ON LENGTH FUNCTIONS, TRIVIALIZABLE SUBGROUPS AND CENTRES OF GROUPS

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Introduction. Conditions are given in [4] for a length function l on a group G to be an extension of a length function l_1 on K, a normal subgroup of G, by a length function l_2 on G/K. The length function l_1 is necessarily non-Archimedean, and so its structure is known by [3]. In this paper we consider possibilities for the normal subgroup K which appears in such a decomposition.

In Section 1 an l-trivializable subgroup of G is defined, and it is shown that the maximal l-trivializable subgroup, which we denote by T, consists of elements a with $l(ax)=l(x)$ for all Archimedean elements x. It is shown that the restriction of the length function to any l-trivializable subgroup K can be replaced by any length function on K, subject only to a bound condition. Two length functions on G are then defined to be equivalent if their maximal trivializable subgroups coincide, and if they agree on elements outside of this subgroup. It follows that any length function is equivalent to an extension of a length function l_1 by a length function l_2 on H, such that T_{l_2}, the maximal l_2-trivializable subgroup of H, contains no non-trivial normal subgroup of H. In Section 2 these results are applied to the centre of a group, and it is shown that a length function is equivalent to an extension of a length function l_1 by a length function l_2 on a group with trivial centre, or is an extension of a length function l_1 by a length function l_2 on an abelian group.

1. l-trivializable subgroups and equivalent length functions. A *length function* l on a group G assigns to each element $x \in G$ a real number $l(x)$ such that, if

$$d(x,y)=\tfrac{1}{2}(l(x)+l(y)-l(xy^{-1})),$$

then $l(x)$ and $d(x,y)$ satisfy the following axioms for all $x,y,z \in G$:

A1′. $l(1)=0$,

A2. $l(x)=l(x^{-1})$,

A4. $d(x,y)<d(x,z)$ implies $d(y,z)=d(x,y)$.

The numbering of the axioms is that of Lyndon [2]. A4 states that of the three numbers $d(x,y)$, $d(x,z)$, $d(y,z)$, two are equal with the third no smaller. It follows from A1′ and A2 that $d(x,1)=d(1,y)=0$, so that by A2 and A4, $d(x,y)=d(y,x)\geqslant 0$, and putting $y=x$, $l(x)=d(x,x)\geqslant 0$.

Received July 6, 1982.

The first author was supported in part by NSERC A4495. The second author expresses his gratitude to the British Council and the Royal Society for financial support during this research.

Michigan Math. J. 30 (1983).

An element $x \in G$ is *Archimedean* if $l(x^2) > l(x)$, and is *non-Archimedean* otherwise. The set of non-Archimedean elements of G is denoted by N, and a length function is said to be *Archimedean* if $N = \{1\}$, and *non-Archimedean* if $N = G$. It follows from Propositions 2.1 and 3.4 of [4] that N is a normal subset of G, and $x \in N$ if and only if the set $\{l(x^n);\ n \text{ an integer}\}$ is bounded. We will also make frequent use of the following result.

LEMMA 1.1 [4, Proposition 3.3]. *If $a, b, ab \in N$, then two of $l(a)$, $l(b)$, $l(ab)$ are equal, with the third no greater.*

A proper subgroup K of G is *l-minimal* if $l(a) \leqslant l(x)$ for all $a \in K, x \notin K$. Since the lengths of powers of Archimedean elements are unbounded, it follows that any l-minimal subgroup is contained in N.

A proper subgroup K of G is *l-trivializable* if $l(ax) = l(x)$ for all $a \in K, x \notin K$. It follows from Axiom A2 that $l(xa) = l(x)$ for all $a \in K, x \notin K$.

When K is a normal subgroup of G then by Theorem 4.2 of [4], K is l-trivializable if and only if it is l-minimal. An arbitrary subgroup K may be l-minimal but not l-trivializable. For example if $G = G_1 * G_2$, the free product of groups G_1 and G_2, and l is the associated length function as described by Lyndon [2], then G_1 and G_2 are both l-minimal, but neither is l-trivializable. However the following result is still valid.

