# SURFACES IN MINKOWSKI 3-SPACE ON WHICH $H$ AND $K$ ARE LINEARLY RELATED

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**1. Introduction.** In this paper, we study surfaces in Minkowski 3-space $M^3$ on which mean curvature $H$ and extrinsic curvature $K$ satisfy a non-trivial linear relation $\alpha+\beta H+\gamma K\equiv 0$. Most results are based on formalisms developed in [3], which extend to the case of indefinite metric complex analytic techniques one might have expected to apply only in the Riemannian case.

On spacelike or timelike surfaces in $M^3$ with $\alpha+\beta H+\gamma K\equiv 0$ and $\beta^2\neq 4\alpha\gamma$, we show the existence of a certain holomorphic quadratic differential associated with the geometry of the immersion. This allows the introduction of special coordinates, and identifies three different flat metrics, among them the exotic metric $\Gamma=\alpha\mathrm{I}+\beta\mathrm{II}+\gamma\mathrm{III}$ studied by J. A. Wolf in [10]. That $\Gamma$ is flat on similar surfaces in Euclidean 3-space $E^3$ was observed by Darboux in [2], a fact we learned recently from Wolf. The use of flat metrics here yields some information in-the-large about the surfaces in question.

There is a rich variety of surfaces in $M^3$ on which $H$ or $K$ is constant. (See [1], [4], [6] and [9] for examples.) Moreover, $H$ and $K$ are linearly related on any surface equidistant in $M^3$ from a surface on which $H$ or $K$ is constant. We show below that a spacelike or timelike surface in $M^3$ on which $\alpha+\beta H+\gamma K\equiv 0$ with $\beta^2\neq 4\alpha\gamma$ is equidistant from at least one surface with $H$ or $K$ constant. In addition, we extend to $M^3$ the classical theorem of Bonnet (see [3]) which associates to a surface of constant $H\neq 0$ (resp. $K>0$), an equidistant surface of constant $K>0$ (resp. $H\neq 0$). This extension is known to geometers, but seems not to be in the literature.

We assume $C^\infty$ smoothness wherever possible. The symbols $\alpha,\beta,\gamma$ and $c$ always denote constants.

**2. Formal preliminaries.** Suppose that $S$ is an oriented surface, and that $A=E\,dx^2+2F\,dx\,dy+G\,dy^2$ and $B=L\,dx^2+2M\,dx\,dy+N\,dy^2$ are real quadratic forms with $\det A\neq 0$. Compute the curvatures $H=H(A,B)$, $K=K(A,B)$ and $H'=H'(A,B)$ by setting

$$2H=\operatorname{tr}_A B,\qquad K=\det B/\det A,\qquad 2H'=\sqrt{H^2-K}$$

with $iH'<0$ in case $H^2<K$. Denote the intrinsic curvature of $A$ by $K(A)$. Wherever $H'\neq 0$, define the skew forms $A'=A'(A,B)$ and $B'=B'(A,B)$ by

$$H'A'=B-HA,\qquad H'B'=HB-KA.$$

Anywhere on $S$, the form $W=W(A,B)$ is given by

$$\sqrt{|\det A|}\;W=\begin{vmatrix} dy^2 & -dx\,dy & dx^2\\ E & F & G\\ L & M & N\end{vmatrix}.$$

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For a detailed discussion of $A', B'$ and $W$, see §2 in [6].

We denote by $R_Z$ the Riemann surface determined on $S$ by a definite real quadratic form $Z$. Given any real quadratic form $Y = a\,dx^2 + 2b\,dx\,dy + c\,dy^2$ on $S$, the quadratic differential $\Omega = \Omega(Y, R_Z) = \phi\,dz^2$ is given by setting $\phi = (a - c - 2ib)$, using only $R_Z$ conformal parameters $z = x + iy$, in terms of which $Z = \lambda(dx^2 + dy^2)$. In case $\phi$ is complex analytic for all conformal parameters on $R_Z$, $\Omega$ is holomorphic.

