ADMISSIBLE LIMITS OF M-SUBHARMONIC FUNCTIONS

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1. Introduction. In this paper we study the boundary behavior of M-subharmonic functions (i.e. subsolutions to the invariant Laplacian) on the unit ball in $\mathbf{C}^n$. In his thesis [7], Ullrich showed that such functions which satisfy

$$\sup_{r<1} \int_S |u(r\zeta)|\, d\sigma(\zeta) < \infty \tag{1}$$

have radial limits almost everywhere on the unit sphere.

It is well known that Fatou's theorem on radial limits of bounded holomorphic functions extends to harmonic functions on the unit disc which satisfy (1), and that non-tangential limits exist almost everywhere.

For subharmonic functions the situation is more complicated. Littlewood [2] proved that a subharmonic function satisfying (1) has radial limits on a set of full measure. Privalov [3] extended this result to subharmonic functions on the ball in $\mathbf{R}^n$. He also gave an incorrect proof of the existence of non-tangential limits. Examples of Zygmund and others [5; 6] show that subharmonic functions satisfying (1) need not have an angular limit at any point on the boundary. This is true even if the Riesz measure μ is absolutely continuous with respect to Lebesgue measure so that $d\mu = f dA$ $(f>0)$. However, if u is subharmonic on the unit disc and f satisfies

$$\int_{|z|<1} (f(z))^p (1-|z|)^{2p-1} dA(z) < \infty \quad (p>1) \tag{2}$$

u will have non-tangential limits; this was proved by Arsove and Huber [1].

Our main result shows that M-subharmonic functions which satisfy (1) and a condition analogous to (2) have admissible limits (in the sense of Koranyi) at almost every point of the unit sphere. Reduced to the one-variable case our proof differs from that of [1]; where they use a normal family argument we use the invariance of certain integrals under the Möbius group of the disc.

Similar problems have been studied in half spaces of $\mathbf{R}^n$ by Widman [8] and Wu [9]. Wu shows that by varying the exponents in the analogue of (2), various kinds of convergence other than non-tangential can be obtained.

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2. Notation and definitions. The notation and definitions are for the most part those given in Rudin [4]. For z, w in $\mathbf{C}^n$, set $\langle z, w\rangle = \sum_{j=1}^n z_j \bar{w}_j$ and let $B = \{z \in \mathbf{C}^n : |z| < 1\}$, $S = \partial B$ with $d\nu$ and $d\sigma$ the Lebesgue measure on B and S respectively. For $z, a \in B$, $(a \neq 0)$ define

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$$\phi_a(z) = \frac{(a - P_a z - (1-|a|^2)^{1/2} Q_a z)}{(1 - \langle z, a\rangle)},$$

where $P_a z = \langle z, a\rangle(\langle a, a\rangle)^{-1} a$ is the orthogonal projection of z onto the space spanned by $\{a\}$ and $P_a z + Q_a z = z$. Each ϕ_a is a holomorphic automorphism of B and each automorphism ψ can be written as $\psi = U \circ \phi_a$, where U is a unitary transformation on $\mathbf{C}^n$. Note that ϕ_a is an involution. The following two identities are found in Rudin [4], p. 26.

$$1 - \langle \phi_a(z), \phi_a(w)\rangle = \frac{(1 - \langle a, a\rangle)(1 - \langle z, w\rangle)}{(1 - \langle z, a\rangle)(1 - \langle a, w\rangle)} \tag{3}$$

$$1 - |\phi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1 - \langle z, a\rangle|^2}. \tag{4}$$

The measure

$$d\lambda(z) = (1-|z|^2)^{-(n+1)}\, d\nu(z)$$

is invariant under the automorphism group; thus for each $f \in L^1(d\lambda)$ and automorphism ψ the equality $\int_B f(z)\, d\lambda(z) = \int_B f \circ \psi(z)\, d\lambda(z)$ holds.

