A MATHEMATICAL MODEL FOR A WAKE

David E. Tepper

1. Introduction. This paper concerns a type of free boundary problem which gives a mathematical model for a wake around an obstacle in the flow of a two-dimensional fluid which is incompressible, irrotational and inviscid. To formulate the problem, we begin with an infinite striplike region $S=\{x+iy: \phi_1(x)<y<\phi_2(x), -\infty<x<\infty\}$, ϕ_1 and ϕ_2 continuous. If α is a Jordan curve with interior Δ contained inside of S, then a wake around α is a doubly connected region $\omega\subset S$ such that $\Delta\cap\omega=\varnothing$, $\partial\omega=\partial S\cup\gamma$, where $\partial S\cap\gamma=\varnothing$, and there is a harmonic function V_ω in ω satisfying:

(a) $V_\omega(z)=0$ for $z\in\partial S$,

(b) $V_\omega(z)=1$ for $z\in\gamma$, and

(c) $|\text{grad } V_\omega(z)|=p$ for $z\in\gamma-\alpha$.

The number p is a constant. (See Figure 1.)

The set $\gamma-\alpha$ is made up of *free stream lines* [6] and is called the free boundary. We will show that there is a region ω when ∂S and α are starlike. The methods in [5] can be used to prove existence. In particular, a solution to this problem is obtained by considering

$$J(v)=\iint_{S-\Delta}(|\nabla v|^2+I_{(v>0)}p^2)\,dx\,dy,$$

where I_A is the characteristic function of A. If $K=\{v: v=1 \text{ on } \partial S,\ v\geqslant 0 \text{ in } S-\Delta\}$, then u_p, a minimum for $J(v)$, will give a wake around α by letting $\omega=\{u_p>0\}$ and $V_\omega=1-u_p$. However, Beurling's paper [4] and various qualitative results in [8], [9] and [10] can be used to get existence with the additional information that the free boundaries are starlike. Furthermore, we obtain values for the constant p where the solution will be non-degenerate, i.e., $\gamma-\alpha\neq\varnothing$. The main idea is to first formulate the problem for compact regions and then approximate S by a sequence of these compact regions. In the compact case we are able to use results in [1], [4], [8], [9] and [10] to find wakes. However, the results in these papers are formulated for doubly connected regions and must be extended to the simply connected case. Before beginning, we mention that wakes are also studied in [10], which includes a survey of classical methods to attack the problem of an infinite wake.

2. COMPACTNESS. Suppose D is simply connected on the compact Riemann sphere and Γ is the boundary of D which is compact in the open plane but does not reduce to a point. Let C be the class of all doubly connected regions $\omega\subset D$ such that $\partial\omega=\Gamma\cup\gamma$ where $\gamma\cap\Gamma=\varnothing$ and γ is compact in the open plane. We call γ the free boundary of ω. (See Figure 2.)

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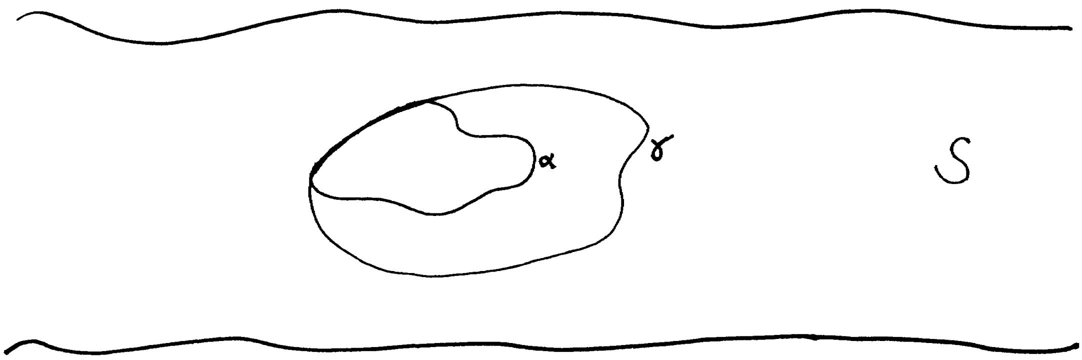

Figure 1. A typical wake around α is shown. The free boundary γ may intersect α but this need not be the case. If $\alpha=\gamma$, then the wake will be called trivial.

