THE EULER-LAGRANGE EQUATIONS FOR EXTREMAL HOLOMORPHIC MAPPINGS OF THE UNIT DISK

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1. Introduction. Extremal problems for holomorphic mappings of the unit disk were considered by various authors ([3], [5], [6]), but only for the class of univalent functions. For a class of mappings to a domain $D \subset \mathbf{C}^n$ these problems were studied only when D is the unit disk. One reason for this limitation, I believe, was an absence of really interesting functionals. However, some years ago, Royden [4] introduced the functional $\|f'(0)\|$ for mappings of the unit disk to a domain $D \subset \mathbf{C}^n$. The supremum of this functional gives us the infinitesimal norm for the Kobayashi metric at a point $z = f(0)$. But extremal mappings in this case are much more interesting because they are invariant under biholomorphic transformations and, hence, are connected with some invariants. In particular, their boundary values may coincide with so called Moser chains [1]. But for a proof of the last conjecture we should know, at least, that boundary values lie on the boundary of the domain. We prove here this property for large classes of functionals and domains.

The standard tool for the study and computation of extremals in the calculus of variations are the Euler-Lagrange equations. In our paper, we deduce them in the case of pseudoconvex domains. In the last section we show how these equations help to find extremals for some types of domains. The author hopes that further studies in the complex calculus of variations will give us a clearer understanding of biholomorphic invariants.

Similar results were proved by different methods by Lempert [2] for Royden's functional and strongly linear convex domains of class C^∞.

2. Notations and preliminary results. Let $\Delta_r = \{\zeta \in \mathbf{C} : |\zeta| < r\}$ be the disk of radius r on the complex plane and $\Delta = \Delta_1$. As usual we shall denote by H or H^p the spaces of all holomorphic functions or of those whose boundary values lie in L^p. We define H_n, H_n^p, L_n^p as a direct sum of n copies of H, H^p, L^p. If $D \subset \mathbf{C}^n$, then $H(\Delta, D)$ is the set of all holomorphic mappings of Δ to D. We denote by A the subspace of H, consisting of functions continuous up to the boundary; A_n means a direct sum of n copies of A.

In addition, we shall use the following notations: $S_r = \partial \Delta_r$, $S = S_1$; if $f = (f_1, \ldots, f_r)$, $h = (h_1, \ldots, h_n)$, then $(f, h) = \Sigma f_j h_j$, $|f| = \Sigma |f_j|$; $\rho(A, B)$ is the distance between sets A and B; $\bar{A}$ means the closure of A and if u is a function, then

$$\Delta u = \left(\frac{\partial u}{\partial z_1}, \frac{\partial u}{\partial z_2}, \ldots, \frac{\partial u}{\partial z_n} \right).$$

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For $f \in H_n^p$ the notation $\|f\|_p$ means the norm f in L_n^p and $\|f\|_{p,r}$ is the norm of $f \mid S_r$ in L_n^p.

In this paper we shall consider real-valued functionals Φ on H_n, satisfying the following conditions:

(A) Φ is differentiable at each point $f \in H_n$, i.e., there is a unique linear continuous functional $\Phi'(f,g): H_n \to \mathbf{R}$ such that $\Phi(f+g)=\Phi(f)+\Phi'(f,g)+r(f,g)$, where $|r(f,h)|\cdot\|h\|_{1,r}^{-1} \to 0$ when $\|h\|_{1,r} \to 0$ and r is greater than some $r_0<1$.

(B) $\Phi'(f,h) = \operatorname{Re}\sum_{j=1}^{n} \int_S \omega_j h_j \, d\theta$ where ω_j are holomorphic on $\mathbf{C}\setminus\bar{\Delta}_r$, $r<1$.

Evidently these functionals are continuous on H_n.

Now we can formulate our variational problem (P): Suppose we are given a bounded domain $D \subset \mathbf{C}^n$, functionals Φ_j on H_n, $0 \leqslant j \leqslant N$, and real numbers a_j, $1 \leqslant j \leqslant N$. We want to find $f_0 \in H(\Delta, D)$ such that $\Phi_0(f_0) \geqslant \Phi_0(f)$ among all f satisfying the following restrictions:

(1) $\Phi_i(f)=a_i$ $(1 \leqslant i \leqslant N)$,

(2) $f \in H(\Delta, D)$.

A solution of problem (P) is called an extremal mapping for (P) or, simply, an extremal. Sometimes this problem has no solution.

EXAMPLE 1. Let $\Phi_0(f)=|f(0)|^2$. Then the problem has no solution.

The next theorem gives us sufficient conditions for the existence of a solution.

THEOREM 1. *Let us suppose that in problem* (P), *the functionals* Φ_i *are continuous with respect to the convergence of holomorphic functions on compact sets, and* $D=D'\setminus P$, *where* $D'=\{z \in \mathbf{C}^n : u(z)<0\}$ *is plurisubharmonic in a neighborhood of* D' *and* P *is an analytic subset of* D'. *If for any* f *satisfying the restrictions of problem* (P) *there is* $\zeta \in K \subset\subset \Delta$ *(where* K *does not depend on* f*) such that* $f(\zeta) \subset D_1 \subset\subset D$ *then a solution of* (P) *exists.*

Proof. If we denote by A the set of all f satisfying the restrictions of problem (P), and if $\Omega=\sup\Phi_0(f)$, $f \in A$, then we can choose a sequence $\{f_k\}$, $f_k \in A$, such that $\Omega=\lim\Phi_0(f_k)$. We can assume that $f_k \to f_0$ uniformly on compact sets. Therefore, $\Phi_0(f_0)=\Omega$, $\Phi_j(f_0)=a_j$, $1 \leqslant j \leqslant N$, and $f_0(\zeta_1) \in D_1$ for some $\zeta_1 \in K$. It is clear that $f_0(\zeta) \in \bar{D}$ for any $\zeta \in \Delta$. But if $f_0(\zeta_2) \in \partial D'$, then the function $u_1(\zeta)=u(f_0(\zeta))$ is a non-positive subharmonic function on Δ and $u_1(\zeta_2)=0$. Hence, by the maximum principle it follows that $u \equiv 0$ and $f_0(\zeta) \in \partial D'$ for any $\zeta \in \Delta$. But this contradicts the fact that $f_0(\zeta_1) \in D$ and, therefore, $f_0(\zeta) \in D$ for any $\zeta \in \Delta$. The same reasoning shows that $f_0(\zeta)$ cannot belong to P. □

If $f \in L_n^\infty$, then we define $\|f\|$ as $\max_j \sup_{\zeta \in S}|f_j(\zeta)|$ and, if $f \in L_n^1$, as $\int_S \sum |f_j| \, d\theta$.

The next lemma is well known.

LEMMA 1. *Let* $F(f)=\operatorname{Re}\int_S (f,\omega)\,d\theta$ *be a linear functional on* H_n^∞ *and* $\omega(\theta) \in L_n^1$. *Then there is an extension* Φ *of* F *on* L_n^∞ *such that*

$$\Phi(h)=\operatorname{Re}\int_S (f,\omega+g)\,d\theta,$$

$$\|F\|=\|\Phi\|, \quad g \in H_n^1, \quad g(0)=0$$

and

$$\|\omega+g\| \leqslant \|\omega+g_1\|$$

for any $g_1 \in H_n^1$, $g_1(0)=0$.

Later we shall prove that extremals are almost proper mappings, i.e., boundary values lie on the boundary of the domain for almost all points of S. In Section 5 we prove it for our functionals and domains with C^1 boundary. But if the functionals are bad, it may not be true, as Example 2 shows.

