFINITION 1.15. $x \in H^*(A_{p^n})$ is sum indecomposable if x cannot be written as $x_1 + x_2$ with $\theta^*_{i,n}(x_1) \neq 0$, $\theta^*_{i,n}(x_2) = 0$ and $\theta^*_{j,n}(x_1) = 0$, $\theta^*_{j,n}(x_2) \neq 0$ for some $i \neq j$.

We now classify all sum indecomposable multiple image classes in $H^*(A_{p^n})$. Briefly a sum indecomposable multiple image class in $H^*(A_{p^n})$ must be in the image, under ϕ^*, of a sum indecomposable multiple image class in $H^*(S_{p^n})$. These classes were classified in [10]. This is a formal consequence of the structure of U_n, W_n and the Steenrod algebra $A_{(p)}$. However, for completeness we state the result precisely.

DEFINITION 1.16. M_n is the subalgebra of U_n generated by:

$$1, M_{j,n} M_{l,n} L_n^{p-3}, Q_{i,n},$$

with $1 \leq i \leq n-1$, $0 \leq j \leq l \leq n-1$.

Note that $M_n \subset W_n \subset U_n$.

DEFINITION 1.17. Given $x_{m,j} \in M_j$, we define $x_{m,j-1} \in U_{j-1}$ as follows:

(a) If $x_{m,j}=1$ then $x_{m,j-1}=1$.

(b) If $x_{m,j}=Q_{i,j}$ then $x_{m,j-1}=Q_{i-1,j-1}$ for $2 \leq j \leq n$ and $1 \leq i \leq j-1$, with the convention that $Q_{0,j-1}=L_{j-1}^{p-1}$.

(c) If $x_{m,j}=M_{i,j}M_{l,j}L_j^{p-3}$ then $x_{m,j-1}=-M_{i-1,j-1}M_{l-1,j-1}L_{j-1}^{p-3}$ for $3 \leq j \leq n$, $0<i<l \leq j-1$.

(d) If $x_{m,j}=x'_{m,j}x''_{m,j}$ then $x_{m,j-1}=x'_{m,j-1}x''_{m,j-1}$.

Note that (a)–(d) define a unique class $x_{m,j-1}$ for every $x_{m,j} \in M_j$.

THEOREM 1.18. *Suppose $x \in H^*(A_{p^n})$ is a sum indecomposable multiple image class. Further suppose i is the largest integer such that $\vartheta_{i,n}^*(x) \neq 0$. Then there exists $x_{m,i} \in M_i$ for $1 \leq m \leq p^{n-i}$ such that $\vartheta_{i,n}^*(x)=A\langle x_{1,i}, \ldots, x_{p^{n-i},i}\rangle$ and $\vartheta_{i-1,n}^*(x)=A\langle x_{1,i-1}, \ldots, x_{1,i-1}, \ldots, x_{p^{n-i},i-1}, \ldots, x_{p^{n-i},i-1}\rangle$, where each $x_{m,i-1}$ occurs p times in $\vartheta_{i-1,n}^*(x)$. If $i-1>2$ and each $x_{m,i-1}$ is in M_{i-1} (not just U_{i-1}), then $\vartheta_{i-2,n}^*(x) \neq 0$ and may be obtained from $\vartheta_{i-1,r}^*(x)$ precisely as $\vartheta_{i-1,n}^*(x)$ was obtained from $\vartheta_{i,n}^*(x)$. In fact this iteration continues r times until either $i-r=2$ or $x_{m,i-1} \notin M_{i-r}$ when $\vartheta_{i-(r+t),n}^*(x)=0$ for all $t>0$. Thus x has $r+1$ non-trivial images in the detecting groups $H^*(T_{i-s,n})$ for $0 \leq s \leq r$.*

Here $A\langle x_1, \ldots, x_{p^{n-i}}\rangle$ is the $A_{p^{n-i}}$ invariant class generated by $x_1x_2 \ldots x_{p^{n-i}}$.

To complete our computation of $H^*(A_{p^n})$ as a graded ring we point out if $\vartheta_n^*(x_1)=(\vartheta_{1,n}^*(x_1), 0, 0, \ldots, 0)$ and $\vartheta_n^*(x_2)=(0, \vartheta_{2,n}^*(x_2), 0, \ldots, 0)$ then $\vartheta_n^*(x_1 \cdot x_2)=0$, but it does not necessarily follow that $x_1x_2=0$. The next theorem summarizes the situation.