We call $A, B$ a Codazzi pair and write $\operatorname{Cod}(A, B)$ if and only if $B$ satisfies the classical Codazzi-Mainardi equations

$$
\begin{aligned}
L_y - M_x &= L\Gamma^1_{12} + M(\Gamma^2_{12} - \Gamma^1_{11}) - N\Gamma^2_{11} \\
M_y - N_x &= L\Gamma^1_{22} + M(\Gamma^2_{22} - \Gamma^1_{12}) - N\Gamma^2_{12},
\end{aligned}
\tag{1}
$$

where the Christoffel symbols $\Gamma^i_{jk}$ are computed for $A$.

Our first observation takes its place among the facts listed in §3 of [6]. The pair $\hat{A}, \hat{B}$, in (2) is related to the pair $A, B$ as the first two fundamental forms on an equidistant surface are related to those on a given surface in $E^3$ or $M^3$. If $H^2 < K$ for $A, B$ then (3) shows that $\det \hat{A} \neq 0$ for all $t$. Thus the fact below associates to any Codazzi pair $A, B$ for which $H^2 < K$ an infinite family of Codazzi pairs $\hat{A}, \hat{B}$ for which $\hat{H}^2 < \hat{K}$.

FACT. *If* $1 - 2tH + Kt^2 \neq 0$ *for a real constant* $t$, *and if*

$$
\begin{aligned}
\hat{A} &= (1 - Kt^2)A + 2t(Ht - 1)B \\
\hat{B} &= tKA + (1 - 2tH)B
\end{aligned}
\tag{2}
$$

*then* $\operatorname{Cod}(A, B)$ *is equivalent to* $\operatorname{Cod}(\hat{A}, \hat{B})$.

*Proof.* Since $A, B$ are achieved as $\hat{\hat{A}}, \hat{\hat{B}}$ for the constant $-t$, it is enough to show that $\operatorname{Cod}(A, B)$ implies $\operatorname{Cod}(\hat{A}, \hat{B})$. Curvatures computed for the pair $\hat{A}, \hat{B}$ will be hatted. Here

$$
\begin{aligned}
\det \hat{A} &= \det A(1 - 2Ht + t^2K)^2 \\
K &= \hat{K}(1 - 2Ht + t^2K) \\
H - tK &= \hat{H}(1 - 2Ht + t^2K) \\
H' &= \hat{H}'|1 - 2Ht + t^2K|,
\end{aligned}
\tag{3}
$$

so that when $1 - 2Ht + t^2K \neq 0$, $\det A \cdot \det \hat{A} > 0$ and $H'\hat{H}'$ is real.

Wherever $H^2 > K$, use coordinates doubly orthogonal for $A, B$ and thus for $\hat{A}, \hat{B}$. Then $\operatorname{Cod}(A, B)$ is expressed by $L_y = E_y H$, $N_x = G_x H$, and computation yields $\hat{L}_y = \hat{E}_y \hat{H}$, $\hat{N}_x = \hat{G}_x \hat{H}$, giving $\operatorname{Cod}(\hat{A}, \hat{B})$. Wherever $H^2 < K$, use $R_W$ conformal parameters $z = x + iy$ so that

$$
A = \mathcal{E}\,dz^2 + \bar{\mathcal{E}}\,d\bar{z}^2, \qquad B = \mathcal{L}\,dz^2 + \bar{\mathcal{L}}\,d\bar{z}^2
$$

with similar expressions for $\hat{A}$ and $\hat{B}$. Then $\operatorname{Cod}(A, B)$ is expressed by $\mathcal{L}_{\bar{z}} = \mathcal{E}_{\bar{z}} H$ with $2\partial/\partial z = \partial/\partial x + i\partial/\partial y$, and computation yields $\hat{\mathcal{L}}_{\bar{z}} = \hat{\mathcal{E}}_{\bar{z}} \hat{H}$, giving $\operatorname{Cod}(\hat{A}, \hat{B})$.