A function $u: B \to [-\infty, +\infty)$ is said to be M-subharmonic (M-sh) if it is upper semicontinuous and

$$u(a) \le \int_S u(\phi_a(r\zeta))\, d\sigma(\zeta)$$

for each $a \in B$, $r < 1$. The function u is M-harmonic if equality holds. For $u \in C^2(B)$, u is M-sh if and only if $\tilde{\Delta} u \ge 0$ in B, where $\tilde{\Delta}$ is the invariant Laplacian

$$\tilde{\Delta} u(a) = 4(1-|a|^2) \sum_{i,k=1}^{n} (\delta_{ik} - a_i \bar{a}_k) D_i \bar{D}_k u(a).$$

For $z \in B$ and $\zeta \in S$ the invariant Poisson kernel is defined by

$$P(z, \zeta) = \frac{(1-|z|^2)^n}{|1 - \langle z, \zeta\rangle|^{2n}}.$$

We define the Green function $G(z, w)$ by setting

$$g(r) = c_n \int_r^1 \frac{(1-t^2)^{n-1}}{t^{2n-1}}\, dt \quad \text{and} \quad G(z, w) = g(|\phi_z(w)|).$$

If u is M-harmonic in B and satisfies (1) then u can be represented as the Poisson integral of a finite Borel measure τ,

$$u(z) = \int_S P(z, \zeta)\, d\tau(\zeta).$$

For u M-sh, the following analogue of the Riesz decomposition appears in Ullrich [7].

PROPOSITION A. *If u is M-sh in B and satisfies* (1) *then*

$$u(z)=h(z)-\int_B G(z,w)\,d\mu(w), \tag{5}$$

where h is the least M-harmonic majorant of u, and μ is a positive Borel measure. The measure $d\mu=\tilde{\Delta}u\,d\lambda$ in the sense of distributions and satisfies

$$\int_B (1-|w|^2)^n\,d\mu(w)<+\infty. \tag{6}$$

Conversely, if μ is a positive measure and satisfies (6), *then the Green potential*

$$V(z)=-\int_B G(z,w)\,d\mu(w)$$

is M-sh on B.

Ullrich's analogue of Littlewood's theorem is

PROPOSITION B [7]. *Suppose u is M-sh on B and*

$$\sup_{r<1}\int_S |u(r\zeta)|\,d\sigma(\zeta)<\infty.$$

Then $\lim_{r\to 1} u(r\zeta)$ *exists for almost every* $\zeta\in S$.

For $\alpha>1$ and $\zeta\in S$, the domain, introduced by Koranyi,

$$D_\alpha(\zeta)=\{z\in\mathbf{C}^n: |1-\langle z,\zeta\rangle|<(\alpha/2)(1-|z|^2)\}$$

in B is called an admissible domain. A function V on B is said to have an admissible limit L at ζ if

$$\lim_{\substack{|z|\to 1\\ z\in D_\alpha(\zeta)}} V(z)=L.$$

Our third theorem below involves the concept of an admissible limit in L^p, introduced by Ziomek [10] in $\mathbf{R}^n$. Define the truncated region $D_{\beta,\rho}(\zeta)$ by

$$D_{\beta,\rho}(\zeta)=D_\beta(\zeta)\cap\{z\in B: |z|>\rho\}.$$

A function u on B is said to have an admissible limit A in L^p at ζ if

$$\lim_{\rho\to 1}\frac{1}{\nu(D_{\beta,\rho}(\zeta))}\int_{D_{\beta,\rho}(\zeta)} |u(z)-A|^p\,d\nu(z)=0.$$

Our results are as follows.

THEOREM 1. *Suppose u is M-sh in B and satisfies* (1). *Suppose that $\tilde{\Delta}u$ is absolutely continuous with respect to $d\lambda$. If*

$$\int_B (\tilde{\Delta}u(z))^p(1-|z|^2)^n\,d\lambda(z)<\infty \tag{7}$$

for some $p>n$, then u has admissible limits a.e. on S.

REMARK. When $n=1$ we have

$$\tilde{\Delta}u(z)=\Delta u(z)(1-|z|^2)^2$$
$$d\lambda(z)=(1-|z|^2)^{-2}dA(z),$$

and Theorem 1 is equivalent to the theorem of Arsove and Huber [1]; see (2).

THEOREM 2. *Suppose* $f(z)\geq 0$, $n<p<\infty$, *and*

$$\int_B (f(z))^p\,d\lambda(z)<\infty.$$

Then the Green potential

$$v(z)=-\int_B G(z,w)f(w)\,d\lambda(w) \tag{8}$$

is continuous on the closed ball.