THEOREM 1.19.

(1) *$H^*(A_{p^n})$ is generated by classes that map non-trivially under ϑ_n^*.*

(2) *Suppose $\vartheta_n^*(u)=0$, where $u=\prod_{i=1}^r u_i$ with $u_i \in H^*(A_{p^n})$, $\vartheta_n^*(u_i) \neq 0$, and let l_i be the smallest power of p such that $I_{p^n,l_i}^*(u_i) \neq 0$ ($I_{p^n,l_i}: A_{l_i} \to A_{p^n}$ is the natural inclusion). Then $u \neq 0$ in $H^*(A_{p^n})$ unless:*

(a) *$u_{i_1}=u_{i_2}$ is an odd-dimensional exterior in $H^*(A_{l_{i_1}})$ for some $1 \leq i_1 < i_2 \leq r$, $l_{i_1}<p^n$;*

(b) *$u_{i_1}=u_{i_2}=\cdots=u_{i_p}$ is an even-dimensional exterior class in $H^*(A_{l_{i_1}})$ for some $1 \leq i_1 < i_2 < \cdots < i_p \leq r$; or*

(c) *$\prod_{j=1}^r l_{i_j} > p^n$.*

See [10] for the analog of 1.19 for S_{p^n}.

In applications it is sometimes useful to restate 1.19 as follows:

THEOREM 1.20. *Let K_n: $\times^p A_{p^{n-1}} \to A_{p^{n-1}} \wr \mathbf{Z}_p \to A_{p^n}$ be the natural inclusion. Then we have the injection*

$$0 \to H^*(A_{p^n}) \xrightarrow{\vartheta_n^*, K_n^*} U_n \times \left(\bigotimes_{l=1}^{p} H^*(A_{p^{n-1}}) \right)^{A_p}$$

where the multiple image classes are given, as before, by 1.18.

Now let m be an arbitrary integer. Then m may be uniquely written as $\sum_{j=0}^{r} a_j p^j$ with $0 \leq a_j \leq p-1$, $a_r \neq 0$. Then any p-Sylow subgroup of A_m is isomorphic to $A_{m,p} = \times^{a_r}(\int^r \mathbf{Z}/p) \times \cdots \times \times^{a_1}(\mathbf{Z}/p)$ and we have the following commutative diagram of inclusions:

$$\begin{array}{ccccc}
\times^{a_r}(S_{p^r}) \times \cdots \times \times^{a_1}(S_p) & \xrightarrow{\bar{J}_m} & S_m & \xrightarrow{\bar{I}_{p^{r+1},m}} & S_{p^{r+1}} \\
\uparrow \phi & & \uparrow \phi & & \uparrow \phi \\
\times^{a_r}(A_{p^r}) \times \cdots \times \times^{a_1}(A_p) & \xrightarrow{J_m} & A_m & \xrightarrow{I_{p^{r+1},m}} & A_{p^{r+1}}. \\
\uparrow & \nearrow & & & \\
A_{m,p} & & & &
\end{array}$$

THEOREM 1.21.
(1) $I^*_{p^{r+1},m}, \bar{I}^*_{p^{r+1},m}$ *are surjections.*
(2) $J^*_m, \bar{J}^*_m$ *are injections.*
(3) $v \in \text{Image } J^*_m$ *if and only if*

$$v = \sum A\langle v_{r,1}, \ldots, v_{r,a_r}\rangle \otimes \cdots \otimes A\langle v_{i,1}, \ldots, v_{1,a_1}\rangle,$$

with $v_{t,l} \in H^*(A_{p^t})$ *for each* l.

We now turn to the action of the Steenrod algebra, $A_{(p)}$, on $H^*(A_m)$. By 1.21 it suffices to consider $m = p^n$. As

$$P^j(b^{p^k}) = \begin{cases} b^{p^k} & \text{if } j = 0, \\ b^{p^{k+1}} & \text{if } j = p^k, \\ 0 & \text{otherwise,} \end{cases}$$

it is easy to compute the action of $A_{(p)}$ on U_n and hence on $H^*(A_{p^n})$. The following result is trivial to verify.

