A UNIVALENCY CRITERION

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Dedicated to George Piranian

1. Introduction. Let f be a meromorphic and locally univalent function in the upper half-plane U, that is, $f'(z)\neq 0$ and any pole of f is simple. It is natural, when looking for criteria which imply the univalence of f, to introduce the Schwarzian derivative $S(f,z)$, defined by

$$S(f,z)=\left(\frac{f''}{f'}\right)'(z)-\frac{1}{2}\left(\frac{f''}{f'}\right)^2(z).$$

We shall use the notation

$$U=\{z:\operatorname{Im} z>0\},\qquad L=\{z:\operatorname{Im} z<0\},\qquad B(z,r)=\{w:|w-z|\le r\}.$$

If f can be extended to a local homeomorphism F defined on the whole sphere $\bar{\mathbf{C}}$ then f will be univalent in U. This method for establishing univalence was emphasized by Ahlfors in [1], where he gave extensions and alternative derivations of many known criteria for univalence. If F is differentiable at $z=z_0$, say, the condition $|F_{\bar z}|<|F_z|$ for $z=z_0$ ensures that the Jacobian of F is not zero at z_0 and hence that F is a local homeomorphism at z_0. The stronger condition $|F_{\bar z}|\le k|F_z|$ for all $z\in L$, where $0<k<1$, says that f has a k-quasiconformal extension to L. This is not the standard terminology, but agrees with that used by Ahlfors in [1]. Thus for $0<k<1$, a k-quasiconformal mapping is one whose maximal dilatation does not exceed $(1+k)/(1-k)$. Ahlfors has proved the following result [1, p. 29].

THEOREM A. *Suppose that* $0<k<1$, $|c-1|\le k$ *and* $y=\operatorname{Im} z$. *If f is meromorphic and locally univalent in U and such that*

$$\left|2y^2S(f,z)-c(c-1)\left(\frac{\bar z+it}{z+it}\right)^2\right|\le k|c| \tag{1.1}$$

for all $z\in U$ and some $t>0$, then f is univalent in U and has a k-quasiconformal extension to $\bar{\mathbf{C}}$.

The case $c=1$ is the half-plane version of the well-known criterion of Nehari [4] and Ahlfors and Weill [2]. As Ahlfors remarks [1, p. 29], the criterion (1.1), depending as it does on establishing that the values of $y^2S(f,z)$ lie in a variable disk, seems too complicated to be useful. Ahlfors let $t\to\infty$ in (1.1) and asked if

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the corresponding condition implies the existence of a k-quasiconformal extension for f. This is indeed the case, as the following result shows.

THEOREM 1. *Suppose that f is meromorphic and locally univalent in U and satisfies there the inequality*

$$|2y^2 S(f,z) - c(c-1)| \le k|c|, \tag{1.2}$$

where $0<k<1$ and $|c-1|\le k$. Then f is univalent in U and has a k-quasiconformal extension to $\bar{\mathbf{C}}$. If $|c-1|<1$ and (1.2) *holds for all $z\in U$ with $k=1$, then f is univalent in U.*

The case $k=1$ can be dealt with by modifying the proof for $0<k<1$, or by using the standard method of extending the result from $0<k<1$ to cover the case $k=1$ (see Lehto [3]). These methods are discussed in Section 5. Moreover, the condition (1.2) is sharp for $c=1$ in the sense that the theorem fails if $k>1$: the function $f(z)=z^{i\delta}$, $\delta>0$, is not univalent in U but $|2y^2 S(f,z)| = y^2|z|^{-2}(1+\delta^2) \le 1+\delta^2$. However, the condition (1.2) cannot be sharp in the same sense for c close to zero since the condition $|2y^2 S(f,z)|\le 1$ both implies univalence and is implied by (1.2) for $k>1$, if $|c|$ is small.