For each region $\omega \in C$, we define the stream function of ω to be the harmonic function V satisfying

(1) $$V(z)=0, \quad \text{for } z \in \Gamma$$

(2) $$V(z)=1, \quad \text{for } z \in \gamma.$$

Suppose Q is a continuous bounded positive function in D. When D is doubly connected in the complex plane, Beurling in [4] gave necessary and sufficient conditions for there to exist a region $\omega \in C$ such that

(3) $$|\text{grad } V(z)| = Q(z) \quad \text{for } z \in \gamma.$$

Such a region ω is called a solution (for the function Q). Although Beurling's existence theorem is stated for the case where D is doubly connected in the finite plane, we show that the result holds for the case where D is simply connected.

THEOREM 1. *If there exists $\omega_1 \in C$ with stream function V_1 such that for all ζ on the free boundary of ω_1 we have*

(4) $$\limsup_{\substack{z \to \zeta \\ z \in \omega_1}} \frac{|\text{grad } V_1(z)|}{Q(z)} < 1,$$

then there exists a solution $\omega_0 \subset \omega_1$. If in addition there exists $\omega_2 \in C$ with stream function V_2 such that $\omega_2 \subset \omega_1$, and for all ζ on the free boundary of ω_2 we have

(5) $$\liminf_{\substack{z \to \zeta \\ z \in \omega_2}} \frac{|\text{grad } V_2(z)|}{Q(z)} \geqslant 1,$$

then there is a solution ω_0, $\omega_2 \subset \omega_0 \subset \omega_1$. Furthermore, if ever there is more than one solution, then there is a particular pair of solutions ω' and ω'' with $\omega' \subset \omega''$.

Proof. For the case where D is doubly connected in the finite plane, the result is shown in [4]. When D is doubly connected we first suppose there is a point z_0 interior to $D-\omega_1$. If f is a conformal mapping of $\{w: |w|>1\}$ to D such that z_0 corresponds to the infinite point, then Beurling's theorem for the doubly connected case will apply in $\{w: |w|>1\}$ for the function $Q(f(w))|f'(w)|$. The