THEOREM 1.22. *The following relations and the Cartan formula describe the* $A_{(p)}$ *action on* U_n.
(a) $\beta(M_{0,n}) = L_n$.
(b) $P^{p^{l-1}}(Q_{l,n}) = Q_{l-1,n}$, *with* $Q_{0,n} = L_n^{p-1}$.
(c) $P^{p^{j-1}}(M_{j,n} L_n^{(p-3)/2}) = M_{j-1} L_n^{(p-3)/2}$.

2. Analysis of A_{p^n} at p. We begin this section with a p-primary analysis of A_{p^n}. Recall that if K is a finite group and L is a subgroup of S_n then the wreath product $K \int L$ is a semi-direct product $1 \to \times^n K \to K \int L \to L \to 1$ where L acts on $\times^n K$ by permuting the factors. There are two natural inclusions $\times^n K \to K \int L$ and $K \times L \to K \int L$, and it is clear that the inclusions $\theta_{i,n}$ and $\psi_{i,n}$ of diagram (1.1) are obtained by iterating inclusions of these two types. The regular representation R_n mentioned in the introduction is precisely $\psi_{n,n}$ [10].

Now let N_n be the normalizer of $T_{n,n}$ in A_{p^n} and $\bar{N}_n$ be the normalizer of $T_{n,n}$ in S_{p^n}. Of course $N_n \subset \bar{N}_n$ and we have the homomorphism first considered by Cárdenas [1]

(2.1) $$\psi : \bar{N}_n \to \mathrm{GL}(n, \mathbf{Z}/p)$$

defined as follows: For $x \in N_n$ we have $xa_i x^{-1} = a_1^{r_{1i}} \dots a_n^{r_{ni}}$, where a_i generates the ith factor of $\times^n \mathbf{Z}/p$. We let $\psi(x)$ be the $n \times n$ matrix whose (i, j)th entry is $r_{j,i}$. Clearly $\psi(x)$ is non-singular, and

PROPOSITION 2.2 ([16], [4], [10]). *The sequence* $1 \to T_{n,n} \to \bar{N}_n \xrightarrow{\psi} \mathrm{GL}(n, \mathbf{Z}/p) \to 1$ *is split exact.*

If $w: \mathrm{GL}(n, \mathbf{Z}/p) \to \mathbf{Z}/2 \to 1$ is defined by

$$w(U) = (\det U)^{(p-1)/2} \quad \text{and} \quad \mathrm{GL}^+(n, \mathbf{Z}/p) = \ker w,$$

we note that ψ restricted to N_n maps to $\mathrm{GL}^+(n, \mathbf{Z}/p)$.

COROLLARY 2.3. *The sequence* $1 \to T_{n,n} \to N_n \xrightarrow{\psi} \mathrm{GL}^+(n, \mathbf{Z}/p) \to 1$ *is split exact.*

In a similar fashion we obtain the following propositions for subgroups of A_{p^n}. Let $N_{j,n}, N_{j_1,\dots,j_m}$ and M_j be the normalizers of $T_{n,j}$, $\times_{r=1}^m (\times^{j_r} \mathbf{Z}/p)$ and $\times^{p^{n-j}} A_{p^j}$ in A_{p^n}, and let $\bar{N}_{j,n}, \bar{N}_{j_1,\dots,j_m}$ and $\bar{M}_j$ be the normalizers of $T_{j,n}$, $\times_{r=1}^m (\times^{j_r} \mathbf{Z}/p)$ and $\times^{p^{n-j}} S_{p^j}$ in S_{p^n}. Further, $S_{(j_1,\dots,j_m)}$ is the subgroup of S_m generated by the transpositions (a, c), where $j_a = j_c$.

PROPOSITION 2.4. *The following commutative diagrams have short split exact rows:*

$$\begin{array}{ccccccccc} 1 & \to & \times^{p^{n-i}} N_i & \to & N_{i,n} & \leftrightarrows & A_{p^{n-i}} & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \times^{p^{n-i}} N_i & \to & \bar{N}_{i,n} & \leftrightarrows & S_{p^{n-i}} & \to & 1, \end{array} \tag{2.5}$$

$$\begin{array}{ccccccccc} 1 & \to & \times_{r=1}^m N_{j_r} & \to & N_{j_1,\dots,j_m} & \leftrightarrows & A_{(j_1,\dots,j_m)} & \to & 1 \\ & & \downarrow & & & & & & \\ 1 & \to & \times_{r=1}^m \bar{N}_{j_r} & \to & \bar{N}_{j_1,\dots,j_m} & \leftrightarrows & S_{(j_1,\dots,j_m)} & \to & 1, \end{array} \tag{2.6}$$

$$\begin{array}{ccccccccc} 1 & \to & \times^{p^{n-i}} A_{p^i} & \to & M_i & \leftrightarrows & A_{p^n} & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \times^{p^{n-i}} S_{p^i} & \to & \bar{M}_i & \leftrightarrows & S_{p^n} & \to & 1. \end{array} \tag{2.7}$$