EXAMPLE 2. If $D=\Delta$ and $f_0: \Delta \to \Delta$ is a conformal mapping such that $|f|=1$ on some set $E \subset S$ and $|f|<1$ on $S \setminus E$, then we define

$$\omega(e^{i\theta}) = \begin{cases} \overline{f_0}(e^{i\theta}), & e^{i\theta} \in E \\ 0, & e^{i\theta} \notin E. \end{cases}$$

It is clear that the problem (P) for such D and $\Phi_0(f)=\int f\omega\, d\theta$ has a solution f_0, but $f_0(\zeta) \notin S$ on $S \setminus E$.

3. The variational lemma. In this section we shall construct a set of admissible variations of an extremal mapping.

Let $P \subset S$ be a measurable set with Lebesgue measure $m(P)>0$. We define a function $\lambda(\theta, \epsilon)$ as follows

$$\lambda(\theta, \epsilon) = \begin{cases} 1, & e^{i\theta} \notin P \\ \epsilon^{-1}, & e^{i\theta} \in P \end{cases}$$

for any $\theta \in [0, 2\pi]$ and $\epsilon>0$. We define a new norm on H_n^∞

$$\|f\|_\epsilon = \operatorname*{ess\ sup}_{\theta \in [0,2\pi]} \left\{ \max_{1 \leqslant j \leqslant n} \lambda(\theta, \epsilon) |f_j|(\theta) \right\}$$

and let $B_{\epsilon,P}(r)$ be a ball of radius r with respect to this norm in H_n^∞. It is clear that the new norm is equivalent to the standard one.

LEMMA 2. *Suppose that* $\Phi_j(f) = \operatorname{Re} \int (f, \omega_j)\, d\theta$, $\omega_j \in L_n^1$, $0 \leqslant j \leqslant N$, *are linear functionals on* H_n^∞. *Then for*

$$\|\Phi_0\|_\epsilon = \sup\{|\Phi_0(f)| : \|f\|_\epsilon \leqslant 1,\ \Phi_j(f)=0,\ 1 \leqslant j \leqslant N\}$$

we have

$$\varliminf_{\epsilon \to 0} \|\Phi_0\|_\epsilon / \epsilon < \infty$$

if and only if there are real numbers λ_k, $1 \leqslant k \leqslant N$, *and* $g \in H_n^1$, $g(0)=0$ *such that* $\omega_0 - \sum_{k=1}^N \lambda_k \omega_k = g$ *on* $S \setminus P$.

Proof. We introduce the following notations:

$$\tilde{H}_n^\infty = \{f \in H_n^\infty : \Phi_j(f)=0,\ 1 \leqslant j \leqslant N\}, \qquad \tilde{B}_{\epsilon,P}(r) = B_{\epsilon,P}(r) \cap \tilde{H}_n^\infty.$$

At first let us prove that our condition is sufficient. If $\omega_0 - \sum_{k=1}^N \lambda_k \omega_k = g$ on $S \setminus P$, then for any $f \in \tilde{B}_{\epsilon,P}(1)$,

$$\Phi_0(f)=\Phi_0(f)-\sum_{k=1}^{N}\lambda_k\Phi_k(f)=\operatorname{Re}\int_S(f,\omega_0-\sum\lambda_k\omega_k)\,d\theta$$

$$=\operatorname{Re}\int_S(f,g)\,d\theta+\operatorname{Re}\int_P(f,\omega_0-\sum\lambda_k\omega_k-g)\,d\theta$$

and, since $\|f\|\leqslant\epsilon$ on P,

$$|\Phi_0(f)|\leqslant\epsilon\int_P|\omega_0-\sum\lambda_k\omega_k-g|\,d\theta\leqslant C\epsilon.$$

Hence, $\|\Phi_0(f)\|\leqslant C\epsilon$ and our statement is proved.

To prove the necessity of our condition, we introduce the analytic function g defined by the formula

$$g(z)=-(2\pi)^{-1}\int_P\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta.$$

Then

$$|e^{\alpha g}(\zeta)|=\begin{cases}e^{-\alpha}, & \zeta\in P\\ 1, & \zeta\notin P.\end{cases}\tag{1}$$

The last equality is true only almost everywhere, and the left side of (1) must be considered as a boundary value. In our paper all functions on S are defined almost everywhere and in the future we shall not recall it specifically.

We define the new functional $\tilde{\Phi}_\alpha(f)=\operatorname{Re}\int_S(f,\omega_0)e^{\alpha g}\,d\theta$ on the space $\tilde{H}=\{f\in H_n^\infty : fe^{\alpha g}\in\tilde{H}_n^\infty\}$. Let us extend $\tilde{\Phi}_\alpha$ on H_n^∞, conserving its standard norm. It is clear that

$$\hat{\Phi}_\alpha(f)=\operatorname{Re}\int\left(f,\omega_0-\sum_{k=1}^{N}\lambda_{k,\alpha}\omega_k\right)e^{\alpha g}\,d\theta$$

furnishes such an extension.

By Lemma 1 it follows that an extension Φ_α^1 of $\hat{\Phi}_\alpha$ on L_n^∞, conserving its norm, can be given by the formula

$$\Phi_\alpha^1(f)=\operatorname{Re}\int_S(f,\omega_0-\sum\lambda_{k,\alpha}\omega_k-g_\alpha)e^{\alpha g}\,d\theta$$

where $g_\alpha\in H_n^1$, $g_\alpha(\theta)=0$.

Since $\underline{\lim}_{\epsilon\to 0}\|\Phi_0\|_\epsilon\cdot\epsilon^{-1}<\infty$, there is a sequence $\{\epsilon_j\}$, $\epsilon_j\to 0$, such that $\|\Phi_0\|_{\epsilon_j}\leqslant K\epsilon_j$. But, if $\|\Phi_\alpha^1\|=\lim_{k\to\infty}|\Phi_\alpha^1(f_{\alpha k})|$, where $f_{\alpha k}\in B_{1,P}(1)\cap\tilde{H}$, then $e^{\alpha g}f_{\alpha k}\in\tilde{B}_{\epsilon,P}(1)$ for $\epsilon=e^{-\alpha}$, and hence, $\|\Phi_{\alpha_j}^1\|\leqslant\|\Phi_0\|_{\epsilon_j}\leqslant K\epsilon_j$ for $\alpha_j=-\ln\epsilon_j$. Therefore $\|e^{\alpha_j}\Phi_{\alpha_j}^1\|\leqslant K$. But

$$\begin{aligned}\|e^{\alpha_j}\Phi_{\alpha_j}^1\|&=\int_S|e^{\alpha_j(g+1)}|\cdot|\omega_0-\sum\lambda_{k,\alpha_j}\omega_k-g_{\alpha_j}|\,d\theta\\&=\int_P|\omega_0-\sum\lambda_{k,\alpha_j}\omega_k-g_{\alpha_j}|\,d\theta+e^{\alpha_j}\int_{S\setminus P}|\omega_0-\sum\lambda_{k,\alpha_j}\omega_k-g_{\alpha_j}|\,d\theta\end{aligned}\tag{2}$$

and we see that

$$(3)\qquad \int_{S\setminus P} |\omega_0 - \sum \lambda_{k,\alpha_j}\omega_k - g_{\alpha_j}|\, d\theta \to 0 \quad \text{as } j\to\infty.$$

We can assume that the functionals Φ_j, $j\geqslant 1$, are linearly independent on H_n^∞ and that $|\lambda_{1,\alpha_j}|\geqslant|\lambda_{k,\alpha_j}|$ for some subsequence of $\{\alpha_j\}$. Under the last assumption we can choose a new subsequence of $\{\alpha_j\}$ such that

$$\mu_{k,\alpha_j} = \lambda_{k,\alpha_j}\cdot\lambda_{1,\alpha_j}^{-1} \to \mu_k.$$

If $\lambda_{1,\alpha_j}\to\infty$ for this subsequence, then by (2)

$$\int_S |\omega_0\lambda_{1,\alpha_j}^{-1} - \sum \mu_{k,\alpha_j}\omega_k - g_{\alpha_j}\lambda_{1,\alpha_j}^{-1}|\, d\theta \leqslant K\lambda_{1,\alpha_j}^{-1}$$

and therefore

$$\int_S |\sum \mu_k\omega_k - g_{\alpha_j}\cdot\lambda_{1,\alpha_j}^{-1}|\, d\theta \to 0.$$

Since H_n^1 is closed in L_n^1, $\sum \mu_k\omega_k \in H_n^1$, and the functionals $\Phi_j, j\geqslant 1$, are linearly dependent, which contradicts our assumption.