LEMMA 1.2. *If a subgroup K is l-trivializable then it is l-minimal.*

Proof. Let $a \in K, x \notin K$. Then $x^{-1}, x^{-1}a^{-1} \notin K$, and

$$2d(a,x) = l(a) + l(x) - l(ax^{-1}) = l(a),$$

$$2d(a,ax) = l(a) + l(ax) - l(ax^{-1}a^{-1}) = l(a),$$

so that by Axiom A4, $2d(x,ax) = l(x) + l(ax) - l(a) = 2l(x) - l(a) \geqslant l(a)$. Hence $l(x) \geqslant l(a)$, and K is l-minimal. □

PROPOSITION 1.3. *Let l be a length function on G, and let K be an l-trivializable subgroup of G. Then l' defined by*

$$l'(x) = \begin{cases} l_1(x) & \text{if } x \in K \\ l(x) & \text{if } x \notin K \end{cases}$$

is a length function on G if and only if l_1 is a length function on K with the properties

(i) $2d(x,y) \geqslant l_1(a)$ *for all* $a \in K$, $x, y \notin K$ *with* $xy^{-1} \notin K$,

(ii) $l(x) \geqslant l_1(a)$ *for all* $a \in K, x \notin K$.

We note that Condition (i) implies Condition (ii), unless K is of index 2 in G, in which case (i) is vacuous. If K has index greater than 2, then for $x \notin K$, there exists $y \notin K$ with $xy^{-1} \notin K$. Also $x^{-1}, yx^{-1} \notin K$ and we have

$$2d(x,y) = l(x) + (l(y) - l(xy^{-1})),$$

$$2d(x^{-1}, yx^{-1}) = l(x) - (l(y) - l(xy^{-1})).$$

Thus either $2d(x,y)$ or $2d(x^{-1}, yx^{-1})$ is $\leqslant l(x)$, and (i) implies (ii).

Proposition 1.3 may be considered an extension of Proposition 4.1 of [4], although the proofs of the two results are essentially the same. Proposition 4.1 states that if K is a normal subgroup of G, and l_1 and l_2 are length functions on K and $H=G/K$, respectively, with $f\colon G\to H$ the projection homomorphism, then a length function l' on G is defined by

$$l'(x)=\begin{cases} l_1(x) & \text{if } x\in K\\ l_2(f(x)) & \text{if } x\notin K,\end{cases}$$

if and only if (i) $2d_2(u,v)\geqslant l_1(a)$, (ii) $l_2(u)\geqslant l_1(a)$ for all $a\in K$, $u,v\in H$ with $u,v\neq 1$. It is easy to show that a length function l on G is defined by $l(x)=0$ if $x\in K$, with $l(x)=l_2(f(x))$ if $x\notin K$.

For the proof of Proposition 1.3, if l_1 is a length function on K, then Axioms A1$'$ and A2 are easily shown to be satisfied for l'. It remains to show that Axiom A4 is satisfied if and only if Conditions (i) and (ii) hold.

We denote elements of G by a,b,c if they are in K, and by x,y,z if they are not. Writing functions d' and d_1 to correspond to l' and l_1, the possibilities for d' for pairs of elements of G are as follows.

$$2d'(a,b)=2d_1(a,b).$$

$$2d'(a,x)=l_1(a)+l(x)-l(ax^{-1})=l_1(a).$$

$$\text{If } xy^{-1}\notin K,\ 2d'(x,y)=2d(x,y).$$

$$\begin{aligned}\text{If } xy^{-1}=a\in K,\ 2d'(x,y)&=l(x)+l(y)-l_1(a)\\ &=l(x)+l(xa^{-1})-l_1(a)=2l(x)-l_1(a).\end{aligned}$$

For Axiom A4 all possible combinations of three elements of G need to be considered, and these can be split into seven cases. To complete the proof of Proposition 1.3 we refer to the proof of Proposition 4.1 of [4], since the cases, and the arguments for each case, are exactly as appear there.

