THE GAUSS-BONNET THEOREM FOR 2-DIMENSIONAL SPACETIMES

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The purpose of the present note is to extend the classical Gauss–Bonnet formula

$$\int_\Gamma k_g\, ds + \iint_D K\, dA + \sum_i \Theta_i = 2\pi$$

for a region D with boundary Γ on a 2-dimensional Riemannian manifold to the case of a 2-dimensional Lorentzian manifold. Such an extension becomes possible by refining the notion of angle in the Lorentzian plane which was defined in our previous paper [2].

Section 1 deals with the definition and properties of angle in a 2-dimensional spacetime and illustrates a special case of the formula dealing with the term $\sum_i \Theta_i$. In Section 2 we prepare needed facts for the terms $\int_\Gamma k_g\, ds$ and $\iint_D K\, dA$ and state the formula. The proof is given in Section 3 together with a concluding remark.

1. Angle in a spacetime. Following [6, pp. 24–27] we mean by a 2-dimensional spacetime a connected, 2-dimensional, oriented and time-oriented Lorentzian manifold (M, g). Thus M admits a globally defined unit timelike vector field which is future-pointing.

For each point x of M, the tangent space $T_x(M)$ is oriented and has a Lorentzian inner product together with time-orientation. For any unit timelike vector E in $T_x(M)$, we denote by $E^\perp$ the unique unit spacelike vector such that $g(E, E^\perp) = 0$ and such that the ordered basis $\{E, E^\perp\}$ is positively oriented. We say that a Lorentzian coordinate system $\{x_1, x_2\}$ in $T_x(M)$ is allowable if the vector $(0, 1)$ is a unit future-pointing timelike vector and $(0, 1)^\perp = (1, 0)$.

Let X and Y be two unit timelike vectors which are future-pointing (or past-pointing). The angle from X to Y is defined to be the number u such that

$$\begin{bmatrix} \operatorname{ch} u & \operatorname{sh} u \\ \operatorname{sh} u & \operatorname{ch} u \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

where (x_1, x_2) and (y_1, y_2) are the components of X and Y, respectively, with respect to an allowable coordinate system. The number u is independent of the choice of an allowable coordinate system, as can be easily seen. We shall denote the angle from X to Y by (X, Y). The angle (Y, X) is equal to $-(X, Y)$. We have also $(-X, -Y) = (X, Y)$.

We now want to define the angle (X, Y) in the case where X is a future-

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pointing unit timelike vector and Y a past-pointing unit timelike vector (or vice versa). Note that $-Y$ is future-pointing and if $u=(-Y,X)$, that is,

$$\begin{bmatrix} \operatorname{ch} u & \operatorname{sh} u \\ \operatorname{sh} u & \operatorname{ch} u \end{bmatrix}(-Y)=X$$

in the same sense as before, then

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \operatorname{ch}(-u) & \operatorname{sh}(-u) \\ \operatorname{sh}(-u) & \operatorname{ch}(-u) \end{bmatrix}X=Y.$$

We ignore the matrix $\left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ and define the angle (X,Y) to be $-u$. The angle (X,Y) is defined similarly for past-pointing X and future-pointing Y.

We note the properties of the angle function for unit timelike vectors as follows.

LEMMA 1.
(1) $(X,-X)=0$;
(2) $(X,Y)+(Y,Z)=(X,Z)$;
(3) $(X,X)=0$;
(4) $(Y,X)=-(X,Y)$;
(5) $(-X,Y)=(X,Y)$;
(6) $(X,-Y)=(X,Y)$.

Proof. (1) is obvious. To prove (2), we may consider essentially three cases: (i) X, Y, Z are future-pointing; (ii) X, Y are future-pointing and Z past-pointing; (iii) X, Z are future-pointing and Y past-pointing. The verification in each case is easy. The others follow directly from the definition of angle, but we note also that they follow formally from (1) and (2).

Let us consider a region D bounded by a simple timelike closed polygon Γ in the Lorentzian plane L^2. Assume that Γ consists of successive timelike segments $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ and denote by $X_1, X_2, \ldots, X_k$ the unit timelike vectors on $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$, each with the time-orientation compatible with the direction in which each segment is travelled. If θ_i denotes the angle (X_i, X_{i+1}), $1 \leqslant i \leqslant k$, where $X_{k+1}=X_1$, then

$$\sum_i \theta_i=(X_1,X_2)+(X_2,X_3)+\cdots+(X_{k-1},X_k)+(X_k,X_1)=0$$

by repeated use of Lemma 1(2). By calling θ_i the exterior angles of Γ, we can say that *the sum of exterior angles of a simple closed timelike polygon is zero*—this is a very special case of the Gauss-Bonnett formula we shall prove. The case of a timelike triangle was shown in [2].

