DIVISIBILITY IN DOUGLAS ALGEBRAS

Sheldon Axler and Pamela Gorkin

Let D denote the open unit disk in the complex plane. Let L^∞ and H^∞ denote the usual Banach algebras on the unit circle ∂D. A closed subalgebra between H^∞ and L^∞ is called a Douglas algebra. The smallest Douglas algebra properly containing H^∞ is $H^\infty+C$, where C denotes the algebra of continuous complex valued functions defined on ∂D.

The set of nonzero multiplicative linear functionals on a Douglas algebra B is called the maximal ideal space of B and is denoted $M(B)$. With the weak-* topology, $M(B)$ is a compact Hausdorff space. Each $m \in M(H^\infty)$ has a unique extension (also denoted by m) to a linear functional on L^∞ of norm one. Thus $M(B)$ may be identified with a subset of $M(H^\infty)$. Each function $f \in L^\infty$ can be thought of as a continuous function (also denoted by f) on $M(H^\infty)$. In the obvious way, we think of D as a subset of $M(H^\infty)$. With these identifications, when a function $f \in L^\infty$ is thought of as a function on $M(H^\infty)$, then $f \mid D$ is just the usual harmonic extension of f. Furthermore, the map on H^∞ taking f to $f \mid D$ identifies H^∞ with the set of bounded analytic functions on the disk D.

For $u \in L^\infty$, the smallest norm closed subalgebra of L^∞ containing H^∞ and u is denoted $H^\infty[u]$.

Theorem 4 gives a condition which insures that a function h in a Douglas algebra B multiplies all powers of a function u into B. In the case where $B=H^\infty+C$ and $\bar{u}$ is a unimodular function in $H^\infty+C$, this was proved by Guillory and Sarason; see the Theorem on page 176 of [3] and also the last paragraph on page 178 of [3]. Their proof used the construction invented by Carleson to prove the Corona Theorem. Luecking [6] sought a proof that did not use the corona construction, and in doing so he found a nice generalization of Guillory and Sarason's theorem to arbitrary Douglas algebras, while removing the restriction that $\bar{u}$ be in the Douglas algebra under consideration. Luecking considered only unimodular functions u, and in this case Theorem 1 of Luecking's paper [6] is actually equivalent to our Theorem 4, despite the somewhat different appearance. However, Luecking's proof uses two deep theorems—one concerning Blaschke products and the other dealing with Bergman spaces—in a nontrivial way. We believe our proof of Theorem 4 is considerably easier than either Guillory and Sarason's or Luecking's proofs.

Our main tool involves interpolating sequences. A sequence $\{z_n\}$ in D is called an interpolating sequence if for each bounded sequence $\{w_n\}$ of complex numbers, there exists a function f in H^∞ such that $f(z_n)=w_n$ for each n. A Blaschke product whose zero sequence is an interpolating sequence is called an interpolating Blaschke product.

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For B a Douglas algebra, QB denotes the set of functions f in B such that the complex conjugate $\bar{f}$ is also in B. For the special case where $B=H^\infty+C$, the algebra QB is traditionally denoted by QC. An important result of Wolff [8, Theorem 1] states that given $f\in L^\infty$, there is an outer function $g\in QC$ such that $fg\in QC$. Wolff breaks his proof into two lemmas; our Theorem 5 is a strengthened version of one of these lemmas. In the case where $B=H^\infty+C$, Wolff's Lemma 1.1 [8] gives a Blaschke sequence (not necessarily interpolating) satisfying the conditions of Theorem 5. The proof of Theorem 5 uses Theorem 4 rather than the dyadic VMO techniques used by Wolff.

We now discuss some facts that we will need. We make frequent use of the Chang–Marshall Theorem, which states that every Douglas algebra is generated by H^∞ and the complex conjugates of some collection of interpolating Blaschke products; see [2, Chapter IX].

If W is a subset of the disk D, the closure of W means the closure of W in the space $M(H^\infty)$. An interpolating Blaschke product b with zero sequence $\{z_n\}$ has the property that if $m\in M(H^\infty)$ and $m(b)=0$, then m is in the closure of $\{z_n\}$; see [4, p. 206].

The proof of Theorem 4 will require three lemmas.