THEOREM 3. *Suppose u is M-sh in B and satisfies* (1). *If* $q<n/(n-1)$ *then u has admissible limits in L^q for almost every* $\zeta\in S$.

3. Some geometric lemmas. We use the following lemmas in the proof of Theorem 1.

LEMMA 1. *Let* $\alpha>1$, $0<r<1$. *Suppose* $a\in D_\alpha(\zeta)$. *Then*

$$\phi_a(rB)\subseteq D_\beta(\zeta)\quad \text{for any } \beta\geq\alpha\left(\frac{1+r}{1-r}\right).$$

REMARK. We can also show that if $a\in D_\alpha(\zeta)$ and $a'=|a|\zeta$ then $a\in\phi_{a'}(rB)$ for any $r>\sqrt{1-\alpha^{-2}}$.

Proof. One can use the definition of the automorphisms ϕ_a and the fact that the P_a and Q_a occurring in that expression are self adjoint projections to establish the equality

$$(1-\langle z,a\rangle)(1-\langle\phi_a(z),\zeta\rangle)=(1-\langle a,\zeta\rangle)(1-\langle z,\phi_a(\zeta)\rangle).$$

Now if $z\in rB$ and $a\in D_\alpha(\zeta)$ we have

$$\begin{aligned}|1-\langle\phi_a(z),\zeta\rangle| &= \frac{|1-\langle a,\zeta\rangle|\,|1-\langle z,\phi_a(\zeta)\rangle|}{|1-\langle z,a\rangle|}\\ &\leq \frac{\alpha}{2}\left(\frac{1+r}{1-r}\right)\frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z,a\rangle|^2}\\ &\leq\frac{\beta}{2}(1-|\phi_a(z)|^2).\end{aligned}$$

This is the required inequality. □

LEMMA 2. *Suppose* $z\in B$ *and* $\alpha>1$. *Define* $\tilde{D}_\alpha(z)\subseteq S$ *by*

$$\tilde{D}_\alpha(z)=\{\zeta\in S: z\in D_\alpha(\zeta)\}.$$

There exists a constant $C = C(\alpha, n)$ *such that*

$$\sigma(\tilde{D}_\alpha(z)) \leq C(1-|z|^2)^n.$$

Proof. As in §5.1 of 4, we define the non-isotropic balls in S by

$$Q(\zeta, \delta) = \{\eta \in S : |1-\langle\zeta, \eta\rangle|^{1/2} < \delta\}.$$

With $\zeta \in \tilde{D}_\alpha(z)$ (i.e. $z \in D_\alpha(\zeta)$), we set $\eta = z/|z|$. Now applying Lemma 5.4.3 of [4] we see that

$$|1-\langle\zeta, \eta\rangle| < 4\alpha|1-\langle z, \eta\rangle| \leq 4\alpha(1-|z|^2).$$

We deduce that

$$\tilde{D}_\alpha(z) \subseteq Q(\eta, 2(\alpha(1-|z|^2))^{1/2}).$$

Appealing to Proposition 5.1.4 of [4], there exists a constant $A_0(n)$ such that

$$\sigma(Q(\eta, 2(\alpha(1-|z|^2))^{1/2}) \leq A_0[4\alpha(1-|z|^2)]^n. \qquad \square$$

We conclude this section with

LEMMA 3. *Suppose* $p \geq 1$ *and*

$$\int_B f(z)^p (1-|z|^2)^n \, d\lambda(z) < \infty.$$

Then

$$\int_{D_\beta(\zeta)} f(z)^p \, d\lambda(z) < \infty$$

and hence

$$\lim_{\rho\to 1} \int_{D_{\beta,\rho}(\zeta)} f(z)^p \, d\lambda(z) = 0$$

for almost every $\zeta \in S$.