2. Geodesic curvature and Gaussian curvature. Let Γ be a smooth timelike curve on a 2-dimensional spacetime M parametrized with proper time s so that the tangent vector $T=T(s)$ at each point is a unit timelike vector. We choose the unit normal vector $T^\perp(s)$ along Γ. The geodesic curvature $k_g(s)$ is defined by $k_g=g(\nabla_s T, T^\perp)$ where ∇_s denotes covariant differentiation along the curve. We have then $\nabla_s T=k_g T^\perp$ and $\nabla_s T^\perp=k_g T$, as well as $k_g=-g(\nabla_s T^\perp, T)$.

Now let Z be a unit timelike vector field parallel along Γ. By assuming that Z at the initial point is future-pointing we know it is so at every point, since parallel displacement preserves time-orientation. It is easy to see that the vector field $Z^{\perp}$ along Γ is also parallel. We consider the angle $\alpha=(T, Z)$, which is a function of s. We prove:

LEMMA 2. $d\alpha/ds=-k_g$.

Proof. If Γ is future-pointing (i.e. T is at every point), then we have

$$T=\operatorname{ch}\alpha Z-\operatorname{sh}\alpha Z^{\perp} \quad \text{and} \quad T^{\perp}=-\operatorname{sh}\alpha Z+\operatorname{ch}\alpha Z^{\perp}.$$

If Γ is past-pointing, then

$$T=-\operatorname{ch}\alpha Z+\operatorname{sh}\alpha Z^{\perp} \quad \text{and} \quad T^{\perp}=\operatorname{sh}\alpha Z-\operatorname{ch}\alpha Z^{\perp}.$$

Based on these equations we get

$$\nabla_s T=(\operatorname{sh}\alpha)(d\alpha/ds)Z-(\operatorname{ch}\alpha)(d\alpha/ds)Z^{\perp}$$

or

$$\nabla_s T=-(\operatorname{sh}\alpha)(d\alpha/ds)Z+(\operatorname{ch}\alpha)(d\alpha/ds)Z^{\perp}.$$

In either case, we get

$$k_g=g(\nabla_s T, T^{\perp})=(-\operatorname{ch}^2\alpha+\operatorname{sh}^2\alpha)(d\alpha/ds)=-d\alpha/ds.$$

Now let X be a future-pointing unit timelike vector field globally defined on M, whose existence is assured by time-orientability of M. The corresponding connection form ω is defined by $\omega(V)=g(\nabla_V X, X^{\perp})$ for any tangent vector V.

LEMMA 3. $d\omega=KdA$, *where K is the Gaussian curvature and dA is the volume element of M.*

Proof. We compute, for any vector fields V and W,

$$\begin{aligned}
d\omega(V, W)&=V\omega(W)-W\omega(V)-\omega([V, W])\\
&=Vg(\nabla_W X, X^{\perp})-Wg(\nabla_V X, X^{\perp})-g(\nabla_{[V,W]}X, X^{\perp})\\
&=g(\nabla_V\nabla_W X, X^{\perp})+g(\nabla_W X, \nabla_V X^{\perp})\\
&\quad -g(\nabla_W\nabla_V X, X^{\perp})-g(\nabla_V X, \nabla_W X^{\perp})-g(\nabla_{[V,W]}X, X^{\perp})\\
&=g(R(V, W)X, X^{\perp})+g(\nabla_W X, \nabla_V X^{\perp})-g(\nabla_V X, \nabla_W X^{\perp}).
\end{aligned}$$

Here $g(\nabla_W X, \nabla_V X^{\perp})=0$, because $\nabla_W X$ is orthogonal to X and hence a scalar multiple of $X^{\perp}$, and on the other hand $\nabla_V X^{\perp}$ is a scalar multiple of X. Similarly, $g(\nabla_V X, \nabla_W X^{\perp})=0$.