Therefore, $|\lambda_{k,\alpha_j}|\leqslant M<\infty$, and we can choose some subsequence of $\{\alpha_j\}$ for which $\lambda_{k,\alpha_j}\to\lambda_k$. If $h=\omega_0-\sum\lambda_k\omega_k$, then by (2)

$$\int_S |h-g_{\alpha_j}|\, d\theta \leqslant K + \int_S |\sum (\lambda_{k,\alpha_j}-\lambda_k)\omega_k|\, d\theta \leqslant 2K$$

for sufficiently large j, and we see that

$$(4)\qquad \int_S |g_{\alpha_j}|\, d\theta \leqslant 2K + \int_S |g_0| \leqslant K_1.$$

This means that $g_{\alpha_j}\to h_1$ uniformly on compact subsets of Δ for some subsequence of $\{\alpha_j\}$. But, by (3), $\int_{S\setminus P}|g-g_{\alpha_j}|\, d\theta\to 0$ and it follows by Khinchin-Ostrovsky's theorem that $h=h_1$ on $S\setminus P$, which proves our lemma. □

REMARK. It would be interesting to change the inequality

$$\varliminf_{\epsilon\to 0} \|\Phi_0\|_\epsilon\cdot\epsilon^{-1} < \infty$$

in Lemma 2 to

$$\varliminf_{\epsilon\to 0} \|\Phi_0\|_\epsilon\cdot\exp(-(2\pi)^{-1}\operatorname{mes} P\cdot\ln\epsilon) < \infty.$$

For example, if $\Phi_0(f)=\operatorname{Re} f(0)$, the second inequality follows by the two constants theorem, and obvious calculations show that it is sharp.

We shall call linear functionals $F_k(f)=\operatorname{Re}\int (f,\omega_k)\, d\theta$ linearly independent on $P\subset S$ if $\sum\lambda_k\omega_k = g$ on P, $g\in H_n^1$ when and only when $g\equiv 0$ and $\lambda_k=0$ for all k.

A VARIATIONAL LEMMA. *Suppose that the linear functionals $F_k(f)=\operatorname{Re}\int (f,\omega_k)\, d\theta$, $0\leqslant k\leqslant N$, are linearly independent on $P\subset S$ and let $C_k(\epsilon)$, $1\leqslant k\leqslant$*

N be functions on $\mathbf{R}^+$, $|C_k(\epsilon)| \leqslant K$. Then for sufficiently small $\epsilon > 0$ there are functions $f_\epsilon \in B_{\epsilon, P}(1)$ such that $F_k(f_\epsilon) = C_k(\epsilon) \cdot \epsilon$, $k \geqslant 1$, and $\underline{\lim}_{\epsilon \to 0} |F_0(f_\epsilon)| \cdot \epsilon^{-1} = \infty$.

Proof. Since F_k, $0 \leqslant k \leqslant N$ are linearly independent on P, by Lemma 2, it follows that for any $k \geqslant 1$ and sufficiently small $\epsilon > 0$ there are functions $f_{k,\epsilon} \in B_{\epsilon, P}(1)$ such that
(1) $F_k(f_{k,\epsilon}) = (N+1) C_k(\epsilon) \epsilon$;
(2) $F_j(f_{k,\epsilon}) = 0$ for $j \neq k$, $j \geqslant 0$.
For $k = 0$ we can find, by the same Lemma, functions $f_{0,\epsilon} \in B_{\epsilon, P}(1)$ such that
(1) $\underline{\lim}_{\epsilon \to 0} |F_0(f_{0,\epsilon})| \cdot \epsilon^{-1} = \infty$;
(2) $F_k(f_{0,\epsilon}) = 0$ for $k \geqslant 1$.
Let us take $f_\epsilon = (N+1)^{-1} \sum_{k=0}^N f_{k,\epsilon}$. Then
(1) $F_k(f_\epsilon) = C_k(\epsilon) \cdot \epsilon$ for $k \geqslant 1$;
(2) $\underline{\lim}_{\epsilon \to 0} |F_0(f_\epsilon)| \cdot \epsilon^{-1} = \infty$; and
(3) $f_\epsilon \in B_{\epsilon, P}(1)$.
The variational lemma is proved. □

4. Extremals of linear functionals for holomorphic mappings in ρ-pseudoconvex domains. In this section we shall prove that extremals of linear functionals are almost proper when the domain $D \subset \mathbf{C}^n$ is ρ-pseudoconvex.

A domain $D \subset \mathbf{C}^n$ is called ρ-pseudoconvex if there is a plurisubharmonic function $u \in C^0(\bar{D})$, such that $u|_{\partial D} = 0$, $u < 0$ on D and for some $\gamma > 0$, $\rho(z, \partial D) \geqslant \gamma |u(z)|$. Besides strongly pseudoconvex domains (which obviously are ρ-pseudoconvex) this class contains, e.g., analytic polyhedra, since for an analytic polyhedron $P = \{z \in \mathbf{C}^n : |f_j(z)| < 1\}$ we can take $u(z) = \max_j \{|f_j|^2 - 1\}$. Therefore this class is sufficiently large.

Condition B of Section 2 suggests considering linear functionals $F(f)$, represented as $\operatorname{Re} \int (f, \omega) \, d\theta$, where $\omega = (\omega_1, \dots, \omega_n)$, ω_j are holomorphic on $\mathbf{C} \setminus \bar{\Delta}_r$, $r < 1$.

We need the following lemmas.

LEMMA 3. *If $\omega \in H(\mathbf{C} \setminus \bar{\Delta}_r)$, $r < 1$ and $\omega = f$, $f \in H^1(\Delta)$, on some set $E \subset S$ with positive measure then ω extends analytically on Δ.*

It follows by this lemma that if functionals

$$F_k(f) = \operatorname{Re} \int (f, \omega_k) \, d\theta, \quad k = 1, \dots, N,$$

satisfying Condition B of Section 2, are linearly independent on S then they are linearly independent on each set P with positive measure.

The next lemma is well known.

LEMMA 4. *Suppose that u is a negative subharmonic function in Δ. Then $u(z) \leqslant -C(1 - |z|)$, where $C > 0$ does not depend on z.*

LEMMA 5. *Suppose that $f \in H_n^\infty$ and $\omega \in H(\mathbf{C} \setminus \bar{\Delta}_r)$, $r < 1$. Then, for $1 > t > (1+r)/2$,*

$$\left|\int (f,\omega)\,d\theta-\int (f_t,\omega)\,d\theta\right|\leqslant C(1-t),$$

where $f_t(z)=f(tz)$ *and* C *depends only on* $\|f\|_\infty$ *and* ω.