Proof. Observe first that by Lemma 2 $\sigma(\tilde{D}_\beta(z)) \leq C(1-|z|^2)^n$. Hence,

$$\int_S \left(\int_{D_\beta(s)} f(z)^p \, d\lambda(z)\right) d\sigma(s) = \int_B \int_S \chi_{\tilde{D}_\beta(z)}(\zeta) \, d\sigma(\zeta) f(z)^p \, d\lambda(z)$$

$$\leq C \int_B (1-|z|^2)^n f(z)^p \, d\lambda(z).$$

Our assumption is that this last integral is finite, so an application of Fubini's Theorem yields the conclusion of the Lemma. $\square$

4. Proof of Theorem 1. Let u be M-sh in B and satisfy (1). As in (5), u can be written as the sum of a M-harmonic function h and a Green potential. The function h satisfies (1) and thus by Koranyi's theorem [4] has admissible limits a.e. $\zeta \in S$. It suffices therefore to prove that the Green potential has admissible limit zero almost everywhere.

We define

$$g_0(t)=g(t)\chi_{(0,1/2)}(t) \qquad g_1(t)=g(t)-g_0(t),$$
$$G_0(z,w)=g_0(|\phi_z(w)|) \qquad G_1(z,w)=g_1(|\phi_z(w)|).$$

We now write the Green potential as

$$\int_B G(z,w)\tilde{\Delta}u(w)\,d\lambda(w)=\psi_0(z)+\psi_1(z)$$

with

$$\psi_0(z)=\int_B G_0(z,w)\tilde{\Delta}u(w)\,d\lambda(w)$$

and

$$\psi_1(z)=\int_B G_1(z,w)\tilde{\Delta}u(w)\,d\lambda(w).$$

We will show both ψ_0 and ψ_1 have admissible limits almost everywhere.

We begin with ψ_1; here our estimates are essentially the same as in [7]. From the definition of G_1 we see that $G_1(z,w)\le C(1-|\phi_z(w)|^2)^n$. From this and the identity (4) we deduce

$$\begin{aligned}\psi_1(z)&\le C\int_B (1-|\phi_z(w)|^2)^n\tilde{\Delta}u(w)\,d\lambda(w)\\&=C\int_B\left(\frac{1-|z|^2}{|1-\langle z,w\rangle|^2}\right)^n(1-|w|^2)^n\tilde{\Delta}u(w)\,d\lambda(w).\end{aligned}$$

We define a measure τ on S by

$$\tau(A)=\int_{\tilde{A}}(1-|w|^2)^n\tilde{\Delta}u(w)\,d\lambda(w),$$

where $\tilde{A}=\{w\in B: w/|w|\in A\}$. Since the measure $\tilde{\Delta}u(w)\,d\lambda(w)$ satisfies (6), τ is a finite measure.

From the inequality $|1-\langle z,w/|w|\rangle|<2|1-\langle z,w\rangle|$ we conclude that

$$\left(\frac{1-|z|^2}{|1-\langle z,w\rangle|^2}\right)^n\le 2^nP\left(z,\frac{w}{|w|}\right),$$

and hence

$$\psi_1(z)\le C'\int_S P(z,\zeta)\,d\tau(\zeta).$$

We define

$$[M_\alpha\psi_1](\zeta)=\sup\{\psi_1(z): z\in D_\alpha(\zeta)\}.$$

From the weak 1-1 boundedness of the admissible maximal function for the Poisson integral of τ [4, Theorem 5.4.5],

$$\sigma\{M_\alpha\psi_1>t\}<C\frac{\|\tau\|}{t}.$$

By standard methods ψ_1 has admissible limits a.e. ζ.

We now consider ψ_0. Observe that $G_0(z,\cdot)$ is supported on $\phi_z(B/2)$, and thus for $1/p+1/q=1$

$$\psi_0(z)\leq\left(\int_B [G_0(z,w)]^q\,d\lambda(w)\right)^{1/q}\left(\int_{\phi_z(B/2)} [\tilde{\Delta}u(w)]^p\,d\lambda(w)\right)^{1/p}.$$

By the invariance of $d\lambda$,

$$\int_B [G_0(z,w)]^q\,d\lambda(w)=\int_B [g_0(|\phi_z(w)|)]^q\,d\lambda(w)$$

$$=\int_B [g_0(|w|)]^q\,d\lambda(w).$$

From the estimates

$$g_0(t)\leq\begin{cases}\log(1/t) & 0<t<\frac{1}{2},\ n=1\\ ct^{2-2n} & 0<t<\frac{1}{2},\ n\geq 2\\ 0 & \frac{1}{2}\leq t<1\end{cases}$$

we deduce that $\|G_0(z,\cdot)\|_{L^q d\lambda(w)}=C_{q,n}$ for some finite $C_{q,n}$ independent of z if $q<n/(n-1)$. Hence, if $p>n$,

$$\psi_0(z)\leq C\left[\int_{\phi_z(B/2)} [\tilde{\Delta}u(w)]^p\,d\lambda(w)\right]^{1/p}.$$