Proof. Minor modification of arguments found in [16], [4], or [10]. □

We now summarize facts found in [2], [23], [1], [16], [4], and [10] required in Section 3. (Some quoted results were proved only for S_n but minor modification extends these results to A_n.)

PROPOSITION 2.8. *Let K_p be a p-Sylow subgroup of K. Then the transfer*

$$t(K, K_p): H^*(K_p) \to H^*(K)$$

is an epimorphism and the injection

$$i(K_p, K): H^*(K) \to H^*(K_p)$$

is a monomorphism whose image consists of stable elements of $H^*(K_p)$. Furthermore, $H^*(K_p) \cong \mathrm{Im}\, i(K_p, K) \oplus \mathrm{Ker}\, t(K, K_p)$.

PROPOSITION 2.9. *For $x \in S_{p^i}$ the homomorphism*

$$ad_x : H^*(T_{n,n}) \to H^*(xT_{n,n}x^{-1})$$

is induced by the inner automorphism $y \to xyx^{-1}$. Furthermore, for $E = \sum_{j=1}^n \alpha_j e_j$ and $B = \sum_{j=1}^n \beta_j b_j$ in $H^*(T_{n,n})$, $ad_x(E) = \psi(x)E$ and $ad_x(B) = \psi(x)B$ where $\psi : \bar{N}_n \to \mathrm{GL}(n, \mathbf{Z}/p)$ is the homomorphism 2.4.

Since ad_x operates on $L_n, Q_{i,n}, M_{j,n}$ via multiplication by the determinant function we have:

COROLLARY 2.10. *U_n is contained in $H^*(T_{n,n})^{\mathrm{GL}^+(n,\mathbf{Z}/p)}$. W_n is contained in $H^*(T_{n,n})^{\mathrm{GL}(n,\mathbf{Z}/p)}$.*

PROPOSITION 2.11. *For $x \in H^*(A_{p^n})$, $\vartheta_{n,n}^*(x) \in H^*(T_{n,n})^{\mathrm{GL}^+(n,\mathbf{Z}/p)}$, and for $x \in H^*(S_{p^n})$, $\psi_{n,n}^*(x) \in H^*(T_{n,n})^{\mathrm{GL}(n,\mathbf{Z}/p)}$.*

PROPOSITION 2.12. *If K and H are subgroups of a finite group G then*

$$i(H,G) \circ t(G,K) = \sum t_x i_x ad_x$$

where i_x is the inclusion map $H^*(xKx^{-1}) \to H^*(xKx^{-1} \cap H)$, t_x is the transfer $H^*(xKx^{-1} \cap H) \to H^*(H)$, and the sum runs over a set of double coset representatives $K \times H$.

PROPOSITION 2.13. *In 2.12 if $G = A_{p^n}$ or S_{p^n}, $K = A_{p^{n-1}} \int \mathbf{Z}/p$ or $S_{p^{n-1}} \int \mathbf{Z}/p$, and $H = T_{n,n}$, then in the sum $\sum_x t_x i_x ad_x$ each double coset representative x may also be chosen to be in N_n or $\bar{N}_n$ respectively.*

3. Proofs of results. All our proofs and in fact the entire structure of $H^*(A_{p^n})$ is based on the construction and structure of the Steenrod algebra $A_{(p)}$. We begin by recollecting one way Steenrod defined his pth power operations ([23]). Let X be a finite regular cell complex. Steenrod considered the following spaces and maps:

$$X^p \xrightarrow{j} W_{\mathbf{Z}/p} \times_{\mathbf{Z}/p} X^p \xleftarrow{1 \times \Delta} W_{\mathbf{Z}/p} \times_{\mathbf{Z}/p} X = B_{\mathbf{Z}/p} \times X,$$

where j is the inclusion and Δ is the diagonal map. Given any $u \in H^*(X)$ there exists a unique natural class $\mathcal{P}(u)$ in $H^*(W_{\mathbf{Z}/p} \times_{\mathbf{Z}/p} X^p)$ such that:

(1) $j^*(\mathcal{P}(u)) = u \otimes \cdots \otimes u = u^{\otimes p}$.