Proof. Since (ω, f) is holomorphic, when $|z|>r$, then

$$\int_S (f,\omega)\,d\theta=\int_S (f_t,\omega_t)\,d\theta.$$

Therefore,

$$\left|\int_S (f,\omega)\,d\theta-\int_S (f_t,\omega)\,d\theta\right|=\left|\int_S (f_t,\omega_t-\omega)\,d\theta\right|$$

$$\leqslant\|f\|_\infty\cdot\int_S |\omega_t-\omega|\,d\theta\leqslant C(1-t).$$

Now we can prove

THEOREM 2. *Let* Φ_k, $0\leqslant k\leqslant N$, *be linear functionals on* H_n^∞, *satisfying conditions* A *and* B *of Section 2. If* $f_0\colon\Delta\to\mathbf{C}^n$ *is an extremal for the variational problem* (P), *where* D *is* ρ*-pseudoconvex, then* f_0 *is almost proper.*

Proof. Suppose that $f_0(e^{i\theta})\in K\subset\subset D$ for $e^{i\theta}\in P\subset S$, $\operatorname{mes}P>0$. Then, by Egorov's theorem, there is a $P_1\subset P$, $\operatorname{mes}P_1>0$ and $r_1<1$ such that $f_0(re^{i\theta})\in K_1\subset\subset D$ for $e^{i\theta}\in P_1$ and $r>r_1$. Since $u(f_0(\zeta))<0$ on Δ, then by Lemma 4, $|u(f_0(\zeta))|\geqslant C_1(1-|\zeta|)$, and, by the definition of ρ-pseudoconvex domains, $\rho(f_0(\zeta),\partial D)\geqslant C(1-r)$. If $f_{0r}(\zeta)=f_0(r\zeta)$ and $a_{kr}=\Phi_k(f_0-f_{0r})$, then by Lemma 5 $|a_{kr}|\leqslant C_2(1-r)$.

Let us consider the ball $B_{\epsilon,P_1}(C_3)$ where $\epsilon=C(1-r)$ and $C_3=\rho(K_1,\partial D)$. For any function $f\in B_{\epsilon,P_1}(C_3)$ the function $h=f_{0r}+f$ maps Δ to D, since $h(e^{i\theta})\in D$ for almost all $e^{i\theta}\in S$. By the variational lemma we can choose a function $g_r\in B_{\epsilon,P_1}(C_3)$, such that

$$\Phi_k(g_r)=a_{kr},\quad 1\leqslant k\leqslant N,\qquad\text{and}\qquad \varlimsup_{r\to1}|\Phi_0(g_r)|(1-r)^{-1}=\infty.$$

Then the functions $f_r=f_{0r}+g_r$ satisfy the conditions of our variational problem and, evidently, $\Phi_0(f_r)>\Phi_0(f_0)$ for some r. This contradiction proves our lemma. □

5. The extremal principle and the Euler-Lagrange equations. In this section we shall study the case of domains with boundaries of class C^1. Let us suppose that $D=\{z\in\mathbf{C}^n\colon u(z)<0\}$, where u is plurisubharmonic in $\bar{D}$, $u\in C^1(\bar{D})$ and $\nabla u\neq0$ on ∂D. If f_0 is some extremal for our problem (P), then we shall denote by F_k the first derivative of Φ_k at f_0 and by p the real function on L_n^1, defined as follows

$$p(h)=\int_A (\operatorname{Re}(\nabla u(f_0),h))^+\,d\theta$$

where the notation φ^+ means the maximum number from φ and 0; A is the set of $\zeta\in S$, for which $f(\zeta)\in\partial D$. Evidently $p(h)\geqslant 0$, $p(\alpha h)=\alpha p(h)$, $\alpha\geqslant 0$, and $p(h+g)\leqslant p(h)+p(g)$.

The next lemma is basic for our studies.

LEMMA 6. *With notations as above for problem* (P) *there are* j, $0\leqslant j\leqslant N$, $T>0$ *and* δ_j *equal to* 1 *or* -1, *when* $j\geqslant 1$, $\delta_0=1$, *such that* $\delta_j F_j(h)\leqslant Tp(h)$ *for any* $h\in X_j=\{h\in H_n^1: F_l(h)=0,\ l\neq j\}$.

Proof. Let us suppose that our lemma is not true. Then for each j, $0\leqslant j\leqslant N$, and $m\in\mathbf{Z}^+$ there are $h_{0m}^+\in X_0\cap A_n$, $h_{jm}^\pm\in X_j\cap A_n$, $j\geqslant 1$, such that

$$F_j(h_{jm}^+)\geqslant mp(h_{jm}^+),\qquad -F_j(h_{jm}^-)\geqslant mp(h_{jm}^-)$$

and

$$F_j(h_{jm}^+)=1,\qquad F_j(h_{jm}^-)=-1$$

For any $q=(q_0^+,q_1^+,q_1^-,\dots,q_N^+,q_N^-)\in\mathbf{R}_+^{2N+1}$ we define the function

$$f_{qm}(z)=f_0+q_0^+h_{0m}^++\sum_{j=1}^N(q_j^+h_{jm}^+-q_j^-h_{jm}^-)=f_0+h_{qm}$$

and the linear mapping of $\mathbf{R}_+^{2N+1}$ to $\mathbf{R}^{N+1}$, $A(q)=(q_0^+,q_1^+-q_1^-,\dots,q_N^+-q_N^-)$. Let us consider a domain $D_1, D\subset\subset D_1$, such that $u\in C^1(\bar D_1)$ and u is plurisubharmonic in D_1. If $z\in\bar D$, $z+w\in D_1$ then

$$u(z+w)=u(z)+2\operatorname{Re}(\nabla u(z),w)+v(z,w). \tag{5}$$

It is easy to see that

$$v(x)=\sup\{|v(z,w)|: z\in\bar D,\ \|w\|\leqslant x\}=\bar o(x).$$

LEMMA 7. *Suppose that u is a non-positive subharmonic function in* Δ *and* Δu *is the Riesz measure of u. If for some compact* $K\subset\Delta$, $\Delta u(K)>a>0$ *or for some set* $P\subset S$, *the upper radial limits of u at* $\zeta\in P$ *do not exceed* $-a<0$, *then* $u(\zeta)\leqslant -C(1-|\zeta|)$, *where* $C>0$ *depends only on K, a and P.*

Proof. Denoting by $\varphi(\zeta)$ the value of the upper radial limit at $\zeta\in S$, we have the obvious inequality

$$u(z)\leqslant\operatorname{Re}\int_S\frac{\zeta+z}{\zeta-z}\varphi(\zeta)\,d\zeta+\iint_\Delta\ln\left|\frac{z-\zeta}{1-\bar z\zeta}\right|\Delta u_\zeta.$$

If $\Delta u(K)>a>0$ for some $K\subset\Delta_r$, $r<1$, then

$$u(z)\leqslant\iint_\Delta\ln\left|\frac{z-\zeta}{1-\bar z\zeta}\right|\Delta u_\zeta \tag{5'}$$

and since, if $|\zeta|<r$

$$\left|\frac{z-\zeta}{1-\bar z\zeta}\right|\leqslant\frac{|z|+r}{1+r|z|}\leqslant 1-\frac{(1-r)}{2}(1-|z|),$$

then

$$\ln\left|\frac{z-\zeta}{1-\bar{z}\zeta}\right| \leqslant -C_1(1-|z|),$$

where $C_1>0$ depends only on r, and, by (5′), $u(z)\leqslant -C_1 a(1-|z|)\leqslant -C(1-|z|)$. If $\varphi(\zeta)\leqslant -a<0$ on $P\subset S$, then

$$u(z)\leqslant \operatorname{Re}\int_S \frac{\zeta+z}{\zeta-z}\varphi(\zeta)\,d\zeta \leqslant -a\operatorname{Re}\int_P \frac{\zeta+z}{\zeta-z}\,d\zeta \leqslant -C(1-|z|)$$

where $C>0$ depends only on a and P. The lemma is proved. □

In the future we shall denote by P the subset of S where radial limits of $u_0(\zeta)=u(f_0(\zeta))$ are negative. Obviously, $A=S\backslash P$.