Let $\alpha>1$ be fixed and set $\beta=3\alpha$. If $|\xi|<\frac{1}{2}$ we estimate $|\phi_z(\xi)|^2\geq 2|z|^2-1$, and from Lemma 1 we obtain the inclusion

$$\phi_z(B/2)\subset D_{\beta,\rho}(\zeta)\quad(\rho\leq 2|z|^2-1). \tag{9}$$

Thus

$$\psi_0(z)\leq C\left[\int_{D_{\beta,\rho}(\zeta)} [\tilde{\Delta}u(w)]^p\,d\lambda(w)\right]^{1/p}$$

so with $f(w)=\tilde{\Delta}u(w)$ in Lemma (3) and $z\in D_\alpha(\zeta)$ we have $\lim_{z\to\zeta}\psi_0(z)=0$ (a.e. ζ). Since $u=h+\psi_0+\psi_1$ and each of the terms has admissible limits a.e. ζ, u also has admissible limits a.e.

5. Proofs of Theorem 2 and Theorem 3. To prove Theorem 2 assume $n<p$ and then the conjugate q satisfies $1<q<n/(n-1)$. Then $G(0,w)$ is in $L^q(d\lambda)$ and by invariance of λ

$$\|G(z,w)\|_{L^q(d\lambda)}=\|G(0,w)\|_{L^q(d\lambda)}.$$

Thus the Green potential (8) is a continuous function, being the convolution of an $L^p(d\lambda)$ function with an $L^q(d\lambda)$ function.

For $0<r<1$, let $f_r(z)=X_{rB}(z)\cdot f(z)$ and let $V_r(z)$ be the Green potential of $f_r(z)$. Since f_r has compact support, $V_r(z)\to 0$ uniformly as $|z|\to 1$. By Holder's inequality

$$\|V-V_r\|_\infty\leq C\|f-f_r\|_p$$

so that V_r tends to V uniformly on the ball as $r\to 1$. Hence, $V(z)\to 0$ uniformly as $|z|\to 1$.

To prove Theorem 3 we write $u(z)=h(z)-(\psi_0(z)+\psi_1(z))$ as in Theorem 1. The proof of Theorem 1 shows $\psi_1(z)$ and $h(z)$ have admissible limits a.e. (ζ) and hence admissible limits in L^q a.e. (ζ). Thus, it suffices to show ψ_0 has admissible limits in L^q a.e. (ζ).

Let μ be the measure of Proposition A. We observe that our assumption (1) and Lemma 3 (with $d\mu$ instead of $fd\lambda$) imply that

$$\lim_{r\to 1}\int_{D_{\beta,r}(\zeta)} d\mu(w)=0 \text{ a.e. } \zeta\in S.$$

Let α be fixed and set $\beta=3\alpha$. For $\rho>\sqrt{2}/2$ set $r=2\rho^2-1$. Then by Lemma 1 and the estimate (9), $G_0(z,w)$ is supported in $D_{\beta,r}(\zeta)$ for all $z\in D_{\alpha,\rho}(\zeta)$.

Thus,

$$\int_{D_{\alpha,\rho}(\zeta)}\psi_0(z)\,d\lambda(z)=\int_{D_{\alpha,\rho}}\int_B G_0(z,w)\,d\mu(w)$$

$$\le\int_B\int_B G_0(z,w)\chi_{D_{\beta,r}(\zeta)}(w)\,d\mu(w).$$

By Young's inequality,

$$\left(\int_{D_{\alpha,\rho}(\zeta)}\psi_0(z)^q\,d\lambda(z)\right)^{1/q}\le\left(\int_B (G_0(z,w))^q\,d\lambda(w)\right)^{1/q}\left(\int_{D_{\beta,r}(\zeta)}d\mu(w)\right).$$