Where $H^2 \equiv K$, either $B \propto A$ or else $A$ and $B$ share exactly one null direction. On any open, connected set where $B \propto A$, Fact 1 from [6] gives $B \equiv cA$, so $\mathrm{Cod}(A,B)$ and $\mathrm{Cod}(A',B')$ both hold trivially. On any open connected set where $A$ and $B$ share exactly one null direction, use coordinates so that $E \equiv G \equiv L \equiv 0$. Then $\mathrm{Cod}(A,B)$ is expressed by $FM_x = MF_x$, $F(M_y - N_x) = MF_y$, and computation gives $\mathrm{Cod}(\hat{A},\hat{B})$. A continuity argument completes the proof. □

Suppose now that $\alpha + \beta H + \gamma K \equiv 0$ for $A,B$ with $\alpha^2 + \beta^2 + \gamma^2 \neq 0$. If $\beta\gamma = 0$, either $H$ or $K$ is constant. These situations are explored in §4 of [6]. If $\beta\alpha \neq 0$ and $\beta^2 \equiv 4\alpha\gamma$, $H' \geqslant 0$ and at least one of the principal curvatures $H \pm H'$ is constant on $S$.

In Lemmas 1, 2 and 3, we work with the forms $X = \beta A + 2\gamma B$ and $X' = \beta A' + 2\gamma B'$, given any constants $\gamma \neq 0$ and $\beta$. Of course, $X'$ is defined only where $H' \neq 0$ for $A,B$.

LEMMA 1. *If* $\mathrm{Cod}(A,B)$ *then* $\alpha + \beta H + \gamma K \equiv 0$ *with* $\beta^2 \neq 4\alpha\gamma$ *and* $\gamma \neq 0$ *is equivalent to*

(i) $\Omega(A, R_X)$ *holomorphic in case* $X$ *is definite, to*

(ii) $\Omega(A', R_{X'})$ *holomorphic in case* $X$ *is indefinite with* $H^2 > K$, *and to*

(iii) $\Omega(H'X', R_W)$ *holomorphic in case* $X$ *is indefinite with* $H^2 < K$.

*Proof.* For any constants $\gamma \neq 0$ and $\beta$, we compute $\tilde{H} = H(A,X)$, $\tilde{K} = K(A,X)$, $\tilde{H}' = H'(A,X)$, $\tilde{A}' = A'(A,X)$, $\tilde{X}' = X'(A,X)$ and $\tilde{W} = W(A,X)$, obtaining

$$\tilde{H} = \beta + 2\gamma H, \qquad \tilde{K} = \beta^2 + 4\gamma(\beta H + \gamma K), \qquad \tilde{H}' = 2|\gamma| H' \tag{4}$$
$$\tilde{A}' = \pm A', \qquad \tilde{X}' = X', \qquad \tilde{W} = 2\gamma W$$

where $\pm$ is the sign of $\gamma$. Thus $\tilde{K}$ is a constant $\neq 0$ if and only if there is a constant $\alpha$ with $\alpha + \beta H + \gamma K \equiv 0$ and $\beta^2 \neq 4\alpha\gamma$. Since $\beta$ and $\gamma$ are constants, $\mathrm{Cod}(A,B)$ and $\mathrm{Cod}(A,X)$ are equivalent. Thus Lemmas 5, 6 and 7 of [3] applied to the pair $A,X$ give the result. □

Overlooked above is the case in which $H^2 \equiv K$ with $X$ indefinite. Then $\alpha + \beta H + \gamma K \equiv 0$ holds with $\beta^2 \neq 4\alpha\gamma$ if and only if both $H$ and $K$ are constants. However, even if $\mathrm{Cod}(A,B)$ holds, it need not follow that $B \propto A$, as the example

$$A = 2F\,dx\,dy, \qquad B = 2F\,dx\,dy + dy^2 \tag{5}$$

indicates. Indeed, one can even find $F$ so that the intrinsic curvature $K(A)$ in (5) takes on any constant value.