2. The quasiconformal extension. We may assume that there are no poles of f or zeros of f'' on the positive imaginary axis—if $f''(z)\equiv 0$, there is nothing to prove. We use the following notation. For $0<r<1$, set

$$R = \frac{2r}{1-r^2}, \qquad z_0 = i(1+R^2)^{1/2},$$

so that

$$R < (1+R^2)^{1/2} = \frac{1+r^2}{1-r^2}.$$

We also set

$$D_1 = D_1(r) = \{z : |z-z_0| < R\},$$
$$D_2 = D_2(r) = \{z : z=\infty \text{ or } |z-z_0| > R\},$$

and put $\partial D_1 = \Gamma$. Clearly $D_1 \subset U$, $D_2 \supset L$ and, if $0<r_1<r_2<\cdots<r_n\to 1$ as $n\to\infty$, then

$$\bigcup_{n=1}^{\infty} D_1(r_n) = U.$$

Finally, we use the notation

$$u(z) = f(z)(f'(z))^{-1/2}, \qquad v(z) = (f'(z))^{-1/2},$$

so that

$$u'v - v'u \equiv 1, \qquad u''v - v''u \equiv 0, \qquad u''v' - v''u' = \tfrac{1}{2}S(f,z).$$

Note that $(f'(z))^{1/2}$ is properly defined since $f'(z)\ne 0$ and all the poles of f', if any, are double poles.

The anticonformal mapping $\zeta \to z(\zeta)$ given by

$$z(\zeta)-z_0=R^2(\bar{\zeta}-\bar{z}_0)^{-1}$$

maps D_2 onto D_1, D_1 onto D_2, and has $z(\zeta)=\zeta$ for $\zeta\in\Gamma$. Moreover,

$$\frac{dz}{d\bar{\zeta}}=-R^2(\bar{\zeta}-\bar{z}_0)^{-2},\quad \frac{dz}{d\zeta}=0.$$

For a fixed r, $0<r<1$, we define

$$\tilde{f}(\zeta)=f(\zeta),\quad \zeta\in D_1\cup\Gamma,$$

(2.1)
$$\tilde{f}(\zeta)=\frac{u(z)+((\zeta-z)/c)u'(z)}{v(z)+((\zeta-z)/c)v'(z)},\quad \zeta\in D_2,$$

where $z=z(\zeta)$ and u and v are as above. Clearly, $\tilde{f}$ will depend on r, but we do not emphasize this.

Theorem 1 for $0<k<1$ will follow if we show that, for a sequence $\{r_n\}$ of values of r tending to 1, the above $\tilde{f}$ is a k-quasiconformal mapping of $\bar{\mathbf{C}}$ onto $\bar{\mathbf{C}}$. In fact we shall show that $\tilde{f}$ is a k-quasiconformal mapping for all r outside a countable exceptional set which arises when Γ contains poles of f. For such a sequence $\{r_n\}$, $r_n\to 1-$ as $n\to\infty$, we consider the corresponding k-quasiconformal maps, denoted by $\tilde{f}_n$. The set $\{\tilde{f}_n\mid n\in\mathbf{N}\}$ forms a normal family and by passing to a subsequence, if necessary, we obtain a limit function F such that

(a) $\tilde{f}_n(\zeta)\to F(\zeta)$ as $n\to\infty$, locally uniformly in $\mathbf{C}$,
(b) $F(\zeta)=f(\zeta)$ for $\zeta\in U$,
(c) F is a k-quasiconformal map of $\bar{\mathbf{C}}$ onto $\bar{\mathbf{C}}$.

Since

$$z(\zeta)=i\,\frac{\bar{\zeta}(1+r^2)+i(1-r^2)}{\bar{\zeta}(1-r^2)+i(1+r^2)},$$

we see that, as $r\to 1-$,

$$z(\zeta)\to\bar{\zeta},\quad \zeta-z(\zeta)\to\zeta-\bar{\zeta}.$$

Thus, no matter how the subsequence $\{r_n\}$ is chosen, it is clear from (2.1) that for $\zeta\in L$ we have

$$F(\zeta)=\frac{u(\bar{\zeta})+((\zeta-\bar{\zeta})/c)u'(\bar{\zeta})}{v(\bar{\zeta})+((\zeta-\bar{\zeta})/c)v'(\bar{\zeta})}.$$

Suppose that $k=1$. In order to prove Theorem 1 in this case it suffices to show that for a sequence $\{r_n\}$ of values of r tending to 1 as $n\to\infty$, the corresponding mappings $\tilde{f}_n$ are locally univalent and hence univalent in $\bar{\mathbf{C}}$. Since $\tilde{f}_n(z)=f(z)$ in $D_1(r_n)$, it follows that f is univalent in $D_1(r_n)$ for all n. Hence f is univalent in U.