LEMMA 1. *Let B be a Douglas algebra. Let h be a function in B and let b be an interpolating Blaschke product. If*

$$\{m\in M(B): m(b)=0\}\subset\{m\in M(B): m(h)=0\}$$

then $h/b\in B$.

Proof. Let ϵ be a positive number. By the Chang–Marshall Theorem there is a function $f\in H^\infty$ and an inner function q with $\bar{q}\in B$ such that $\|h-f\bar{q}\|<\epsilon$. Let

$$W_1=\{z\in D: b(z)=0 \text{ and } |f(z)|<\epsilon\} \quad \text{and}$$

$$W_2=\{z\in D: b(z)=0 \text{ and } |f(z)|\geqslant\epsilon\}.$$

Let b_j be the interpolating Blaschke product whose zero set is W_j. Thus $b=b_1b_2$.

We claim that $\bar{b}_2\in B$. If not, then b_2 is not invertible in B and thus there exists $m\in M(B)$ with $m(b_2)=0$. Thus $m(b)=0$ and so by hypothesis $m(h)=0$. Since m is in the closure of W_2, we have $|m(f)|\geqslant\epsilon$. Recall that $\|h-f\bar{q}\|<\epsilon$, so

$$\epsilon>|m(h-f\bar{q})\,m(q)|=|m(f\bar{q})\,m(q)|=|m(f)|\geqslant\epsilon,$$

a contradiction. Thus our claim that $\bar{b}_2\in B$ is verified.

It follows easily from the definition of interpolating sequence and the Open Mapping Theorem that there is a constant K such that the distance from g to bH^∞ (abbreviated $\operatorname{dist}(g,bH^\infty)$) is less than or equal to $K\sup\{|g(z)|: z\in D$ and $b(z)=0\}$. Now

$$\begin{aligned}\operatorname{dist}(h/b,B)&\leqslant\epsilon+\operatorname{dist}(f\overline{qb},B)\\&=\epsilon+\operatorname{dist}(f,bB)\\&=\epsilon+\operatorname{dist}(fb_2,bB)\end{aligned}$$

$$\leqslant \epsilon + \operatorname{dist}(fb_2, bH^\infty)$$
$$\leqslant \epsilon + K \sup\{|f(z)| : z \in W_1\}$$
$$\leqslant \epsilon + K\epsilon.$$

Since ϵ is arbitrary we have $h/b \in B$, as desired. □

We note that the next lemma can be false if b is a Blaschke product whose zeroes do not form an interpolating sequence. For example, take $B = H^\infty + C$, let q be a Blaschke product with infinitely many zeroes, and let b be a Blaschke product such that $m(b)=0$ for all $m \in M(H^\infty + C)$ such that $|m(q)| < 1$. A Blaschke product b with this property is constructed in [7, p. 441]. Since b equals zero on an open subset of $M(H^\infty + C)$, the conclusion of Lemma 2 cannot hold. Note that $b(1-|\bar{q}|)=0$ on $M(H^\infty + C)$, so by Theorem 4, b/q^N is in $H^\infty + C$ for every integer N.

LEMMA 2. *Let B be a Douglas algebra. Let $m \in M(B)$ and let b be an interpolating Blaschke product. Then there is a sequence $\{m_n\}$ in $M(B)$ such that $m_n \to m$ and $m_n(b) \neq 0$ for every n.*

Proof. If $m(b) \neq 0$, then take $m_n = m$, and we are done. So we can assume that $m(b)=0$. Thus m is in the closure of the zero sequence of b. There exists a continuous function $L: D \to M(H^\infty)$ such that $L(0)=m$, $f \circ L$ is analytic on D for every $f \in H^\infty$, and $b \circ L$ is not constant on D. The existence of a mapping L with these properties is shown by Hoffman [5, p. 80], see also [2, p. 198]. Let m_n equal $L(1/n)$. Since the zeroes of a nonconstant analytic function are isolated, for n sufficiently large we have $m_n(b) = (b \circ L)(1/n) \neq 0$.