STATEMENT 1. *If* $\operatorname{mes} P=0$, *then there is* $K\subset\subset\Delta$ *such that for* $u_0(\zeta)$, $\Delta u_0(K)>a>0$ *and for each* $m\in\mathbf{Z}^+$ *there is* $t_m>0$ *such that for* $u_{qm}(\zeta)=u(f_{qm}(\zeta))$, $\Delta u_{qm}(K)>a/2>0$ *when* $\|q\|\leqslant t_m$.

Proof. The first part of the statement follows by the fact that $u_0\equiv 0$ a.e. on S and $|u_0|<M$. The second part is trivial.

Now we fix some $\epsilon>0$ and denote by $P_\epsilon\subset P$ the set of all $\zeta\in S$ where radial limits of u_0 don't exceed $-\epsilon$. For each $m\in\mathbf{Z}^+$ we can find $t_m>0$ such that, when $\|q\|\leqslant t_m$,

(1) $f_{qm}(\zeta)\in D_1$, $\zeta\in\Delta$;

(2) $u_{qm}(\zeta)\leqslant -\epsilon/2$, $\zeta\in P_\epsilon$, or, if $\operatorname{mes}P=0$, $\Delta u_{qm}(K)<-a<0$ for some $K\subset\subset\Delta$.

Therefore, by Lemma 7, when $|\zeta|>2^{-1}$,

$$u_{qm}(\zeta)\leqslant v_{qm}(\zeta)=C\ln|\zeta|+\int_{S\backslash P}\Psi_{qm}^{+}P(\zeta,\theta)\,d\theta$$

where $P(\zeta,\theta)$ is the Poisson kernel, and

$$\Psi_{qm}(\zeta)=2\operatorname{Re}(\nabla u(f_0(\zeta)),h_{qm}(\zeta))+v(f_0(\zeta),h_{qm}(\zeta)).$$

Let us introduce some new notations: $\Delta_{qm}=\{\zeta\in\Delta\colon v_{qm}(\zeta)<0\}$ and

$$g_{qm}(\zeta)=\zeta\cdot\exp\left\{C^{-1}\int_{S\backslash P}\Psi_{qm}^{+}S(\zeta,\theta)\,d\theta\right\}.$$

Here $S(\zeta,\theta)$ is the Schwartz kernel.

STATEMENT 2. (a) Δ_{qm} *is connected.* (b) g_{qm} *maps* Δ_{qm} *conformally onto* Δ.

Proof. (a) Since v_{qm} is harmonic outside of 0 and $v_{qm}(e^{i\theta})\geqslant 0$ then any connected component of Δ_{qm} must contain 0.

(b) At first, we prove that g_{qm} maps Δ onto Δ. Actually, if $\zeta\notin\operatorname{Im}g_{qm}$ we may take the curve $\gamma(t)=t\zeta_0$, $t\in[0,1]$, and its lifting $\gamma^*(t)$, i.e., a curve such that $g_{qm}(\gamma^*(t))=\gamma(t)$. Since $|g_{qm}(\zeta)|\geqslant|\zeta|$, the curve $\gamma^*(t)$ must lie in the disk of radius $1-|\zeta_0|$. Therefore, the lifting exists and $\zeta_0=g_{qm}(\gamma^*(1))$. Since $v_{qm}(\zeta)=$

$\ln|g_{qm}(\zeta)|$ we see that Δ_{qm} is the preimage of Δ and g_{qm} maps Δ_{qm} onto Δ. To prove that g_{qm} is one-to-one on Δ_{qm} we may note that any point $\zeta \in \Delta$ is assumed as often as zero and the last point is assumed only once. □

We define $g_{qm}^{-1}: \Delta \to \Delta_{qm}$ as the inverse function for g_{qm}. In the future we shall need the following notations:

$$\tilde{f}_{qm}(\zeta) = f_{qm}(g_{qm}^{-1}(\zeta))$$

$$A_m(q) = (F_0(\tilde{f}_{qm}) - F_0(f_0), \ldots, F_N(\tilde{f}_{qm}) - F_N(f_0))$$

$$\tilde{A}_m(q) = (\Phi_0(\tilde{f}_{qm}) - \Phi_0(f_0), \ldots, \Phi_N(\tilde{f}_{qm}) - \Phi_N(f_0)).$$

Note that $\tilde{f}_{qm}(\zeta) \in D$ for $\zeta \in \Delta$ and $A_m(0) = \tilde{A}_m(0) = 0$.

STATEMENT 3. *The mappings A_m and $\tilde{A}_m$ are continuous in q when $\|q\| < t_m$.*

Proof. If $q_k \to q$ then $\Psi_{q_k m} \to \Psi_{qm}$ uniformly on S. Hence, $g_{q_k m} \to g_{qm}$ uniformly on compact sets of Δ. It is evident after the last assertion that $g_{q_k m}^{-1} \to g_{qm}^{-1}$ and $\tilde{f}_{q_k m} \to \tilde{f}_{qm}$ uniformly on compact sets, too. Since Φ_j and F_j are continuous with respect to this convergence, we obtain our statement. □

STATEMENT 4. *For each $m \in \mathbf{Z}^+$ and $b > 0$ there is $q_m^1 > 0$ such that*

$$\|A_m(q) - \tilde{A}_m(q)\| \leqslant b\|q\|$$

when $\|q\| < q_m^1$.

Proof. Let us denote by C_m the maximum of $\|h_{jm}^{\pm}\|_\infty$. Then

$$\|h_{qm}\|_\infty \leqslant (2N+1)C_m\|q\| \quad \text{and} \quad \|\Psi_{qm}\|_\infty \leqslant K_m\|q\| \tag{6}$$

when $\|q\|$ is sufficiently small. Since Φ_j satisfies condition A, there is $r_j < 1$ such that

$$\Phi_j(\tilde{f}_{qm}) - \Phi_j(f_0) = F_j(\tilde{f}_{qm}) - F_j(f_0) + G(f_0, \tilde{f}_{qm})$$

and, for any $\delta > 0$

$$|G(f_0, \tilde{f}_{qm})| \leqslant \delta \|\tilde{f}_{qm} - f_0\|_{1, r_j} \tag{7}$$

if $\|\tilde{f}_{qm} - f_0\|_{1, r_j}$ is sufficiently small.