3. The function $\tilde{f}$. We have to show that the function $\tilde{f}$ given by (2.1) is locally homeomorphic at ∞ and on $\Gamma=\partial D_1$, and that

(3.1)
$$\tilde{f}_\zeta\neq 0,\quad |\tilde{f}_{\bar{\zeta}}|\le k|\tilde{f}_\zeta|$$

for $\zeta\in D_2\backslash\{\infty\}$. Note that if (3.1) holds, then $\tilde{f}$ is a local homeomorphism at all points $\zeta\in D_2\backslash\{\infty\}$. Hence $\tilde{f}$ is locally univalent and hence univalent in $\bar{\mathbf{C}}$. Since

$\tilde{f}$ is clearly absolutely continuous on lines, (3.1) then implies that $\tilde{f}$ is k-quasiconformal in $\bar{\mathbf{C}}$.

Henceforth we consider $\tilde{f}(\zeta)$ only for $\zeta \in D_2 \cup \Gamma$. We have

$$\tilde{f}(\zeta)=f(z)+\frac{\zeta-z}{c}f'(z)\left(1-\frac{1}{2}\frac{\zeta-z}{c}\frac{f''(z)}{f'(z)}\right)^{-1},$$

where $z=z(\zeta)$. We may choose the sequence $\{r_n\}$, and hence the fixed r now considered, so that Γ contains no poles of f. Since $f'(z)\neq 0$ for $z\in U$ and $f'(z)\neq\infty$ except when $f(z)=\infty$, the derivatives above remain finite on Γ. Thus, for $\zeta_0\in\Gamma$ and δ sufficiently small,

$$\tilde{f}(\zeta_0+\delta)=f(\zeta_0)+\delta f'(\zeta_0)+O(\delta^2)$$

if $\zeta_0+\delta\in D_1\cup\Gamma$, while if $\zeta_0+\delta\in D_2$ we have

$$\begin{aligned}\tilde{f}(\zeta_0+\delta)&=f(\zeta_0)+f'(\zeta_0)\left\{\delta\frac{dz}{d\bar{\zeta}}(\zeta_0)+\frac{1}{c}\Big[(\zeta-\zeta_0)-(z-\zeta_0)\Big]\right\}+O(\delta^2)\\ &=f(\zeta_0)+f'(\zeta_0)\left\{\frac{\delta}{c}+\frac{\bar{\delta}(c-1)}{c}\frac{dz}{d\bar{\zeta}}(\zeta_0)\right\}+O(\delta^2).\end{aligned}$$

Since $|(dz/d\bar{\zeta})(\zeta_0)|=1$ and $|c-1|\le k<1$ (or $|c-1|<1$ if $k=1$), it follows that $\tilde{f}$ is homeomorphic in a sufficiently small neighborhood of ζ_0. (When $k=1$, the assumption $|c-1|<1$ is made to ensure this.)