To complete the proof we need only show that $m_n \in M(B)$. By the Chang-Marshall Theorem, it suffices to show that if q is inner and $\bar{q} \in B$, then $|m_n(q)| = 1$. Since $\bar{q} \in B$ and $m \in M(B)$, we have $|m(q)|=1$. However, $q \circ L$ is an analytic map from D to $\bar{D}$ satisfying $|q \circ L(0)| = |m(q)| = 1$. Thus by the Maximum Modulus Theorem, $q \circ L$ is constant, so $1 = |(q \circ L)(1/n)| = |m_n(q)|$, completing the proof. □

A final lemma is needed before proving Theorem 4.

LEMMA 3. *Let B be a Douglas algebra. Let h be a function in B and let b be a finite product of interpolating Blaschke products. If $|h| \leqslant |b|$ on $M(B)$, then $h/b \in B$.*

Proof. Suppose b is the product of n interpolating Blaschke products. The proof will be by induction on n. The case $n=1$ follows immediately from Lemma 1.

Now suppose $n>1$ and $b = b_1 \dots b_n$, where each b_j is an interpolating Blaschke product. Thus $|h| \leqslant |b_1 \dots b_n| \leqslant |b_n|$ on $M(B)$, and so by the $n=1$ case, we have $h/b_n \in B$. We claim that $|h/b_n| \leqslant |b_1 \dots b_{n-1}|$ on $M(B)$. Once the claim is verified, we are done by induction.

To prove the claim let $m \in M(B)$. Then

$$\begin{aligned}|m(h/b_n)m(b_n)| &= |m(h)| \\ &\leqslant |m(b_1 \ldots b_n)| \\ &= |m(b_1 \ldots b_{n-1})||m(b_n)|.\end{aligned}$$

If $m(b_n)$ is nonzero, then we obtain the desired result by dividing both sides of the inequality above by $m(b_n)$. If $m(b_n)=0$, then use Lemma 2 to approximate m by elements of $M(B)$ on which b_n does not vanish. Thus the claim is verified and the proof is completed. □

As motivation for Theorem 4, consider the special case where u is a unimodular function and $\bar{u} \in B$. Suppose the conclusion of Theorem 4 holds, so $hu^N \in B$ for every positive integer N. Let $g_N = hu^N$, so $g_N \in B$. Now for each $m \in M(B)$ we have

$$|m(h)| = |m(\bar{u})^N m(g_N)| \leqslant |m(u)|^N \|h\|.$$

Letting $N \to \infty$, we see that if $|m(u)|<1$, then $m(h)=0$. Thus the converse of Theorem 4 holds in this case.

THEOREM 4. *Let B be a Douglas algebra. Let h be a function in B and let u be a function in L^∞ with $\|u\| \leqslant 1$. If $h(1-|u|)=0$ on $M(B)$, then $hH^\infty[u] \subset B$.*

Proof. Without loss of generality we may assume that $\|h\|<1$. If b is a finite product of interpolating Blaschke products with $\bar{b} \in H^\infty[u]$, we claim that $h/b \in B$. By Lemma 3 we need only verify that $|h| \leqslant |b|$ on $M(B)$. So let $m \in M(B)$. If $|m(b)|=1$, we are done by the normalization of h. So suppose $|m(b)|<1$. Since $\bar{b} \in H^\infty[u]$, we see that $m \notin M(H^\infty[u])$. Thus $|m(u)|<1$ (otherwise u would be constant on the support of m, which would imply that m is multiplicative on $M(H^\infty[u])$). Since $h(1-|u|)=0$ on $M(B)$, we must have $m(h)=0$. Thus $|m(h)| \leqslant |m(b)|$, and the claim is verified.

To complete the proof fix a positive integer N and let ϵ be positive. Use the Chang–Marshall Theorem to choose a function $g \in H^\infty[u]$ such that $\|u^N - g\| < \epsilon$, where g is a finite sum of functions of the form f/b, with $f \in H^\infty$ and b a finite product of interpolating Blaschke products invertible in $H^\infty[u]$. The paragraph above shows that $hg \in B$. Since $\|hu^N - hg\| < \epsilon$, we see that the distance from hu^N to B can be made arbitrarily small. Thus hu^N is in B, and so $hH^\infty[u] \subset B$, as desired. □