If $q<n/(n-1)$ the first integral is finite and the second integral tends to zero a.e. (ζ) as r tends to one as we noted above. Thus,

$$\lim_{\rho\to 1}\int_{D_{\alpha,\rho}(\zeta)}[\psi_0(z)]^q\,d\lambda(z)=0 \text{ a.e. } (\zeta).$$

We note that there are positive constants C_1 and C_2 which depend only on α and n such that

$$C_1(1-\rho)^{n+1}\le\nu(D_{\alpha,\rho}(\zeta))\le C_2(1-\rho)^{n+1}.$$

Recall that $d\lambda=(1-|z|^2)^{-(n+1)}\,d\nu$, and hence

$$\lim_{\rho\to 1}\frac{1}{\nu(D_{\alpha,\rho}(\zeta))}\int_{D_{\alpha,\rho}(\zeta)}[\psi_0(z)]^q\,d\nu(z)=0.$$

6. Sharpness of the results.

EXAMPLE 1. The condition $p>n$ is best possible in Theorem 1. Choose sequences $\{z_j\}$, $\{r_j\}$, $\{\epsilon_j\}$, and $\{a_j\}$ such that

(i) $z_j\in B$, $\lim_{j\to\infty}|z_j|=1$ and $D_\alpha(\zeta)\cap\{z_j\}$ is infinite for each $\zeta\in S$,

(ii) $0<r_j<1/2$ and the family $\{\phi_{z_j}(r_jB)\}$ is pairwise disjoint,

(iii) $\epsilon_j>0$ and $\sum_{j=1}^{n}(1-|z_j|^2)\epsilon_j^{1/n}<\infty$, and

(iv) $a_j>0$ and $\lim_{j\to\infty}a_j=\infty$.

One can choose $f_j\in C_c^\infty(B)$ with $f_j\ge 0$, support $f_j\subseteq\phi_{z_j}(r_jB)$, $\int f_j^n\,d\lambda<\epsilon_j$ and $\int f_j(w)G(z_j,w)\,d\lambda(w)>a_j$. Let $f=\sum f_j$. Then $f\in C^\infty(B)$ and by the above choices f satisfies

$$\int_B f(w)(1-|w|^2)^n\,d\lambda(w)<\infty.$$

Thus the Green potential $v(z)$ defined by (8) is M-sh, $\tilde{\Delta}v=f$, and by Proposition B $\lim_{r\to 1} v(r\zeta)=0$ for a.e. $\zeta\in S$. Also, v satisfies (7) of Theorem 1 with $p=n$.

However,

$$v(z_j)\le -\int_B f_j(w)G(z_j,w)\,d\lambda(w)\le -a_j,$$

hence for a.e. $\zeta\in S$

$$\limsup_{\substack{|z|\to 1\\ z\in D_\alpha(\zeta)}} v(z)=0 \quad\text{and}\quad \liminf_{\substack{|z|\to 1\\ z\in D_\alpha(\zeta)}} v(z)=-\infty.$$

REMARK. Since $G(0,\cdot)\notin L^{n/(n-1)}$, it is easy to construct functions which satisfy $\int_B (f(z))^n\,d\lambda(z)<\infty$ and for which the Green potential (8) assumes the value $-\infty$ at points on the open ball. Hence the assumption $p>n$ in Theorem 2 is sharp.

EXAMPLE 2. The assumption $q<n/(n-1)$ is best possible in Theorem 3. Pick $\{z_j\}$ as before, so every $D_\alpha(\zeta)$ contains infinitely many z_j's ($\epsilon_j>0$), so that $\sum(1-|z_j|^2)^n\epsilon_j<\infty$, and let $u(z)=-\sum_j \epsilon_j G(z_j,z)$. Then u satisfies the hypotheses of Theorem 3 but does not have an admissible limit in L^q for $q=n/(n-1)$ at any $\zeta\in S$. Indeed, given $\zeta\in S$, $\alpha>1$, choose j and $r>0$ such that $\phi_{z_j}(rB)\subset D_{\alpha,\rho}(\zeta)$. Then

$$\int_{D_{\alpha,\rho}(\zeta)}|u|^q\,d\nu\ge\epsilon_j\int_{\phi_{z_j}(rB)}(G(z_j,w))^q\,d\nu(w)=\infty,$$

which certainly implies u has no admissible limit in L^q at ζ.

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