If l' is a length function on G defined as in Proposition 1.3, then it follows that K is l'-trivializable, and that N', the set of non-Archimedean elements for l', is equal to N, the set of non-Archimedean elements for l.

Given a length function l on G, we wish to construct a subgroup which is l-trivializable, and which is maximal with respect to this property.

Define the subset T of G by

$$T=\{a\in G;\, l(ax)=l(x) \text{ for all } x\notin N\}.$$

If $x\notin N$, then $l(x^2)>l(x)$ and so $x\notin T$, showing that $T\subset N$. Clearly if $N=G$, then $T=G$.

THEOREM 1.4. *If $N\neq G$, then T is the maximal l-trivializable subgroup of G.*

Before proving the theorem we establish three lemmas, again under the assumption that $N\neq G$.

LEMMA 1.5. *If $a\in T$ and $x\notin N$, then $l(a)<l(x)$.*

Proof. By Proposition 3.1 of [4], two of $l(ax)=l(x^{-1}a^{-1})$, $l(ax^{-1})=l(xa^{-1})$ and $l(a)+l(x^2)-l(x)$ are equal, with the third no greater. Since $l(x)=l(xa^{-1})=l(x^{-1}a^{-1})$, we have $l(x)\geqslant l(a)+l(x^2)-l(x)$. As $x\notin N$, $l(x^2)>l(x)$ and so $l(x)>2l(x)-l(x^2)\geqslant l(a)$. □

LEMMA 1.6. *If $a\in T$, then $a^{-1}\in T$.*

Proof. We need to show $l(a^{-1}x)=l(x)$, for all $x\notin N$, which, by Axiom A2, is equivalent to $l(xa)=l(x)$, for all $x\notin N$.

If $x\notin N$, then, since N is a normal subset of G, $a^{-1}xa$, $a^{-1}x^{-1}a\notin N$, and by A2, $l(a^{-1}xa)=l(a^{-1}x^{-1}a)$. Thus, since $a\in T$,

$$l(xa)=l(aa^{-1}xa)=l(a^{-1}xa)=l(a^{-1}x^{-1}a)=l(aa^{-1}x^{-1}a)=l(x^{-1}a).$$

There are now three cases.

If $xa\notin N$, then $l(xa)=l(a^{-1}x^{-1})=l(aa^{-1}x^{-1})=l(x^{-1})=l(x)$.

If $x^{-1}a\notin N$, then $l(xa)=l(x^{-1}a)=l(a^{-1}x)=l(aa^{-1}x)=l(x)$.

If $xa, x^{-1}a\in N$, then the conjugate $ax^{-1}=a(x^{-1}a)a^{-1}\in N$. So ax^{-1}, xa and $a^2\in N$, when by Lemma 1.1 two of $l(ax^{-1})$, $l(xa)$ and $l(a^2)$ are equal, with the third no greater. By Lemma 1.5 $l(ax^{-1})=l(x)>l(a)\geqslant l(a^2)$, and hence $l(xa)=l(ax^{-1})=l(x)$.

In each of the cases $l(xa)=l(x)$, and so $a^{-1}\in T$, proving the lemma. □

LEMMA 1.7. *T is an l-minimal subgroup of G.*

Proof. We first show that T is a subgroup of G. For $a,b\in T$, $x\notin N$ there are three cases.

If $bx\notin N$, then $l(abx)=l(bx)=l(x)$.

If $abx\notin N$, then $a^{-1}\in T$ by Lemma 1.6, and so $l(abx)=l(a^{-1}abx)=l(bx)=l(x)$.

If $bx, abx\in N$, then by Lemma 1.1, since $a\in N$, two of $l(a)$, $l(bx)$, $l(abx)$ are equal, with the third no greater. By Lemma 1.5 $l(bx)=l(x)>l(a)$, and hence $l(abx)=l(bx)=l(x)$.

In each of the cases above $l(abx)=l(x)$, and so $ab\in T$, showing that T is a subgroup of G.

To prove T is l-minimal we need to show that $l(a)\leqslant l(x)$ for all $a\in T, x\notin T$, which is equivalent to showing that $b\in T$ if $l(b)<l(a)$ for some $a\in T$.