(2) Under the Kunneth isomorphism $(1 \times \Delta)^*(\mathcal{P}(u)) = \sum w_k \otimes D_k(u)$, where w_k generates $H^k(\mathbf{Z}/p)$ and $D_k : H^q(X) \to H^{pq-k}(X)$ are homomorphisms which define the elements of $A_{(p)}$.

(3) $\beta D_{2k}(u) = D_{2k-1}(u)$, $\beta D_{2k-1}(u) = 0$, and $\beta D_0(u) = 0$.

THEOREM 3.1 ([23]). *If $z \in H^*(W_{\mathbf{Z}/p} \times_{\mathbf{Z}/p} X^p)$, then z is of the form $z = tz_1 + z_2 \cdot \mathcal{P}(z_3)$ with $z_1 \in H^*(X^p)$, $z_2 \in H^*(B_{\mathbf{Z}/p})$, and $z_3 \in H^*(X)$, where t is the transfer. Furthermore, the sequence*

$$H^*(X^p) \xrightarrow{t} H^*(W_{\mathbf{Z}/p} \times_{\mathbf{Z}/p} X^p) \xrightarrow{(1 \times \Delta)^*} H^*(B_{\mathbf{Z}/p} \times X)$$

is exact.

Of course the D_K homomorphisms define the reduced powers.

DEFINITION 3.2 ([23]). Let $u \in H^q(X)$. Then

$$\mathcal{P}^j(u) = a_{j,q} D_{(q-2j)(p-1)}(u) \quad \text{and} \quad \beta\mathcal{P}^j(u) = a_{j,q} D_{(q-2j)(p-1)-1}(u),$$

where $a_{j,q}$ is a non-zero constant in $\mathbf{Z}/p$ dependent on j and q. If $k \neq (q-2j)(p-1)$ or $(q-2j)(p-1)-1$ for some j, then $D_k(u) = 0$.

Of course we may approximate $B_{A_{p^n}}$ by finite regular skeleta in our computations. 2.3, 2.11, 2.13 and 3.1 immediately imply:

THEOREM 3.3. *Image $\vartheta^*_{n,n}$ consists of the elements in $H^*(T_{n,n})^{\mathrm{GL}^+(n,\mathbf{Z}/p)}$ that are in the image of the composition*

$$i^* \circ (1 \times \Delta)^* : H^*(W_{\mathbf{Z}/p} \times_{\mathbf{Z}/p} (B_{A_{p^{n-1}}})^p)$$
$$\cong H^*(A_{p^{n-1}} \int \mathbf{Z}/p) \to H^*(B_{\mathbf{Z}/p} \times B_{A_{p^{n-1}}}) \to H^*(T_{n,n}),$$

where $i: T_{n,n} \to T_{n-1,n-1} \int \mathbf{Z}/p \to A_{p^{n-1}} \int \mathbf{Z}/p$ is the inclusion found in (1.1).

We begin the proof of 1.9 with the following proposition and lemmas.

PROPOSITION 3.4. $H^*(A_p) \cong U_1$.

Proof. Immediate from 2.3 and 2.8 thru 2.13. □

LEMMA 3.5. *There exists $u \in H^*(A_{p^n})$ such that $\vartheta^*_{n,n}(u) = M_{n-1,n} L_n^{(p-3)/2} \in H^*(T_{n,n})$.*

Proof. By induction. The case for $n=1$ is contained in 3.4. Let $n>1$; then we know there exists a u_1 in $H^*(A_{p^{n-1}})$ so that $\vartheta^*_{n-1,n-1}(u_1) = M_{n-2,n-1} L_{n-1}^{(p-3)/2}$. By 3.3 it suffices to show $i^* \circ (1 \times \Delta)^*(\mathcal{P} u_1) = M_{n-1,n} L_n^{(p-3)/2}$. This is a lengthy but totally routine calculation such as is carried out in Lemma 5.4 of [10]. Details are left to the reader. □

LEMMA 3.6. *There exists $u \in H^*(A_{p^n})$ such that $\vartheta^*_{n,n}(u) = L_n^{(p-1)/2} \in H^*(T_{n,n})$.*

Proof. Identical to 3.5. In the inductive step choose $u_1 \in H^*(A_{p^{n-1}})$ so that $\vartheta^*_{n-1,n-1}(u_1) = L_{n-1}^{(p-1)/2}$ and then compute $i^* \circ (1 \times \Delta)^*(b_n \cdot \mathcal{P} u_1)$. □

LEMMA 3.7. $U_n \subset$ *image* $\vartheta^*_{n,n}$.