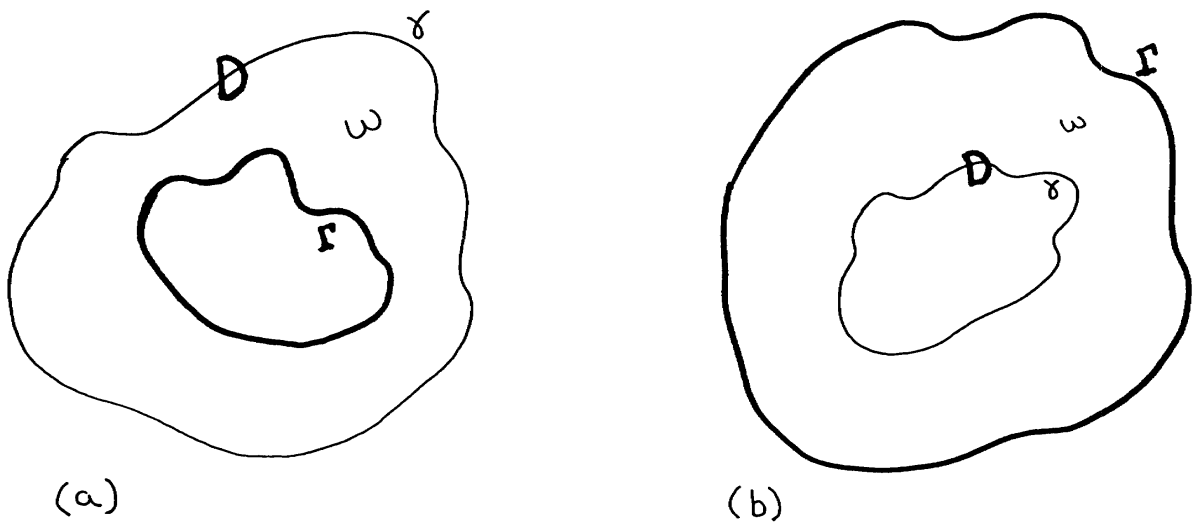

Figure 2. Typical regions ω in C where (a) D is doubly connected in the finite plane and (b) D is simply connected in the finite plane.

image of this solution under the mapping f will be a solution for the function $Q(z)$. If no such point z_0 exists then we replace $Q(z)$ by $Q_\epsilon(z)=Q(z)+\epsilon$, where $\epsilon>0$. For the function Q_ϵ we can find a region $\omega_{1,\epsilon}$ whose stream function will satisfy (4) for the function $Q_\epsilon(z)$ and $\omega_{1,\epsilon}\subset\omega_1$. Furthermore, $D-\omega_{1,\epsilon}$ will have an interior point z_ϵ. It follows that $\omega_{1,\epsilon}$ contains a solution ω_ϵ for the function $Q_\epsilon(z)$ and $\omega=\bigcup_{\epsilon>0}\omega_{1,\epsilon}$ will be a solution for the function $Q(z)$. Since similar reasoning is used to show (5) and the nested property, we omit the details in this paper. □

A simple application of Theorem 1 is the following corollary.

COROLLARY 2. *If D is simply connected and contains the disk $|z|\leqslant R$, then for $Q(z)\geqslant e/R$ there is at least one solution.*

Proof. Let ω_ϵ have free boundary $|z|=\epsilon$ for $0<\epsilon<R$. If V_ϵ is the stream function of ω_ϵ, then for $|z|=\epsilon$ we have

$$|\text{grad } V_\epsilon(z)|\leqslant \frac{1}{\epsilon\log R/\epsilon}.$$

If we let $R/\epsilon=e$, then for $|z|=\epsilon$ we have

$$|\text{grad } V_\epsilon(z)|\leqslant\frac{1}{\epsilon}=\frac{e}{R}\leqslant Q(z).$$

We see that (4) holds for $\omega_\epsilon\in C$ and therefore ω_ϵ contains a solution ω. □

3. UNIQUENESS. In order to prove uniqueness, we must put conditions on D and Q; in particular, we suppose D is starlike. That is, a set A is starlike if for each $z\in A$, the line segment $[0,z]$ is contained inside of A. We also say that the complement of a starlike set is starlike and the boundary of a starlike set is starlike. The following lemma extends Lemma 11 of [1].

LEMMA 3. *Suppose D is starlike. If D is simply connected in the finite plane and $\rho Q(\rho z)$ is a strictly decreasing function of ρ for each $z \in D$, then there is at most one solution. If D is doubly connected in the finite plane and $\rho Q(\rho z)$ is a strictly increasing function of ρ for each $z \in D$, then there is at most one solution. Furthermore, in either case the free boundary of this solution will be starlike and contain no radial segments.*

Proof. Lemma 11 of [1] shows that the doubly connected case holds. To extend to the simply connected case, we map D to $\tilde{D}$ by the function $1/z$. Γ will map to $\tilde{\Gamma}$ which will also be starlike. The function $\tilde{Q}(w) = Q(1/w)|w|^{-2}$ can have only one solution by the doubly connected case. Since there is a one-to-one correspondence between the solutions in D for the function Q and the solutions in $\tilde{D}$ for the function $\tilde{Q}$, the proof is complete. □

