RANDOM SERIES AND BOUNDED MEAN OSCILLATION

Peter Duren

Dedicated to George Piranian

A function f analytic in the unit disk $\mathbf{D}$ is said to belong to the *Hardy space* H^p, $0<p<\infty$, if its integral means

$$M_p(r,f)=\left\{\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta\right\}^{1/p}$$

remain bounded as r tends to 1. We let $M_\infty(r,f)$ denote the maximum of $|f(z)|$ on the circle $|z|=r<1$. Thus H^∞ is the class of bounded analytic functions in $\mathbf{D}$. A function $f\in H^1$ is said to be in the space BMOA if its boundary function $F(t)=f(e^{it})$ is of *bounded mean oscillation*:

$$\sup_I \frac{1}{|I|}\int_I |F(t)-F_I|\,dt<\infty,$$

where $|I|$ is the length of the interval I and

$$F_I=\frac{1}{|I|}\int_I F(t)\,dt.$$

The *Bloch space* $\mathcal{B}$ consists of all analytic functions f for which

$$\sup_{z\in\mathbf{D}}(1-|z|)|f'(z)|<\infty.$$

The proper inclusions

$$H^\infty\subset \text{BMOA}\subset \mathcal{H}^*=\bigcap_{p<\infty} H^p$$

and $\text{BMOA}\subset\mathcal{B}$ are well known. Moreover,

$$\mathcal{B}\not\subset \mathcal{H}_*=\bigcup_{p>0} H^p.$$

A useful criterion for an analytic function f to belong to BMOA is that

$$d\mu(z)=(1-|z|)|f'(z)|^2\,dx\,dy$$

be a *Carleson measure* on $\mathbf{D}$. See [2] and [3] for further background.

We shall be concerned with *random power series*

$$f(z)=\sum_{n=0}^{\infty}\epsilon_n a_n z^n,\quad \epsilon_n=\pm 1,$$

where the ϵ_n are random signs and $\limsup\sqrt[n]{|a_n|}\le 1$. More precisely, such functions have the form

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$$f(z) = f(z,t) = \sum_{n=0}^{\infty} \phi_n(t) a_n z^n, \quad 0 \le t \le 1,$$

where ϕ_n is the nth *Rademacher function.* (See [2], Appendix A.) Each function f is analytic in $\mathbf{D}$. According to familiar results of Paley and Zygmund [8, 9] and Littlewood [5], random power series are similar to lacunary series: they are very well behaved if the coefficients are square-summable and very badly behaved if not. Specifically, the condition $\sum |a_n|^2 < \infty$ implies $f \in \mathcal{H}^*$ almost surely (a.s.); that is, $f \in \mathcal{H}^*$ for almost every choice of signs, or for almost every $t \in [0,1]$. On the other hand, $\sum |a_n|^2 = \infty$ implies $f \notin \mathcal{H}_*$ a.s. A simple example [8] shows that even under the stronger condition

$$\sum_{n=1}^{\infty} |a_n|^2 \log n < \infty, \tag{1}$$

f need not belong to H^∞ for *any* choice of signs. However, if

$$\sum_{n=1}^{\infty} |a_n|^2 (\log n)^\beta < \infty \tag{2}$$

for some $\beta > 1$, then f is almost surely continuous in $\bar{\mathbf{D}}$. (See [6] for the ultimate refinement of this last condition.)

More recently, Anderson, Clunie, and Pommerenke [1] studied the Bloch space and showed in particular that (1) implies $f \in \mathcal{B}$ a.s., and that the condition is best possible in the following sense. For each prescribed sequence $\{\delta_n\}$ of positive numbers decreasing to 0, the coefficients $a_n > 0$ can be chosen so that $\sum a_n^2 \delta_n \log n < \infty$ but $f \notin \mathcal{B}$ a.s. The proof uses a beautiful theorem of Salem and Zygmund (described below) on the behavior of the maxima of the partial sums of random trigonometric series.

Because in some sense BMOA is the "natural limit" of H^p as $p \to \infty$, it had seemed a reasonable conjecture that $\sum |a_n|^2 < \infty$ implies $f \in$ BMOA a.s. The analogous statement for lacunary series is true and is easily proved using the Carleson measure criterion. Since BMOA $\subset \mathcal{B}$, however, the construction of Anderson, Clunie, and Pommerenke shows that this result does not extend to random series. In fact, no condition weaker than (1) can imply that $f \in$ BMOA a.s. Without initial knowledge of [1], Sledd [13] used the Salem–Zygmund theorem to show that (1) actually implies $f \in$ BMOA a.s.