Proof. For $n=1$, 3.7 is merely 3.4. For $n>1$, 3.5 together with the naturality of the Steenrod operations and 1.24 ensure that all exterior generators of U_n are in image $\vartheta^*_{n,n}$. 1.10 implies $P(L^{p-1}, Q_{1,n}, \ldots, Q_{n-1,n}) \subset \text{image } \psi_{n,n} \subset \text{image } \vartheta^*_{n,n}$ (see [11] for a direct proof of this fact using different techniques). Thus 3.6 finishes the proof. □

To prove 1.9 it remains to prove:

LEMMA 3.8. Image $\vartheta^*_{n,n} \subset U_n$.

At this point the specialized arguments used in [10] for $H^*(S_{p^n})$ which were based on results of [12] do not generalize to $H^*(A_{p^n})$. That is, the relation between the homology of symmetric products and $H^*(S_{p^n})$ allowed a counting argument to prove the S_{p^n} analog of 3.8 without considering elements of

$H^*(T_{n,n})^{GL(n,Z/p)}$ not in W_n. [16] and [4] did not use such a counting argument but rather proved and used the following important result in modular invariant theory not found in [10]. We use their result to finish the proof of 3.8.

THEOREM 3.9 ([16], [4]). *$H^*(T_{n,n})^{SL(n,Z/p)}$ decomposes as*

$$P(L_n, Q_{1,n}, \ldots, Q_{n-1,n}) \oplus M_{s_1,\ldots,s_k} R,$$

where

(1) $R \in P(L_n, Q_{1,n}, \ldots, Q_{n-1,n})$ *and*
(2) $M_{s_1,\ldots,s_k} = M_{s_1,n} M_{s_2,n} \ldots M_{s_k,n}/L_n^{k-1}$, *with* $0 \le s_1 < s_2 < \cdots < s_k \le n-1$.

COROLLARY 3.10 ([16], [4]). *$H^*(T_{n,n})^{GL^+(n,Z/p)}$ decomposes as*

$$P(L_n^{(p-1)/2}, Q_{1,n}, \ldots, Q_{n-1,n}) \oplus (M_{s_1,\ldots,s_k} L_n^{(p-3)/2}) \bar{R},$$

where $\bar{R} \in P(L_n^{(p-1)/2}, Q_{1,n}, \ldots, Q_{n-1,n})$.

Proof of 3.8. By induction on n. Again the case $n=1$ is merely 3.4. Dickson's original classification theorem [6] implies the maximal polynomial subalgebra of $H^*(T_{n,n})^{GL^+(n,Z/p)}$ is precisely the maximal polynomial subalgebra of U_n. As

$$(1 \times \Delta)^*(\mathcal{P}(u+v)) = (1 \times \Delta)^*(\mathcal{P}(u) + \mathcal{P}(v))$$

and

$$(1 \times \Delta)^*(\mathcal{P}(u) \cdot \mathcal{P}(v)) = (1 \times \Delta)^*(\mathcal{P}(uv)),$$

it suffices to show that the elements $M_{s_1, \ldots, s_k} L_n^{(p-3)/2}$ for $k>1$ are not in image $\vartheta_{n,n}^*$. Note the expansion of $i^* \circ (1 \times \Delta)^*(\mathcal{P}(u))$ has the same number of exterior classes $e_{i_1} \ldots e_{i_l}$ in each monomial as $\vartheta_{n-1,n-1}^*(u)$ does. Thus the lemma follows from the observation, for $n>1$, that

$$\text{dimension}(M_{s_1,\ldots,s_k} L_n^{(p-3)/2}) < \text{dimension}((M_{r_1,n-1} L_{n-1}^{(p-3)/2} \ldots M_{r_k,n-1} L_{n-1}^{(p-3)/2}))$$

for $k>1$. □

This concludes the proof of 1.9. The proofs of the remaining results in Section 1 are now routine applications of 1.9 and results from Section 2 and [10], and hence are left to the reader.

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