Let $a\in T$, and suppose $l(b)<l(a)$. By Lemma 1.5 $l(b^{-1})=l(b)<l(x)$ for all $x\notin N$, and so $b^{-1}\in N$. Furthermore $ab^{-1}\in N$, since if not, $a^{-1}\in T$ by Lemma 1.6, and so $l(b^{-1})=l(a^{-1}ab^{-1})=l(ab^{-1})$, contradicting the fact that $l(b^{-1})<l(x)$ for all $x\notin N$. Thus a, b^{-1}, $ab^{-1}\in N$ with $l(b)<l(a)$, and it follows by Lemma 1.1 that $l(ab^{-1})=l(a)$. Then if $x\notin N$,

$$2d(a,x^{-1})=l(a)>l(b)=2d(a,b).$$

Axiom A4 implies $2d(b,x^{-1})=l(b)+l(x)-l(bx)=l(b)$, giving $l(bx)=l(x)$. Thus $b\in T$ completing the proof of Lemma 1.7. □

Proof of Theorem 1.4. T is a subgroup of G by Lemma 1.7, and $l(ax)=l(x)$ for all $a\in T, x\notin N$, by definition. To prove that T is l-trivializable it remains to show that $l(ax)=l(x)$ for all $a\in T, x\in N\backslash T$. If $ax\notin N$ then $l(ax)=l(a^{-1}ax)=$

$l(x)$. If $ax \in N$ then $a, x, ax \in N$ and by Lemma 1.1 two of $l(a)$, $l(x)$, $l(ax)$ are equal with the third no greater. Since T is a subgroup $ax \notin T$, and by Lemma 1.7 T is l-minimal, so that $l(a) \leqslant l(x), l(ax)$. Thus we must have $l(a) \leqslant l(x) = l(ax)$, showing that T is l-trivializable.

For K an l-trivializable subgroup of G, then $K \subset N$, so a necessary requirement is that $l(ax) = l(x)$ for all $a \in K, x \notin N$. It follows that $K \subset T$, and T is the maximal l-trivializable subgroup of G; completing the proof of Theorem 1.4. □

For l a length function on G, then by Proposition 1.3 the restriction of l to T can be replaced by any length function, subject only to the bound conditions (i) and (ii). Any such length function on T will therefore be non-Archimedean, and so its structure will be given by a chain of subgroups, as described in [3]. This suggests the following definition. Where two length functions on a group are being considered or where confusion may arise, we will use the symbol T_l to denote the maximal trivializable subgroup associated with a length function l.

Two length functions l, l' on G are *equivalent* if $T_l = T_{l'}$ and $l(x) = l'(x)$ for all $x \notin T_l = T_{l'}$. This is clearly an equivalence relation.

To conclude this section we consider extensions of length functions. If K is a normal subgroup of G, and $f\colon G \to H = G/K$ is the projection homomorphism, then a length function l on G is an *extension* of a length function l_1 on K by a length function l_2 on H if

$$l(x) = \begin{cases} l_1(x) & \text{if } x \in K \\ l_2(f(x)) & \text{if } x \notin K. \end{cases}$$

By Theorem 4.2 of [4], this occurs precisely when the normal subgroup K is l-minimal, or, equivalently, l-trivializable. Thus $K \subset T_l$ and the length function l_1 is non-Archimedean. It is easy to see that $f(T_l) = T_{l_2}$, the maximal l_2-trivializable subgroup of H.