LEMMA 2. *Suppose that* $\mathrm{Cod}(A,B)$ *and* $\alpha + \beta H + \gamma K \equiv 0$ *on* $S$ *with* $\gamma \neq 0$ *and* $\beta^2 \neq 4\alpha\gamma$. *Then*

(i) *where* $X$ *is definite and* $H^2 \neq K$ *there are local coordinates in terms of which*

$$\pm 2\gamma H' A = \{\beta + 2\gamma(H + H')\}\,dx^2 + \{\beta + 2\gamma(H - H')\}\,dy^2 \tag{6}$$
$$\pm 2\gamma H' X = (\beta^2 - 4\alpha\gamma)(dx^2 + dy^2),$$

(ii) *where $X$ is indefinite and $A$ definite, there are local coordinates in terms of which*

$$A = dx^2 + dy^2 + 2\cos\omega\, dx\, dy$$
$$X = 2\sqrt{|\beta^2 - 4\alpha\gamma|}\ \sin\omega\, dx\, dy, \tag{7}$$

(iii) *where $X$ and $A$ are indefinite with $H^2 > K$ there are local coordinates in terms of which*

$$A = \pm\,(dx^2 + dy^2) + 2\cosh\omega\, dx\, dy$$
$$X = \pm\sqrt{|\beta^2 - 4\alpha\gamma|}\ \sinh\omega\, dx\, dy, \tag{8}$$

(iv) *and where $X$ and $A$ are indefinite with $H^2 < K$ there are local coordinates in terms of which*

$$A = \pm\,(dx^2 - dy^2) + 2\sinh\omega\, dx\, dy$$
$$X = 2\sqrt{|\beta^2 - 4\alpha\gamma|}\ \cosh\omega\, dx\, dy. \tag{9}$$

*Moreover, wherever $H' \neq 0$, the metrics $H'X, H'X'$ and $W$ are all flat.*

*Proof.* Where a holomorphic quadratic differential $\Omega = \phi\, dz^2$ is non-zero on a Riemann surface $R$, there are local conformal parameters $z = x + iy$ in terms of which $\phi$ is any fixed complex constant $\neq 0$. The coordinates in this lemma are obtained by taking $\pm\phi = 2$ or $2i$ for the holomorphic quadratic differential identified in Lemma 1. In terms of the coordinates provided, the forms $H'X, H'X'$ and $W$ all have constant coefficients. □

The following result remains valid if the form $X$ is replaced everywhere by the form $X'$.

LEMMA 3. *Suppose* Cod$(A, B)$ *and that $X$ is a complete Riemannian metric with $\alpha + \beta H + \gamma K \equiv 0$ and $\beta^2 < 4\alpha\gamma$. Then $H$, $K$ and $H'$ are constant, giving $K(A) \equiv K(X) \equiv 0$, if either $K(X) \geqslant 0$ on $S$ with $H$ bounded or if $K(X) \leqslant 0$ on $S$.*

*Proof.* By Lemma 2, $H'X$ is a flat metric on $S$. Since $\tilde{K} = K(A, X) \equiv \beta^2 - 4\alpha\gamma < 0$ is constant, $H' > 0$ is bounded away from zero, as is $\tilde{H}' = H'/2|\gamma|$. Thus $H'X$ is a complete flat metric on $S$, making $R_X$ parabolic. Using the coordinates provided by Lemma 2, one checks that $\log H'$ is subharmonic where $K(X) \geqslant 0$ and superharmonic where $K(X) \leqslant 0$. But a subharmonic (resp. superharmonic) function bounded from above (resp. below) must be constant. Once $H'$ is constant, $\alpha + \beta H + \gamma K \equiv 0$ forces $H$ and thereby $K$ to be constant, and by (6), $K(A) \equiv K(X) \equiv 0$. □

A similar result can be stated in case $\beta^2 > 4\alpha\gamma$ in Lemma 3. But then $H'$ is not automatically bounded away from zero, and this must be assumed.