These results reveal a surprisingly large gap between BMOA and $\mathcal{H}^*$. The purpose of the present paper is to explore the difference of these two spaces by studying the effect of the intermediate condition (2) with $0 < \beta < 1$. Our main result is as follows.

THEOREM. *If $\sum_{n=1}^{\infty} |a_n|^2 (\log n)^\beta < \infty$ for $0 \le \beta \le 1$, then for almost every choice of signs ϵ_n, the function $f(z) = \sum_{n=1}^{\infty} \epsilon_n a_n z^n$ has the property*

$$\int_0^1 (1-r)\left(\log \frac{1}{1-r}\right)^{\beta-1} [M_\infty(r,f')]^2 \, dr < \infty. \tag{3}$$

Before giving the proof, we point out some corollaries. The first is Sledd's theorem. The second generalizes the result of Anderson, Clunie, and Pommerenke that the same condition implies $f \in \mathcal{B}$ a.s.

COROLLARY 1. *If* $\sum_{n=1}^{\infty} |a_n|^2 \log n < \infty$, *then* $f \in$ BMOA *a.s.*

Proof. The property (3) with $\beta = 1$ clearly shows that

$$d\mu(z) = (1-|z|)|f'(z)|^2\,dx\,dy$$

is a Carleson measure on $\mathbf{D}$, which implies that $f \in$ BMOA. In fact, if S is a "square" at the boundary with side-length h, it follows that $\mu(S) = o(h)$ uniformly as h tends to 0.

This actually gives the stronger conclusion that $f \in$ VMOA a.s. In other words, f has *vanishing mean oscillation* (see [12], [3]).

COROLLARY 2. *If* $\sum_{n=1}^{\infty} |a_n|^2 (\log n)^\beta < \infty$ *for* $0 \le \beta \le 1$, *then as* $r \to 1$,

$$M_\infty(r, f') = o\left(\frac{1}{1-r}\left(\log \frac{1}{1-r}\right)^{(1-\beta)/2}\right) \quad a.s.$$

Proof. Since $M_\infty(r, f')$ is nondecreasing and the integral (3) is convergent, we have for r sufficiently large

$$[M_\infty(r, f')]^2 \int_r^1 (1-x)\left(\log \frac{1}{1-x}\right)^{\beta-1} dx < \epsilon.$$

But this last integral is easily seen to be

$$O\left((1-r)^2\left(\log \frac{1}{1-r}\right)^{\beta-1}\right), \quad r \to 1,$$

which gives the desired result.

Choosing $\beta = 1$, we obtain the result of Anderson, Clunie,and Pommerenke that (1) implies $f \in \mathcal{B}$ a.s. In fact, f belongs almost surely to the space $\mathcal{B}_0$ for which $f'(re^{i\theta}) = o((1-r)^{-1})$ as $r \to 1$, uniformly in θ. The method of [1] may be adapted to obtain Corollary 2 directly, but the proof of our theorem is easier. The construction in [1] may also be modified to show that Corollary 2 is best possible: the conclusion fails if the hypothesis (2) is weakened. The same is therefore true for the theorem.

COROLLARY 3. *If* $\sum_{n=1}^{\infty} |a_n|^2 < \infty$, *then*

$$M_\infty(r, f') = o\left(\frac{1}{1-r}\left(\log \frac{1}{1-r}\right)^{1/2}\right) \quad a.s.$$

This special case of Corollary 2 may also be deduced from the result of Paley and Zygmund [8] that

$$M_\infty(r, f) = o\left(\left(\log \frac{1}{1-r}\right)^{1/2}\right) \quad \text{a.s.} \tag{4}$$

if $\sum |a_n|^2 < \infty$. Paley [7] produced a construction to show that (4) is the best possible estimate. In comparing (4) with Corollary 2, one is tempted to conjecture that (2) implies

$$M_\infty(r,f)=o\left(\left(\log\frac{1}{1-r}\right)^{(1-\beta)/2}\right)\quad \text{a.s.}$$

if $0<\beta<1$. However, Paley's construction can be adapted to show that the stronger condition (2) allows an improvement of (4) at best by a factor of the form $[\log\log(1/(1-r))]^{-\alpha}$ for some $\alpha>0$.