To prove our statement, we need to estimate the last norm. Since

$$\tilde{f}_{qm} - f_0 = f_0(g_{qm}^{-1}) + h_{qm}(g_{qm}^{-1}) - f_0$$

then, by (6),

$$\|\tilde{f}_{qm} - f_0\|_{1, r_j} \leqslant (2N+1)C_m\|q\| + \|f_0(g_{qm}^{-1}) - f_0\|_{1, r_j}. \tag{8}$$

But $|g_{qm}(\zeta)| \geqslant |\zeta|$, and therefore, $g_{qm}^{-1}(\Delta_{r_j}) \subset \Delta_{r_j}$ and, letting $\xi = g_{qm}^{-1}(\zeta)$, we see that it is sufficient to estimate $|f_0(\xi) - f_0(g_{qm}(\xi))|$ on Δ_{r_j}. Obviously, this modulus does not exceed $L_m|\xi - g_{qm}(\xi)|$ where $L_m = \max\{|f_0'(\zeta)| : \zeta \in \bar{\Delta}_{r_j}\}$ and the second modulus does not exceed

$$L_m\left|1-\exp\left\{C^{-1}\int_{S\setminus P}\Psi^+_{qm}S(\xi,\theta)\,d\theta\right\}\right|.$$

Using (6), we can easily obtain

(9) $$|f_0(\xi)-f_0(g_{qm}(\xi))|\leqslant H_{mj}\|q\|,\quad \xi\in\bar{\Delta}_{r_j},$$

where H_{mj} depends only on m and r_j. Combining (8) and (9), we obtain the needed estimate

(10) $$\|\tilde{f}_{qm}-f_0\|_{1,r_j}\leqslant G_{mj}\|q\|$$

when $\|q\|$ is sufficiently small. After that, the usual argument gives us the proof of Statement 4. □

STATEMENT 5. *For each* $b>0$ *there is* $m\in\mathbf{Z}^+$ *and* $q_m>0$ *such that* $\|A(q)-A_m(q)\|\leqslant b\|q\|$ *when* $\|q\|\leqslant q_m$.

Proof. It follows from the definition of A and A_m that it is sufficient to prove this statement for $|F_j(\tilde{f}_{qm})-F_j(f_{qm})|$. By Condition B, $F_j(h)=\int_S(h,\omega_j)\,d\theta$ where ω_j are homomorphic in $\bar{\mathbf{C}}\setminus\Delta_{r_0}$, $r_0<1$. Let us fix some $1>r>r_0$. As in the proof of Statement 4 we can show that $\gamma_{qm}=g_{qm}(S_r)\subset\Delta\setminus\Delta_{r_0}$ and $\gamma_{qm}\subset\Delta_{qm}$ when $\|q\|$ is less than some positive q'_m. Then

$$F_j(\tilde{f}_{qm})=\operatorname{Re}\int_S\frac{(\omega_j,\tilde{f}_{qm})}{i\zeta}\,d\zeta=\operatorname{Re}\int_{\gamma_{qm}}\frac{(\omega_j,f_{qm}(g_{qm}^{-1}))}{i\zeta}\,d\zeta.$$

If $\xi=g_{qm}^{-1}(\zeta)$ then

(11) $$\begin{aligned}F_j(\tilde{f}_{qm})&=\operatorname{Re}\int_{S_r}(\omega_j(g_{qm}(\xi),f_{qm}(\xi))g'_{qm}(\xi)\,d\xi\\&=\operatorname{Re}\int_{S_r}\frac{(\omega_j(g_{qm}),f_{qm})\,d\xi}{i\xi}+\operatorname{Re}\int_{S_r}\frac{(\omega_j(g_{qm}),f_{qm})\varphi\,d\xi}{i}\end{aligned}$$

where

$$\varphi(\xi)=C^{-1}\int_{S\setminus P}\Psi^+_{qm}S'_\xi(\xi,\theta)\,d\theta.$$

Let us estimate the second integral in the expression for $F_j(\tilde{f}_{qm})$. By Fubini's theorem, it can be rewritten as

$$C^{-1}\operatorname{Im}\int_{S\setminus P}\Psi^+_{qm}\left[\int_{S_r}(\omega_j(g_{qm}),f_{qm})\cdot\frac{2e^{i\theta}}{(e^{i\theta}-\xi)^2}\,d\xi\right]d\theta$$

and, since $|f_{qm}|<B$, its modulus does not exceed

(12) $$\frac{A}{(1-r)^2}\int_{S\setminus P}\Psi^+_{qm}\,d\theta.$$

Now, let us consider the first integral

$$
\begin{aligned}
\operatorname{Re}\int_{S_r} &\frac{(\omega_j(g_{qm}), f_{qm})\,d\xi}{i\xi} \\
&= \operatorname{Re}\int_{S_r} \frac{(\omega_j(g_{qm}), f_0)\,d\xi}{i\xi} + \operatorname{Re}\int_{S_r} \frac{(\omega_j(g_{qm}), h_{qm})\,d\xi}{i\xi} \\
&= F_j(f_0) + F_j(h_{qm}) + \operatorname{Re}\int_{S_r} (\varphi_j, f_0)\,d\theta + \operatorname{Re}\int_{S_r} (\varphi_j, h_{qm})\,d\theta \\
&= F_j(f_{qm}) + \operatorname{Re}\int_{S_r} (\varphi_j, f_{qm})\,d\theta,
\end{aligned}
$$

where $\varphi_j(\xi) = \omega_j(g_{qm}(\xi)) - \omega_j(\xi)$. As in the proof of Statement 4 we can show that

$$
\begin{aligned}
|\varphi_j(\xi)| \leqslant L_m |g_{qm}(\xi) - \xi| &\leqslant L_m \left| 1 - \exp\left\{ C^{-1} \int_{S\setminus P_\epsilon} \Psi_{qm}^{+} S(\xi, \theta)\,d\theta \right\} \right| \\
&\leqslant B_m \left| \int_{S\setminus P} \Psi_{qm}^{+} S(\xi, \theta)\,d\theta \right|
\end{aligned}
$$

when $\|q\|$ is less than some positive q_m''.

Therefore,

$$
\begin{aligned}
\left| \operatorname{Re}\int_{S_r} (\varphi_j, f_{qm})\,d\theta \right| &\leqslant \frac{B_m}{2\pi} \int_{S\setminus P} \Psi_{qm}^{+} \left[\int_{S_r} \left| \frac{e^{i\theta} + z}{e^{i\theta} - z} \right| |f_{qm}\,dz| \right] d\theta \\
&\leqslant \frac{D_m}{(1-r)} \int_{S\setminus P} \Psi_{qm}^{+}\,d\theta.
\end{aligned}
$$

The last estimate together with (11) and (12) gives us that

$$
\begin{aligned}
|F_j(\tilde{f}_{qm}) - F_j(f_{qm})| &\leqslant \frac{G_m}{(1-r)^2} \int_{S\setminus P} \Psi_{qm}^{+}\,d\theta \\
&\leqslant \frac{2G_m}{(1-r)^2} \int_{S\setminus P} (\operatorname{Re}(\nabla u(f_0), h_{qm}))^{+}\,d\theta \\
&\quad + \frac{G_m}{(1-r)^2} \int_{S\setminus P} v(f_0(\zeta), h_{qm}(\zeta))\,d\theta \\
&\leqslant \frac{2G_m}{(1-r)^2} p(h_{qm}) + \frac{G_m}{(1-r)^2} v(\alpha_{qm})
\end{aligned}
$$

where $\alpha_{qm} = \|h_{qm}\|_\infty$. But, by our assumption,

$$
p(h_{qm}) \leqslant \|q\| \cdot \max\{p(h_{jm}^{\pm}), 0 \leqslant j \leqslant N\} \leqslant \|q\| \cdot m^{-1}
$$

and we can choose q_m so small that

$$
\frac{G_m}{(1-r)^2} v(\alpha_{qm}) \leqslant \frac{b}{2} \|q\|.
$$

If m is so large that

$$\frac{2G_m}{(1-r)^2 m} \leqslant \frac{b}{2},$$

then $|F_j(\tilde{f}_{qm}) - F_j(f_{qm})| \leqslant b\|q\|$ and our statement is proved. □