Lemmas 1, 2 and 3 have been stated for an arbitrary Codazzi pair $A, B$ on $S$. They apply therefore if we take for $A$ and $B$ the first two fundamental forms I and II of an immersion $f: S \to \mathfrak{M}^3$ with $\det \mathrm{I} \neq 0$, taking $S$ into an arbitrary 3-manifold $\mathfrak{M}^3$ of constant curvature. (See [5] and [7] or [8].) In §3, we restrict our attention to the choice $\mathfrak{M}^3 = M^3$, although similar discussions are possible (at least locally) in any $\mathfrak{M}^3$.

**3. Surfaces in $M^3$.** Given the immersion $f: S \to M^3$, we speak of $S$ as a surface in $M^3$, and assume that $\mathrm{I} = df \cdot df$ is nondegenerate, so that classical geometry can be done in the usual way. (See §6 in [6].) Then $\epsilon = \det \mathrm{I} \neq 0$, with $S$ called spacelike if $\epsilon > 0$, and timelike if $\epsilon < 0$.

The unit normal $\nu$ for $S$ is the reflection in the horizontal of the vector product $(\partial f/\partial x) \times (\partial f/\partial y)$ divided by $\sqrt{|\epsilon|}$. By II and III we denote the second and third fundamental forms $-df \cdot d\nu$ and $d\nu \cdot d\nu$. The curvatures $H$, $K$ and $H'$ and the forms I′, II′ and $W$ are computed for the pair I, II.

Suppose that $\alpha + \beta H + \gamma K \equiv 0$ on $S$, with $\gamma \neq 0$ and $\beta^2 \neq 4\alpha\gamma$. Set $X = \beta \mathrm{I} + 2\gamma \mathrm{II}$ and $X' = \beta \mathrm{I}' + 2\gamma \mathrm{II}'$. Then Lemma 1 states that $\Omega(\mathrm{I}, R_X)$ is holomorphic where $X$ is definite, that $\Omega(\mathrm{I}', R_{X'})$ is holomorphic where $X$ is indefinite with $H^2 > K$, and that $\Omega(H'X', R_W)$ is holomorphic where $X$ is indefinite with $H^2 < K$. Thus the identity map from $S$ with metric $X$ to $S$ with metric I is harmonic. This follows from Theorems 3 through 6 and Lemmas 11 and 16 in [6]. Moreover, Lemma 2 provides special local coordinates wherever $H' \neq 0$ on $S$, and since $\pm H'X' = \alpha \mathrm{I} + \beta \mathrm{II} + \gamma \mathrm{III}$, it gives the following result.

THEOREM 1. *If $\alpha + \beta H + \gamma K \equiv 0$ on an $S$ in $M^3$ with $\gamma \neq 0$ and $\beta^2 \neq 4\alpha\gamma$, then the metrics $H'(\alpha \mathrm{I} + 2\gamma \mathrm{II})$, $\alpha \mathrm{I} + \beta \mathrm{II} + \gamma \mathrm{III}$ and $W$ are flat wherever $H' \neq 0$.*

The hyperbolic cylinders mentioned in Theorems 2 and 3 play the role in $M^3$ which the right circular cylinder does in $E^3$, since each has $K \equiv 0$ and $H \equiv c$. We use coordinates $u, v, w$ in $M^3$.