4. WAKES. Suppose D is starlike and contains the disk $|z| \leqslant R$. By Corollary 2, there exists a constant λ_D such that $\lambda_D \leqslant e/R$ and there is a solution for all functions $Q(z) \geqslant \lambda_D$. For the case where $Q(z) \equiv \lambda$ a positive constant greater then λ_D, we denote this solution $\tilde{\omega}_\lambda$. Furthermore, if $D_1 \supset D_2$, then $\lambda_{D_1} \leqslant \lambda_{D_2}$. Suppose α is a smooth Jordan curve contained inside D and Ω is the annulus in C which has α as its free boundary. If V is the stream function of Ω, then let $\mu = \sup|\text{grad } V(z)|$ for $z \in \alpha$. We see that $\mu \geqslant \lambda_D$.

We call $\omega \in C$ a wake around α if the stream function V of ω satisfies, for $z \in (\partial\omega - \alpha) \cap \Omega$, $|\text{grad } V(z)| = p$ for some constant p. We say the wake is non-trivial if $(\partial\omega - \alpha) \cap \Omega \neq \varnothing$. The following theorem gives sufficient conditions for there to exist non-trivial wakes for a range of values of the constant p.

THEOREM 3. *If D and α are starlike, then there exists a constant $\rho_D \leqslant \mu$ such that there are non-trivial wakes for $\lambda > \rho_D$. Furthermore, if $\Omega \neq \tilde{\omega}_\mu$, then $\rho_D < \mu$.*

Proof. For $\lambda \geqslant \mu$ the result follows from Theorem 1. For $\mu > \lambda > \lambda_D$ we give a procedure to construct the wake. Finally, we show that when $\Omega \neq \tilde{\omega}_\mu$ this construction gives a non-trivial wake for values of λ, $\mu - \epsilon < \lambda < \mu$. To do this we let $f(z)$ be a conformal mapping of Ω to $\{w: 1 < |w| < R\}$ such that α corresponds to $|w| = 1$ and Γ to $|w| = R$. Since D is starlike, $|f(z)|$ will increase as $|z|$ increases [9, p. 179]. For positive integers n define:

$$Q_n(z) = \begin{cases} \lambda, & \text{for } 1 + 1/n < |f(z)| < R \\ \lambda/[n(|f(z)| - 1)], & \text{for } 1 + \lambda/n\mu < |f(z)| < 1 + 1/n \\ \mu, & \text{for } 1 < |f(z)| < 1 + \lambda/n\mu. \end{cases}$$

If we define $Q_n(z) = \mu$ for $z \in D - \Omega$, then Q satisfies the hypothesis of Theorem 2. It follows that for each n there exists a solution Ω_n for the function Q_n. It follows that $\omega_\lambda = \bigcup \Omega_n$ will be a wake around α.

To see that there will be non-trivial solutions when $\Omega \neq \tilde{\omega}_\mu$, we observe that if V_n is the stream function of Ω_n, then $|\text{grad } V_n(z)| > \lambda$ for z on the free boundary of Ω_n. Theorem 1 and uniqueness of solutions implies $\tilde{\omega}_\lambda \supset \omega_\lambda$. Therefore, $\omega_\lambda \subset \tilde{\omega}_\lambda \cap \Omega$ which assures us that there will be non-trivial solutions whenever $\tilde{\omega}_\lambda$ does not contain Ω.

For S and α starlike, we obtain a model for a wake by letting $D=D_n=S\cap\{z: |z|\leqslant n\}$ and Ω_n the member of the corresponding class C_n whose free boundary is α. Let U_n be the stream function of Ω_n and $\mu_n=\sup|\text{grad } U_n(z)|$ for $z\in\alpha$. Since μ_n decreases as $n\to\infty$, we see that (a), (b), and (c) hold whenever $p>\lim \mu_n$. We could apply the methods of [10] to relax the condition that S be starlike. In such a case S will have to be the limit of starlike regions. □

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Department of Mathematics
Baruch College
City University of New York
New York, New York 10010