LEMMA 8. *Suppose that* $A: \mathbf{R}^k \to \mathbf{R}^l$ *is a linear mapping and* $K_1 \subset \mathbf{R}^k, K_2 \subset \mathbf{R}^l$ *are open cones such that* $A(K_1) = K_2$. *Let* l *be a ray at* K_2, *beginning at the origin. Then there is* $b > 0$ *such that for any continuous mapping* $F: \bar{K}_1 \to \mathbf{R}^l$ *the set* $F(\bar{K}_1 \cap B(0,r))$ *contains some neighborhood of* 0 *in* $l \cap \bar{K}_2$ *if*

$$\|F(x) - Ax\| \leqslant b\|x\|, \quad x \in B(0,r) \cap \bar{K}_1.$$

Proof. Since $A(K_1) = K_2$, then $k \geqslant l$. If $k = l$, then let us consider the homotopy F_t of the mapping F defined by the formula $F_t(x) = (1-t)F(x) + tA(x)$. Note that: (1) $F_0 = F$, $F_1 = A$; (2) $A(\partial K_1) = \partial K_2$. Therefore, if $x \in \partial K_1 \cap B(0,r)$ then $Ax \in \partial K_2$ and

$$\text{(13)} \qquad \begin{gathered} \|(1-t)F(x) + tAx - Ax\| \leqslant (1-t)\, b\|x\| \leqslant b\|x\| \\ \text{and} \quad \rho(F_t(x), \partial K_2) \leqslant b\|x\|. \end{gathered}$$

If $x \in K_1 \cap S(0, r/2)$ and $\alpha = \min\{\|Ax\|, \|x\| = 1\}$, then

$$\text{(14)} \qquad \|F_t(x)\| \geqslant \|Ax\| - (1-t)\|F(x) - Ax\| \geqslant \frac{(\alpha - b)}{2} r.$$

Let us take $b = \min(\alpha/2, a/2)$. Then if $y \in l \cap K_2 \cap B(0, \alpha r/4)$, (13) and (14) give us that $y \notin F_t(\partial(K_1 \cap B(0, r/2)))$ and, by the lemma on the homotopical invariance of the degree of mappings, $y \in F(K_1 \cap B(0, r/2))$.

If $k > l$ we denote Ker A by L and let l_1 be a ray at K such that $A(l_1) = l$. If L_1 is a linear subspace orthogonal to L and $x_0 \in l_1$, $0 < \|x_0\| < 1/2$, we define $Q = \{x \in \bar{K}_1 : x \in x_0 + L_1, \|x\| \leqslant 1\}$ and let A_1 be the restriction of A on Q. It is clear that A_1 is a homeomorphism of Q and $A_1(x_0) = y_1 \in l$. Hence, there is $r_1 > 0$, such that $B(y_1, r_1) \subset A_1(Q)$. If $b = \rho(S(y_1, r_1/2), A_1(\partial Q))$, then for any continuous mapping $F_1: \bar{Q} \to \mathbf{R}^l$ satisfying the inequality $\|F_1(x) - A_1 x\| \leqslant b$, the ball $B(0, r_1/2)$ lies in $F_1(Q)$, and hence $F_1(x_1) = y_1$ for some $x_1 \in Q$.

Let us introduce the following notations:

$$Q_t = \{tx : x \in Q\}$$

$$F_t(x) : Q \to \mathbf{R}^l, \quad F_t(x) = t^{-1} \cdot F(tx)$$

Then $Q_t \subset K_1 \cap B(0, r/2)$ for sufficiently small t and

$$\|F_t(x) - A_1 x\| = \|t^{-1}F(tx) - t^{-1}Atx\| \leqslant b\|x\| \leqslant b$$

for $x \in \bar{Q}_1$. Therefore $F_t(x_t) = y_0$ for some $x_t \in Q$ and $F(tx_t) = ty_0$. Since $tx_t \in Q_t$, we proved that $ty_0 \in F(K_1 \cap B(0, r/2))$ for sufficiently small t.

Let us return to our Lemma 6. By Statements 3, 4, and 5 it follows that $\tilde{A}_m$ is continuous in $\mathbf{R}_+^{2N+1}$ and for each $b > 0$ there is an $m \in \mathbf{Z}^+$ and $q_m > 0$ such that $\|\tilde{A}_m(q) - A(q)\| \leqslant b\|q\|$. It is easy to see that $A(\mathbf{R}_+^{2N+1}) = H = \{x \in \mathbf{R}^{N+1} : x_1 > 0\}$

and, by Lemma 8, if $l=\{tx: x=(1,0,\ldots,0), t>0\}$ then for some m and any t we can find q_t, which is a solution of the equation $\tilde{A}_m(q_t)=tx$. But, this means that

$$\Phi_0(\tilde{f}_{qm})=\Phi_0(f_0)+t \tag{14}$$

and

$$\Phi_j(\tilde{f}_{qm})=a_j, \quad j\geqslant 1. \tag{15}$$

Since $\tilde{f}_{qm}(\zeta)\in D$, when $\zeta\in\Delta$ (14) and (15) contradict the extremality of f_0, which proves Lemma 6. □

Now we can formulate the extremal principle.

THEOREM 3. *If f_0 is an extremal for problem* (P) *and $F_j=\Phi_j'(f_0,\cdot)$, then there are*

(1) $\lambda_j\in\mathbf{R}$, $0\leqslant j\leqslant N$, *and the λ_j are not equal to zero simultaneously;*

(2) $g\in H_n^\infty$, $g(0)=0$;

(3) $\lambda\in L^\infty$, $0\leqslant\lambda$ *such that for any $h\in L_n^1$,*

$$\operatorname{Re}\int_S\left(\sum_{j=0}^N\lambda_j\omega_j+g,h\right)d\theta=\operatorname{Re}\int_A\lambda(\nabla u(f_0),h)\,d\theta$$

where A is the set of points $\zeta\in S$ such that the radial limits at ζ belong to ∂D.

If F_j are linearly independent then $\operatorname{mes}A=2\pi$.

Proof. By Lemma 6 it follows that there is $T>0$ and j, $0\leqslant j\leqslant N$, such that

$$\delta F_j(h)\leqslant Tp(h) \tag{16}$$

where $\delta=\pm1, \delta=1$ if $j=0$, $h\in X_j$.

Using the Hahn–Banach theorem we can extend δF_j on L_n^1, conserving the inequality (16). If we denote the extension by F then $F(h)\leqslant Tp(h)$. But $p(h)\leqslant c\|h\|_1$, and hence F is continuous on L_n^1. By Riesz's theorem, F can be represented as $F(h)=\operatorname{Re}\int_S(h,\omega)\,d\theta$, where $\omega\in L_n^\infty$.

It is clear that there are λ_k, $0\leqslant k\leqslant N$, which are not equal to zero simultaneously such that $F(h)=\sum_{k=0}^N\lambda_kF_k(h)$ when $h\in H_n^1$. If we denote by G_1 the linear functional on L_n^1 defined by the formula

$$G_1(h)=\operatorname{Re}\int_S\left(\sum_{k=0}^N\lambda_k\omega_k,h\right)d\theta$$

then

$$G_2(h)=F(h)-G_1(h)=\int_S\left(\omega-\sum_{k=0}^N\lambda_k\omega_k,h\right)d\theta$$

and, since $G_2(h)=0$ on H_n^1, by a theorem of F. and M. Riesz it follows that

$$\omega-\sum_{k=0}^N\lambda_k\omega_k=g,\quad g\in H_n^\infty,\quad g(0)=0.$$

Therefore

$$F(h)=\operatorname{Re}\int_S \left(\sum \lambda_k\omega_k+g, h\right) d\theta \leqslant T\int_A (\operatorname{Re}(\nabla u, h))^+ d\theta$$

and

$$\operatorname{Re}\left(\sum_{k=0}^N \lambda_k\omega_k+g\right)\cdot h\leqslant 0 \quad \text{if} \quad \operatorname{Re}(\nabla u(f_0), h)\leqslant 0.$$

Hence

$$\sum_{k=0}^N \lambda_k\omega_k+g=\lambda\nabla u(f_0), \quad \lambda|_{S\setminus A}\equiv 0 \tag{17}$$

where $0\leqslant\lambda\leqslant T$, and we have proved the first part of the theorem.