THEOREM 2. *Suppose $\beta \mathrm{I} + 2\gamma \mathrm{II}$ is a complete Riemannian metric on an $S$ in $M^3$ with $\alpha + \beta H + \gamma K \equiv 0$, $\gamma \neq 0$ and $\beta^2 < 4\alpha\gamma$. Then $S$ is (up to isometries of $M^3$) the timelike hyperbolic cylinder*

$$u^2 - w^2 = 1/4c^2, \quad u > 0,$$

*if $K(\beta \mathrm{I} + 2\gamma \mathrm{II}) \geqslant 0$ with $H$ bounded, or if $K(\beta \mathrm{I} + 2\gamma \mathrm{II}) \leqslant 0$.*

*Proof.* Taking $A = \mathrm{I}$ and $B = \mathrm{II}$, Lemma 3 shows that $K(\mathrm{I}) \equiv 0$ with $H, K$ and $H'$ constant if $K(\beta \mathrm{I} + 2\gamma \mathrm{II}) \geqslant 0$ on $S$ with $H$ bounded, or if $K(\beta \mathrm{I} + 2\gamma \mathrm{II}) \leqslant 0$ on $S$. Because $K(\mathrm{I}, \beta \mathrm{I} + \gamma \mathrm{II}) = \beta^2 - 4\alpha\gamma < 0$ with $\det(\beta \mathrm{I} + \gamma \mathrm{II}) > 0$, $S$ is timelike and $H' \neq 0$. Using the coordinates provided by Lemma 2, we see that I and II are the fundamental forms of the hyperbolic cylinder specified with $H = c$. The fundamental theorem for surfaces thus gives the result. (See [7] or [8].) □

Similar reasoning gives the following.

THEOREM 3. *Suppose $\beta \mathrm{I}' + 2\gamma \mathrm{II}'$ is a complete Riemannian metric on an $S$ in $M^3$ with $\alpha + \beta H + \gamma K \equiv 0$, $\gamma \neq 0$ and $\beta^2 < 4\alpha\gamma$. Then $S$ is (up to isometries of $M^3$) the spacelike hyperbolic cylinder*

$$w^2 - u^2 = 1/4c^2, \quad w > 0,$$

*if $K(\beta \mathrm{I}' + 2\gamma \mathrm{II}') \geqslant 0$ with $H$ bounded, or if $K(\beta \mathrm{I}' + 2\gamma \mathrm{II}') \leqslant 0$.*

Theorems 1, 2 and 3 are similar to results obtained in [6] for surfaces with constant $H$, or constant $K \neq 0$. The discussion of equidistant surfaces which follows provides one explanation of that similarity.

For any real constant $t$, let $\hat{f} = \hat{f}(t) = f + t\nu$ give the surface $\hat{S}$ at distance $t$ from $S$. Because $\hat{\epsilon} = \det \hat{\mathrm{I}} = (1 - 2Ht + Kt^2)^2 \epsilon$ for $\hat{\mathrm{I}} = d\hat{f} \cdot d\hat{f}$, the surface $\hat{S}$ is spacelike (resp. timelike) if $S$ is spacelike (resp. timelike). Moreover, $\hat{S}$ is sure to be regularly immersed wherever $1 - 2Ht + Kt^2 \neq 0$ on $S$, with $\nu = \hat{\nu}$ and

$$
\begin{aligned}
\hat{\mathrm{I}} &= (1 - Kt^2)\mathrm{I} + 2t(Ht - 1)\mathrm{II} \\
\hat{\mathrm{II}} &= tK\mathrm{I} + (1 - 2Ht)\mathrm{II} \\
H - tK &= (1 - 2tH + t^2K)\hat{H} \\
K &= (1 - 2tH + t^2K)\hat{K} \\
H' &= |1 - 2tH + t^2K|\hat{H}'.
\end{aligned} \tag{10}
$$

Note that $\hat{\epsilon} \neq 0$ for all real $t$ in case $H^2 < K$ on $S$. Since $S$ is the surface at distance $-t$ from $\hat{S}$, hatted and unhatted objects in (10) can be consistently reversed if $t$ is replaced by $-t$. We consider $S$ to be equidistant from itself, for $t = 0$.