To prove that $\tilde{f}$ is homeomorphic and sense-preserving also in a neighborhood of ∞ we note that, as $\zeta\to\infty$, we have

$$z(\zeta)=z_0+\frac{R^2}{\bar{\zeta}}+O(\zeta^{-2}).$$

Thus

$$\tilde{f}(\zeta)=A+B_1(\bar{\zeta})^{-1}+B_2\zeta^{-1}+O(\zeta^{-2}),$$

where

$$A=f(z_0)-2(f'(z_0))^2(f''(z_0))^{-1},$$

$$B_1=-R^2\left\{3f'(z_0)-2f^{(3)}(z_0)\left(\frac{f'(z_0)}{f''(z_0)}\right)^2\right\},$$

$$B_2=-4c(f'(z_0))^3(f''(z_0))^{-2}.$$

Now $B_2\neq 0$ and

$$\frac{B_1}{B_2}=-\frac{R^2}{2c}S(f,z_0)=\frac{-R^2}{2c(1+R^2)}y_0^2S(f,z_0),$$

since $y_0=\operatorname{Im} z_0=(1+R^2)^{1/2}$. From (1.2) we know that

$$-\frac{1}{2c}y_0^2S(f,z_0)\in B\left(-\frac{c-1}{4},\frac{k}{4}\right)$$

and hence

$$\left|\frac{B_1}{B_2}\right|\le\frac{R^2}{1+R^2}\cdot\frac{1}{4}(|c-1|+k)\le\frac{k}{2}\le\frac{1}{2}$$

since $|c-1|\le k<1$. Also, $|B_1/B_2|\le\frac{1}{2}$ if $k=1$. Thus $\tilde{f}$ is homeomorphic and sense-preserving also in a neighborhood of infinity.

To prove (3.1) we note that for $\zeta\in D_2\backslash\{\infty\}$,

$$\frac{\partial\tilde{f}}{\partial\zeta}=c^{-1}\left(v(z)+\frac{\zeta-z}{c}v'(z)\right)^{-2},$$

$$\frac{\partial\tilde{f}}{\partial\bar{\zeta}}=c^{-1}\left((c-1)+\frac{(\zeta-z)^2}{2c}S(f,z)\right)\left(v(z)+\frac{\zeta-z}{c}v'(z)\right)^{-2}\frac{dz}{d\bar{\zeta}},$$

where, of course, $z=z(\zeta)$. Also

$$v(z)+\frac{\zeta-z}{c}v'(z)=(f'(z))^{-3/2}\left[f'(z)-\frac{f''(z)}{2c}(\zeta-z)\right],$$

and this expression is finite even at the poles of $f(z)$. Thus $\partial\tilde{f}/\partial\zeta\neq 0$, as we asserted. Moreover, $\tilde{f}$ will be k-quasiconformal at ζ if the complex dilatation μ satisfies

$$|\mu|=|\tilde{f}_{\bar{\zeta}}/\tilde{f}_{\zeta}|\le k,$$

i.e. if the following inequality holds:

$$\left|-\tfrac{1}{2}(\zeta-z)^2S(f,z)-c(c-1)\right|\le k|c|\left|\frac{dz}{d\bar{\zeta}}\right|^{-1}. \tag{3.2}$$

Care must be taken at those points where $f(z(\zeta))$, $\tilde{f}$, $\partial\tilde{f}/\partial\zeta$, or $\partial\tilde{f}/\partial\bar{\zeta}$ becomes infinite. If

$$f(z(\zeta))=\infty \quad\text{and}\quad f(z(\zeta)+h)=A_{-1}h^{-1}+A_0+A_1h+\cdots,$$

then

$$\tilde{f}(\zeta)=A_0+\frac{A_{-1}c}{\zeta-z(\zeta)}\neq\infty.$$

Hence the above analysis remains valid in this case. If $\tilde{f}$, $\partial\tilde{f}/\partial\zeta$, or $\partial\tilde{f}/\partial\bar{\zeta}$ is infinite, then

$$\frac{f''(z)}{f'(z)}=\frac{2c}{\zeta-z}. \tag{3.3}$$

At such points we consider $1/\tilde{f}$ instead of $\tilde{f}$. Now $|\mu(1/\tilde{f})|=|\mu(\tilde{f})|$ and

$$\frac{\partial}{\partial\zeta}\left(\frac{1}{\tilde{f}}\right)=-(\tilde{f})^{-2}\frac{\partial\tilde{f}}{\partial\zeta}=-c^{-1}\left(u(z)+\frac{\zeta-z}{c}u'(z)\right)^{-2}.$$