If the F_j's are linearly independent and $\operatorname{mes} P>0$, $P=S\setminus A$, then by (17)

$$\left(\sum_{k=0}^N \lambda_k\omega_k+g\right)\Big|_P\equiv 0$$

and therefore $\sum_{k=0}^N \lambda_k\omega_k=-g$. This means that the F_j's are not linearly independent and we obtain a contradiction. Our theorem is proved. □

The system of equations

$$\begin{cases} \sum_{k=0}^N \lambda_k\omega_k+g=\lambda\nabla u(f_0) \\ u(f_0)=0 \text{ a.e.} \end{cases}$$

we shall call the Euler–Lagrange equations for the extremal problem (P).

6. The computation of extremals. Let $D_\alpha=\{z\in\mathbf{C}^n: \sum_{j=1}^n |z_j|^\alpha<1\}$ be a domain in $\mathbf{C}^n$ and $\alpha>1$. Then

$$\frac{\partial u}{\partial z_j}=\frac{\alpha}{2}\cdot\frac{|z_j|^\alpha}{z_j}$$

and D_α is of class C^1. We want to find extremals for the Royden problem, i.e., for given $a=(a_1,\dots,a_n)\in D_\alpha$ and $v=(v_1,\dots,v_n)\in\mathbf{C}^n$, $v\neq 0$, to define the mapping $f_0:\Delta\to D_\alpha$ such that $f_0(0)=a$, $f_0'(0)=\lambda v$, $\lambda>0$ and λ has a maximal possible value.

Let us introduce new vectors $v_k=(v_{1k},\dots,v_{nk})$, $1\leqslant k\leqslant n-1$, such that $z=\mu v$, $\mu\in\mathbf{C}$, if and only if $(z, v_k)=0$ for all k. Then our problem can be formulated in the following canonical way:

$$\begin{cases} \dfrac{1}{2\pi}\displaystyle\int_S f_j(\theta)\, d\theta=a_j \\ \dfrac{1}{2\pi}\displaystyle\int_S \sum_{j=1}^n \frac{f_j(\theta)v_{jk}}{\zeta}\, d\theta=0 \\ \operatorname{Re}\displaystyle\int_S \sum_{j=1}^n \frac{f_j(\theta)\bar v_j}{\zeta}\, d\theta\to\max. \end{cases}$$

We see that our functionals are linearly independent and that extremals satisfy the Euler–Lagrange equations:

(17) $$\frac{\mu_j}{\zeta}+g_j=\lambda|f_j|^{\alpha}\cdot f_j^{-1}, \quad \mu_j\in\mathbf{C}, \quad g_j\in H^{\infty}$$

(18) $$\sum_{j=1}^{n}|f_j|^{\alpha}=1 \text{ on } S.$$

Multiplying each of (17) by f_j and adding them together we have

$$\sum_{j=1}^{n}\left(\frac{\mu_j}{\zeta}+g_j\right)f_j=\lambda,$$

and therefore $\lambda=\mu\zeta^{-1}+g$, $g\in H^{\infty}$. Since $0\leqslant\lambda\leqslant T$,

(19) $$\lambda(\zeta)=(C_0\zeta^2+b_0\zeta+\bar{C}_0)\cdot\zeta^{-1}$$

where $b_0\in\mathbf{R}$, $b_0\geqslant 2|C_0|$.

It follows by (17) that $f_j(\mu_j\zeta^{-1}+g_j)=\lambda|f_j|^{\alpha}\geqslant 0$, and, as before, $\lambda|f_j|^{\alpha}=(C_j\zeta^2+b_j\zeta+\bar{C}_j)\cdot\zeta^{-1}$. Using (19) we get that

(20) $$|f_j|^{\alpha}=(C_j\zeta^2+b_j\zeta+\bar{C}_j)(C_0\zeta^2+b_0\zeta+\bar{C}_0)^{-1}$$

(21) $$f_j=(C_j\zeta^2+b_j\zeta+\bar{C}_j)(\mu_j+\zeta g_j)^{-1}.$$

STATEMENT 6. *If* $|f|=(C\zeta^2+b\zeta+\bar{C})\zeta^{-1}$, *on* S, *and* $C\zeta^2+b\zeta+\bar{C}=C(\zeta-\alpha)(\zeta-b)$, $f(\zeta)\neq 0$ *on* Δ, *then* $\alpha=\bar{\beta}^{-1}$ *and* $f(\zeta)=C\bar{\alpha}^{-1}(1-\bar{\alpha}\zeta)^2$, $|\alpha|\leqslant 1$.

This statement can be easily verified by a direct calculation.

Combining this statement with (20) and (21) we see that

(22) $$f_j(\zeta)=e_j\left(\frac{\zeta-\alpha_j}{1-\bar{\alpha}_j\zeta}\right)\left(\frac{1-\bar{\alpha}_j\zeta}{1-\bar{\alpha}_0\zeta}\right)^{2/\alpha}.$$

Now we can solve the following algebraic problem instead of the variational one: To find λ, α_j, α_0 and e_j as solutions of the following system of equations:

(23) $$f_j(0)=-e_j\alpha_j=a_j$$

(24) $$f_j'(0)=e_j(1-\gamma|\alpha_j|^2+2\alpha^{-1}\alpha_j\bar{\alpha}_0)=\lambda v_j, \quad \gamma=\frac{\alpha-2}{\alpha}$$

$$\sum_{j=1}^{n}|f_j|^{\alpha}=\sum_{j=1}^{n}|e_j|^{\alpha}\cdot\frac{|1-\bar{\alpha}_j\zeta|^2}{|1-\bar{\alpha}_0\zeta|^2}=1 \text{ on } S.$$

The last equation can be rewritten as follows:

(25) $$\sum_{j=1}^{n}|e_j|^{\alpha}\cdot\alpha_j=\alpha_0$$

(26) $$\sum_{j=1}^{n}|e_j|^{\alpha}(1+|\alpha_j|^2)=1+|\alpha_0|^2.$$

If $\alpha=2$ then $\gamma=0$, $\alpha_j=a_j/(\bar{\alpha}_0a_j-\lambda v_j)$ and, putting the last formula in (25), we have

$$\alpha_0 \sum_{j=1}^{n} |a_j|^2 - \lambda \sum_{j=1}^{n} a_j \bar{v}_j = \alpha_0.$$

Hence

$$\alpha_0 = \frac{-\lambda(q, \bar{v})}{1-\|a\|^2},$$

and, by (26), it follows that

$$\frac{\|a\|^2 \cdot \lambda^2 \cdot |(a, \bar{v})|^2}{(1-\|a\|^2)^2} + \frac{2\lambda^2|(a, \bar{v})|^2}{1-\|a\|^2} + \lambda^2\|v\|^2 + \|a\|^2 = 1 + \frac{\lambda^2|(a, \bar{v})|^2}{(1-\|a\|^2)^2}$$

and

$$\lambda^2 = \frac{(1-\|a\|)^2}{\|v\|^2(1-\|a\|^2) + |(a, \bar{v})|^2}.$$

Using this formula, we can easily define α_0 and α_j. In the case $\alpha \neq 2$, it is more difficult to find solutions of our algebraic system, but (22) allows us to describe the set of extremals as depending on a finite number of parameters.

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