If $\alpha + \beta H + \gamma K \equiv 0$ on $S$, then on $\hat{S}$, $\hat{\alpha} + \hat{\beta}\hat{H} + \hat{\gamma}\hat{K} \equiv 0$ with $\hat{\alpha} = \alpha$, $\hat{\beta} = (2t\alpha + \beta)$, $\hat{\gamma} = \alpha t^2 + \beta t + \gamma$ and $\hat{\beta}^2 - 4\hat{\alpha}\hat{\gamma} = \beta^2 - 4\alpha\gamma$. Thus, if $H$ and $K$ are linearly related on $S$, they remain so on any equidistant surface $\hat{S}$. Modulo restrictions which make $\hat{\epsilon} \neq 0$, Theorem 4 states that, among the surfaces equidistant from an $S$ in $E^3$ or $M^3$ on which $\alpha + \beta H + \gamma K \equiv 0$ with $\beta^2 \neq 4\alpha\gamma$, there is at least one with constant $\hat{H}$ or $\hat{K}$.

THEOREM 4. *Suppose $\alpha + \beta H + \gamma K \equiv 0$ with $\beta\gamma \neq 0$ and $\beta^2 \neq 4\alpha\gamma$ on $S$ in $E^3$ or $M^3$. If $\alpha = 0$, the surface $\hat{S}$ at distance $t = -\alpha/\beta$ has $\hat{H} \equiv 0$, with $\hat{\epsilon} \neq 0$ where $H \neq \beta/\gamma$. If $\alpha \neq 0$, the surface $\hat{S}$ at distance $t = -\beta/2\alpha$ has constant $\hat{K}$, with $\hat{\epsilon} \neq 0$ where $K \neq 0$. If $\beta^2 - 4\alpha\gamma > 0$, the surface $\hat{S}$ at either distance $t = (-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma})/2\alpha$ has constant $\hat{H}$, with $\hat{\epsilon} \neq 0$ where*

$$2\alpha H\sqrt{\beta^2 - 4\alpha\gamma} \neq K(\beta\sqrt{\beta^2 - 4\alpha\gamma} \mp 1).$$

The following theorem gives more precise information if either $H \neq 0$ or $K > 0$ is constant on $S$. In $E^3$, the identical result is due to Bonnet. (See [1].)

THEOREM 5. *Suppose $S$ is a surface in $M^3$. If $H \equiv c \neq 0$, the surface $S$ at distance $t = 1/2c$ has $\hat{K} \equiv 4c^2$ with $\hat{\epsilon} \neq 0$ where $K \neq 0$, while the surface $\hat{S}$ at distance $t = 1/c$ has $\hat{H} \equiv -c$ with $\hat{\epsilon} \neq 0$ where $K \neq c^2$. Similarly, if $K \equiv 4c^2 \neq 0$, the surface $\hat{S}$ at distance $t = \pm 1/2c$ has $\hat{H} = \mp c$, with $\hat{\epsilon} \neq 0$ where $\pm H \neq 2c$.*

Some cases deserve special attention. Suppose $S$ is a surface in $E^3$ or $M^3$. If $H \equiv 0$ on $S$, then $\hat{H} \equiv -t\hat{K}$ on $\hat{S}$, with $\hat{\epsilon} \neq 0$ where $K \neq 1/t^2$. Here $\hat{H}$ or $\hat{K}$ is constant if and only if $\hat{H}$, $\hat{K}$ and $K$ are all constant. If $K \equiv 0$ on $S$, then $\hat{K} \equiv 0$ on $\hat{S}$, with $\hat{\epsilon} \neq 0$ where $H \neq 1/2t$. Here $\hat{H}$ is constant if and only if $H$ is constant. If $K \equiv c < 0$, there are no other equidistant surfaces with $\hat{H}$ or $\hat{K}$ constant unless $H$ is constant. Finally, if $\alpha + \beta H + \gamma K \equiv 0$ on $S$ with $\beta^2 - 4\alpha\gamma \neq 0$, then the same sort of linear equation relates $\hat{H}$ and $\hat{K}$ on any $\hat{S}$, so that $\hat{H}' \geqslant 0$, and at least one principal curvature $\hat{H} \pm \hat{H}'$ is always constant.

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