But

$$(f'(z))^{1/2}\left[u(z)+\frac{\zeta-z}{c}u'(z)\right]=f(z)\left(1-\frac{\zeta-z}{2c}\frac{f''(z)}{f'(z)}\right)+\frac{\zeta-z}{c}f'(z).$$

Thus, at points where (3.3) holds,

$$\frac{\partial}{\partial\zeta}\left(\frac{1}{\tilde{f}}\right)=-c(\zeta-z)^{-2}(f'(z))^{-1}.$$

Hence even at these points ζ, we see that $\partial(1/\tilde{f})/\partial\zeta$ is non-zero and remains finite. A similar argument applies to $\partial(1/\tilde{f})/\partial\bar{\zeta}$. Thus it suffices to prove (3.2) also for these ζ.

If we set

$$T = -\frac{(\zeta - z)^2}{4y^2}, \quad V = \left|\frac{dz}{d\bar{\zeta}}\right|^{-1},$$

then (3.2) reduces to showing that

$$2Ty^2 S(f, z) \in B(c(c-1), k|c|V),$$

when, by (1.2),

$$2y^2 S(f, z) \in B(c(c-1), k|c|).$$

Thus to show that (3.2) is implied by (1.2), we must show that the disk $B(Tc(c-1), k|c||T|)$ is contained in the disk $B(c(c-1), k|c|V)$. This will be so if and only if the distance apart of the centers plus the smaller radius is equal, at most, to the larger radius. We shall show that $|T| < V$, so we are required to prove that $|Tc(c-1) - c(c-1)| + k|c||T| \le k|c|V$ or that

(3.4) $$|T-1||c||c-1| \le k|c|(V - |T|).$$

Since we must establish this whenever $|c-1| \le k$, we have really to prove the inequality

(3.5) $$|T-1| \le V - |T|.$$

Note that if $k = 1$ and $|c-1| < 1$, and if $|T| < V$ and (3.5) holds, then (3.4) holds as a strict inequality.

4. The inequality. The final step in the proof is the verification of the inequality (3.5). Since T and V depend only on $\zeta \in D_2 \setminus \{\infty\}$ we express them in terms of $z \in D_1 \setminus \{z_0\}$. Set

$$z = z_0 + \lambda e^{i\theta}, \quad 0 < \lambda < R, \quad 0 \le \theta \le 2\pi.$$

We show that, for all $\lambda \in (0, R)$ and all θ, the inequality (3.5) holds. Putting $R/\lambda = \mu > 1$ and $z = x + iy$ we have

(4.1) $$y = \operatorname{Im} z = (1 + R^2)^{1/2} + \lambda \sin\theta$$

and

$$V = \left|\frac{dz}{d\bar{\zeta}}\right|^{-1} = \left|\frac{\bar{\zeta} - \bar{z}_0}{R}\right|^2 = \left|\frac{R}{z - z_0}\right|^2 = \frac{R^2}{\lambda^2} = \mu^2.$$

Since also

$$4y^2|T| = \lambda^2(\mu^2 - 1)^2, \quad |T - 1| = (4y^2)^{-1}|\lambda^2 e^{2i\theta}(\mu^2 - 1)^2 + 4y^2|,$$

the inequality (3.5) reads

(4.2) $$|\lambda^2 e^{2i\theta}(\mu^2 - 1)^2 + 4y^2| \le 4y^2\mu^2 - \lambda^2(\mu^2 - 1)^2.$$

The preliminary inequality, $V > |T|$, which we must establish is equivalent to showing that the right-hand side of (4.2) is positive. Giving y its smallest value $y_0 = (1+R^2)^{1/2} - \lambda$ for a fixed λ, we must show that

$$4\mu^2[(1+R^2)^{1/2} - \lambda]^2 - \lambda^2(\mu^2 - 1)^2 > 0$$

for $0 < \lambda < R$. That this latter inequality is true is most readily seen by replacing $(1+R^2)^{1/2}$ by the smaller quantity R and verifying the ensuing inequality.