For $m \in M(H^\infty)$, the closed support of the probability measure on $M(L^\infty)$ which represents m is denoted by supp m. In the proof of Theorem 5, we will use the following fact twice: If g is a Blaschke product and $m \in M(H^\infty)$ is such that $|m(g)|<1$, then there exists $m_1 \in M(H^\infty)$ such that $\operatorname{supp} m_1 \subset \operatorname{supp} m$ and $m_1(g)=0$. To prove this, let Q be the closure of $H^\infty \mid \operatorname{supp} m$ in the Banach algebra $C(\operatorname{supp} m)$. (Actually $H^\infty \mid \operatorname{supp} m$ is closed in $C(\operatorname{supp} m)$, but we don't need to know that.) It is easy to verify that the maximal ideal space of Q can be identified with $\{m_1 \in M(H^\infty) : \operatorname{supp} m_1 \subset \operatorname{supp} m\}$. In particular $m \in M(Q)$, and since $|m(g)|<1$, the unimodular function g is not invertible in Q. Thus there exists $m_1 \in M(Q)$ such that $m_1(g)=0$.

THEOREM 5. *Let B be a Douglas algebra and let $g \in L^\infty$. Then there exists an interpolating sequence $\{z_n\}$ such that $fg \in QB$ whenever $f \in QB$ and $f(z_n) \to 0$.*

Proof. First we consider the case where g is a Blaschke product. By Theorem 2W of [9] (which is actually a weaker version of part of the Chang–Marshall Theorem) there exists an interpolating Blaschke product b such that

$$\sup\{|b(z)| : z \in D \text{ and } |g(z)| < 1/2\} < 1.$$

Let $\{z_n\}$ be the zero sequence of b. To prove the theorem in the case where g is a Blaschke product, suppose that $f \in QB$ and $f(z_n) \to 0$. We claim that $f(1-|g|) = 0$ on $M(B)$. To verify this claim, suppose $m \in M(B)$ and $|m(g)| < 1$. Thus there exists $m_1 \in M(H^\infty)$ such that $\operatorname{supp} m_1 \subset \operatorname{supp} m$ and $m_1(g) = 0$. The Corona Theorem and the condition defining b now imply that $|m_1(b)| < 1$. Thus there exists $m_2 \in M(H^\infty)$ such that $\operatorname{supp} m_2 \subset \operatorname{supp} m_1$ and $m_2(b) = 0$. Since m_2 is in the closure of $\{z_n\}$, the hypothesis on f implies that $m_2(f) = 0$. Since every QB function is constant on $\operatorname{supp} m$ (the support of any representing measure is always an anti-symmetric set), we must have $m(f) = 0$. Thus $f(1-|g|) = 0$ on $M(B)$, as claimed. Hence $\bar{f}(1-|\bar{g}|) = 0$ on $M(B)$, and so by Theorem 4, $\overline{fg} \in B$. Since f and g are both in B, we have $fg \in B$ and so $fg \in QB$ as desired.

To complete the proof we now assume that g is an arbitrary function in L^∞. By Theorem 1 of [1] there exist Blaschke products b_1, b_2 and functions h_1, h_2 in $H^\infty + C$ such that $g = h_1 \bar{b}_1$ and $\bar{g} = h_2 \bar{b}_2$. Since $b_1 b_2$ is a Blaschke product, the case proved above implies that there exists an interpolating sequence $\{z_n\}$ such that $f(b_1 b_2) \in QB$ whenever $f \in QB$ and $f(z_n) \to 0$.

We now show that the sequence $\{z_n\}$ has the desired properties for g. So suppose $f \in QB$ and $f(z_n) \to 0$. Thus $\overline{fg} = \bar{f} h_2 \bar{b}_2 = (\overline{f b_1 b_2}) h_2 b_1 \in B$. Also, $\bar{f} \in QB$ and $\bar{f}(z_n) \to 0$, so $fg = f h_1 \bar{b}_1 = (f \bar{b}_1 \bar{b}_2) h_1 b_2 \in B$. Thus $fg \in QB$ and the proof is complete. □

NOTE. After the preparation of this paper had been completed, we received a preprint entitled "Interpolating Blaschke products and division in Douglas algebras", by C. Guillory, K. Izuchi, and D. Sarason. They have obtained the divisibility criterion in our Lemma 1 and given applications of it, different from ours, to Douglas algebras.

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Department of Mathematics
Michigan State University
East Lansing, Michigan 48824

and

Department of Mathematics
Bucknell University
Lewisburg, Pennsylvania 17837