The proof of our theorem makes use of three lemmas. Lemma 1 is the basic result of Salem and Zygmund [11] already mentioned. Lemma 2 estimates an integral which may be viewed as a generalization of the classical beta function. (The proof is tedious and will be omitted.) Lemma 3 is Hilbert's inequality (see [4], p. 226). The overall approach is due to Pommerenke [10], who introduced it to give a proof of Corollary 1.

LEMMA 1. *Let* $s_n(\theta,t)=\sum_{k=1}^n \phi_k(t)b_k e^{ik\theta}$, *where* ϕ_k *are the Rademacher functions and* $b_k\in\mathbf{C}$. *Let* $M_n(t)=\max_\theta|s_n(\theta,t)|$ *and* $B_n^2=\sum_{k=1}^n|b_k|^2$. *Then*

$$M_n(t)=O(B_n\sqrt{\log n})\quad \textit{a.s.}$$

LEMMA 2. *For* $\gamma\geq 0$,

$$\int_0^1 x^n(1-x)^3\left(\log\frac{1}{1-x}\right)^{-\gamma}dx=O\left(\frac{1}{n^4(\log n)^\gamma}\right),\quad n\to\infty.$$

LEMMA 3. *For arbitrary* $\lambda_n\in\mathbf{C}$,

$$\left|\sum_{n=1}^\infty\sum_{m=1}^\infty\frac{\lambda_n\lambda_m}{n+m}\right|<\pi\sum_{n=1}^\infty|\lambda_n|^2.$$

Proof of Theorem. Write

$$f(z)=f(z,t)=\sum_{n=1}^\infty \phi_n(t)a_n z^n,$$

and let

$$s_n(\theta,t)=\sum_{k=1}^n k a_k e^{ik\theta}\phi_k(t).$$

Then

$$(1-r)\sum_{n=1}^\infty s_n(\theta,t)r^n=zf'(z,t),\quad z=re^{i\theta}.$$

For almost every $t\in[0,1]$, Lemma 1 gives

$$s_n(\theta,t)=O(B_n\sqrt{\log n}),\quad \text{where}\quad B_n^2=\sum_{k=1}^n k^2|a_k|^2,$$

uniformly in θ. Thus, almost surely,

$$|f'(z)|\leq C(1-r)\sum_{n=1}^\infty B_n\sqrt{\log n}\,r^n=C\psi(r),\tag{5}$$

where C denotes a constant (not necessarily the same at each occurrence) and $\psi(r)$ is the sum of the given series.

In view of Lemma 2 we have, for $\gamma \geq 0$,

$$\begin{aligned}
&\int_0^1 (1-r)\left(\log \frac{1}{1-r}\right)^{-\gamma} [\psi(r)]^2\,dr \\
&= \int_0^1 (1-r)^3\left(\log \frac{1}{1-r}\right)^{-\gamma} \left[\sum_{n=1}^{\infty} B_n \sqrt{\log n}\, r^n\right]^2 \\
&= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} B_n \sqrt{\log n}\, B_m \sqrt{\log m} \int_0^1 r^{n+m}(1-r)^3\left(\log \frac{1}{1-r}\right)^{-\gamma} dr \\
&\leq C \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{B_n \sqrt{\log n}\, B_m \sqrt{\log m}}{(n+m)^4[\log(n+m)]^{\gamma}} \leq C \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n+m} \lambda_n \lambda_m,
\end{aligned}$$

where $\lambda_n = B_n(\log n)^{(1-\gamma)/2} n^{-3/2}$. Now choose $\gamma = 1-\beta$ and apply Lemma 3 to obtain

$$\int_0^1 (1-r)\left(\log \frac{1}{1-r}\right)^{\beta-1} [\psi(r)]^2\,dr \leq C \sum_{n=1}^{\infty} B_n^2 (\log n)^{\beta} n^{-3}. \tag{6}$$

On the other hand,

$$\begin{aligned}
\sum_{n=1}^{\infty} B_n^2(\log n)^{\beta} n^{-3} &= \sum_{n=1}^{\infty} \sum_{k=1}^{n} k^2 |a_k|^2 (\log n)^{\beta} n^{-3} = \sum_{k=1}^{\infty} k^2 |a_k|^2 \sum_{n=k}^{\infty} (\log n)^{\beta} n^{-3} \\
&\leq C \sum_{k=1}^{\infty} |a_k|^2 (\log k)^{\beta} < \infty,
\end{aligned}$$

by hypothesis. Thus a combination of (5) and (6) shows that (3) holds almost surely, and the proof of the theorem is complete.

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Department of Mathematics
University of Michigan
Ann Arbor, Michigan 48109