Thus $d\omega(V, W)=g(R(V, W)X, X^{\perp})$. Taking $V=X$ and $W=X^{\perp}$, we obtain

$$\begin{aligned}
d\omega(X, X^{\perp})&=g(R(X, X^{\perp})X, X^{\perp})=-g(R(X, X^{\perp})X^{\perp}, X)\\
&=K.
\end{aligned}$$

Since $dA(X, X^{\perp})=1$ in view of our orientation, we get $d\omega=KdA$. □

We consider a domain D with compact closure whose boundary Γ is a simple closed curve consisting of a finite number of smooth timelike curves Γ_i, $1 \leqslant i \leqslant k$, which we orient in such a way that the theorem of Green, $\int_\Gamma \gamma = \iint_D d\gamma$, holds for any 1-form γ. Suppose that Γ_i starts at A_i with initial tangent vector T_i and ends at A_{i+1} with terminal tangent vector S_i, $1 \leqslant i \leqslant k$, where $A_{k+1} = A_1$. Let $\theta_1 = (S_1, T_2)$, $\theta_2 = (S_2, T_3), \ldots, \theta_{k-1} = (S_{k-1}, T_k)$ and $\theta_k = (S_k, T_1)$ be the exterior angles at $A_2, \ldots, A_k$ and A_1.

Now we can state:

THEOREM. *For a domain D and its boundary Γ as above in a 2-dimensional spacetime M, we have*

$$\int_\Gamma k_g\, ds + \sum_i \theta_i - \iint_D K\, dA = 0.$$

3. Proof of the theorem. Let Z_1 be a future-pointing unit timelike vector at A_1. Translate it parallelly along Γ to obtain a vector field $Z = Z(s)$ whose values at $A_2, \ldots, A_k$ are denoted by $Z_2, \ldots, Z_k$. When Γ is travelled back to A_1, we get a vector Z_{k+1}, which is generally different from Z_1. As before, let α be the angle (T, Z) at each point of Γ_i, $1 \leqslant i \leqslant k$. Then

$$\int_{\Gamma_1} (d\alpha/ds)\, ds = (S_1, Z_2) - (T_1, Z_1)$$

so that using Lemma 2 we obtain

$$\begin{aligned} -\int_\Gamma k_g\, ds - \theta_1 &= (S_1, Z_2) - (T_1, Z_1) + (T_2, S_1) \\ &= (T_2, Z_2) - (T_1, Z_1) \end{aligned}$$

by virtue of Lemma 1(2). Similarly,

$$\begin{aligned} -\int_{\Gamma_2} k_g\, ds - \theta_2 &= (T_3, Z_3) - (T_2, Z_2) \\ &\vdots \\ -\int_{\Gamma_k} k_g\, ds - \theta_k &= (T_1, Z_{k+1}) - (T_k, Z_k). \end{aligned}$$

Adding up we obtain

$$(*) \qquad -\int_\Gamma k_g\, ds - \sum_i \theta_i = (T_1, Z_{k+1}) - (T_1, Z_1) = (Z_1, Z_{k+1}).$$

In order to evaluate (Z_1, Z_{k+1}), we use a future-pointing unit timelike vector field X globally defined on M. Let $\beta = (Z, X)$ along Γ. We have

$$Z = \operatorname{ch} \beta X - \operatorname{sh} \beta X^\perp.$$

We get, using the connection form ω based on X,

$$0 = \nabla_s Z = (\operatorname{sh} \beta)\{(d\beta/ds) - \omega(T)\}X + (\operatorname{ch} \beta)\{\omega(T) - (d\beta/ds)\}X^\perp$$

so that $d\beta/ds = \omega(T)$. Thus

$$(Z_{k+1}, Z_1) = \int_\Gamma \omega. \tag{**}$$

By Green's theorem and Lemma 3 we have

$$\int_\Gamma \omega = \int_D d\omega = \iint_D K\,dA. \tag{***}$$

From (*), (**) and (***) we finally obtain the formula in the theorem. □

We conclude this note with the following observations.

As is well known, the classical Gauss-Bonnet formula leads to the global theorem relating the Euler characteristic of a compact 2-manifold M to $\iint_M K\,dA$ defined in terms of an arbitrary Riemannian metric on M. Among the connected, compact, orientable 2-manifolds, the torus T^2 is the only one which admits a Lorentzian metric. Our formula leads to $\iint_{T^2} K\,dA = 0$ for any Lorentzian metric on T^2. For a proof, we may assume that the metric is time-oriented, divide T^2 into appropriate regions bounded by timelike curves and apply the formula to each region and its boundary. The integrals $\int k_g\,ds$ add up to 0, since each "side" is travelled twice with different orientations. The exterior angles at each vertex add up to 0 from the properties of Lorentzian angle. What remains is $\iint_{T^2} K\,dA = 0$.

This conclusion will also follow from the classical theorem by the methods in A. Avez [1] and S. S. Chern [4], which establish the global Gauss-Bonnet theorem for pseudo-Riemannian manifolds for higher dimensions. Some results related to our topic are discussed in [3] and [5].

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