Returning now to the inequality (4.2), we note that

$$|\lambda^2 e^{2i\theta}(\mu^2-1)^2 + 4y^2| \le \lambda^2(\mu^2-1)^2 + 4y^2$$

and this latter term is less than $4y^2\mu^2 - \lambda^2(\mu^2-1)^2$ if $2y^2 \ge \lambda^2(\mu^2-1)$. Thus we need to prove (4.2) only in the case when $y < [\frac{1}{2}(R^2-\lambda^2)]^{1/2} < R/\sqrt{2}$ and, in particular, only when $-\pi \le \theta < 0$. Now (4.2) is equivalent to the inequality $G(\theta) \ge 0$, where

$$G(\theta) = [4y^2\mu^2 - \lambda^2(\mu^2-1)^2]^2 - [\lambda^2(\mu^2-1)^2\cos 2\theta + 4y^2]^2 - [\lambda^2(\mu^2-1)^2 \sin 2\theta]^2,$$

and y is given by (4.1). Elementary considerations show that the minimum of $G(\theta)$ in the range $[-\pi, 0]$ occurs at $\theta = -\pi/2$. So it suffices to verify (4.2) for $\theta = -\pi/2$. Then it reads

$$|4((1+R^2)^{1/2} - \lambda)^2 - \lambda^2(\mu^2-1)^2| \le 4\{(1+R^2)^{1/2} - \lambda\}^2\mu^2 - \lambda^2(\mu^2-1)^2.$$

If the expression in the modulus sign is positive, the inequality is obvious, since $\mu > 1$. Otherwise the inequality reads

$$(R^2-\lambda^2)(\mu^2-1) \le 2\{(1+R^2)^{1/2} - \lambda\}^2(\mu^2+1).$$

This final inequality is true for all R and λ, $0 < \lambda < R$, as is again readily seen on replacing $(1+R^2)^{1/2}$ by R. Thus (3.5) is finally established and the proof of Theorem 1 is complete for $k < 1$.

5. The case $k = 1$. Suppose that $k = 1$ and $|c-1| < 1$. We have seen that then $\tilde{f}$ is locally homeomorphic at every point in $D_1 \cup \Gamma \cup \{\infty\}$. As we remarked at the end of Section 3, (3.4) holds as a strict inequality. In fact, this would be true for $0 < \lambda < R$ even if $|c-1| = 1$, $c \ne 0$, since our proof of (3.5) shows that (3.5) holds as a strict inequality. Thus the closed disk $B(Tc(c-1), |cT|)$ is contained in the *interior* of $B(c(c-1), |c|V)$ so that $|\mu(f, \zeta)| = |(\tilde{f}_{\bar\zeta}/\tilde{f}_\zeta)(\zeta)| < 1$ for every $\zeta \in D_2 \backslash \{\infty\}$. Hence $\tilde{f}$ is locally homeomorphic in $D_2 \backslash \{\infty\}$. We conclude that $\tilde{f}$ is a global homeomorphism, so that f is univalent in U. The proof of Theorem 1 is complete. □

We remark that the case $k = 1$ can also be dealt with as follows, by using the method in Lehto's paper [3, p. 606]. If (1.2) holds with $k = 1$ and $|c-1| < 1$, let f_n be a locally univalent meromorphic function in U with

$$S(f_n, z) = (1 - 1/n)S(f, z), \quad z \in U.$$

Since $2y^2 S(f, z) \in B(c(c-1), |c|)$, it follows that $2y^2 S(f_n, z) \in B(c(c-1), k_n|c|)$, where $k_n = 1 - (1 - |c-1|)/n < 1$. By what we have proved, f_n is univalent in U.

Thus, by means of a Möbius transformation, we can normalize f_n so that f_n agrees with f at three given points of U, where f attains distinct values. Hence the functions f_n form a normal family, and a subsequence converges locally uniformly in U to a univalent function g with $S(g,z)=S(f,z)$. Therefore $f\circ g^{-1}$ is a Möbius transformation, so that also f is univalent, as required.

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