If l is any length function of G, then l_0 defined by

$$l_0(x) = \begin{cases} 0 & \text{if } x \in T_l \\ l(x) & \text{if } x \notin T_l, \end{cases}$$

is a length function which is equivalent to l. Any proper normal subgroup of G, contained in $T_{l_0} = T_l$, is l_0-minimal and so satisfies the conditions of Theorem 4.2 of [4] for an extension. In particular the *core* $C = \bigcap\{xT_lx^{-1}; x \in G\}$ is the maximal normal subgroup of G, contained in T_l. Thus l_0 is an extension of l_1 on C by l_2 on $H = G/C$. Since C is the maximal normal subgroup of G, contained in T_l, it follows that $T_{l_2} = f(T_l)$ contains no non-trivial normal subgroup of H. Thus we have proved

THEOREM 1.8. *Any length function l on G is equivalent to a length function which is an extension of a non-Archimedean length function l_1 on C by a length function l_2 on $H = G/C$, where T_{l_2} contains no non-trivial normal subgroup of H.*

2. Length functions and groups with trivial centres. In this section we prove the following theorem, where T is the maximal l-trivializable subgroup of G, and Z is the centre of G.

THEOREM 2.1. *Let l be a length function on G.*

(i) *If $Z\not\subset N$, then l is an extension of a non-Archimedean length function l_1 on N by an Archimedean length function l_2 on $H=G/N$, an abelian group.*

(ii) *If $Z\subset N=T$, then l is an extension of a non-Archimedean length function l_1 on N by an Archimedean length function l_2 on $H=G/N$, where H has trivial centre or is an abelian group.*

(iii) *If $Z\subset N\neq T$, then l is equivalent to an extension of a non-Archimedean length function l_1 on C, the core of T, by a length function l_2 on $H=G/C$, a group with trivial centre.*

By Theorem 5.3 of [4], assuming $N\neq G$, $N=T$ if and only if the lengths of elements of N are bounded. The structure of non-Archimedean length functions is given in [3], and the structure of Archimedean length functions on abelian groups is given in [1].

Some preliminary results are necessary before proving the theorem.

LEMMA 2.2 [4, Lemma 5.2]. *Let $a\in N, x\notin N$. If the set $\{l(x^{-n}ax^n);$ n an integer$\}$ is bounded, then $l(ax)=l(x)$.*

LEMMA 2.3. *If $Z\subset N$, then $Z\subset C$, the core of T.*

Proof. If $a\in Z, x\notin N$, then $l(x^{-n}ax^n)=l(a)$. So the lengths $l(x^{-n}ax^n)$ are bounded, and it follows by Lemma 2.2 that $l(ax)=l(x)$ and $a\in T$. Thus $Z\subset T$, and since Z is a normal subgroup of G, it follows that $Z\subset C$. □

LEMMA 2.4. *If $Z\not\subset N$, then the lengths of the elements of N are bounded.*

Proof. If $a\in N, x\in Z\backslash N$, then $l(x^{-n}ax^n)=l(a)$. The lengths $l(x^{-n}ax^n)$ are therefore bounded, and so by Lemma 2.2, $l(ax)=l(x)=l(ax^{-1})$. Axiom A2 implies $l(x)=l(xa^{-1})=l(x^{-1}a^{-1})$, and since Proposition 3.1 of [4] states that two of $l(xa^{-1})$, $l(x^{-1}a^{-1})$, and $l(a)+l(x^2)-l(x)$ are equal with the third no greater, it follows that $l(x) = l(xa^{-1}) = l(x^{-1}a^{-1}) \geqslant l(a)+l(x^2)-l(x)$. Hence $l(a) \leqslant 2l(x)-l(x^2)$, and the lengths of the elements of N are bounded by $2l(x)-l(x^2)$. □

LEMMA 2.5. *If $xy=yx$ and $l(x)>l(y)$ then $l(xy)+l(xy^{-1})=2l(x)$. Moreover if $y\in N$, then $l(xy)=l(xy^{-1})=l(x)$, and if $y\notin N$, then $x\notin N$ and $l(xy)\neq l(xy^{-1})$ with* $\max(l(xy),l(xy^{-1}))=l(x)+l(y^2)-l(y)=l(y)+l(x^2)-l(x)$.

Proof. Consider

$$2d(xy,x)=l(xy)+l(x)-l(y),$$

$$2d(xy,y)=l(xy)+l(y)-l(x).$$

Since $l(x)>l(y)$ we have $2d(xy,x)>2d(xy,y)$, and so by Axiom A4, $2d(xy,y)=2d(x,y)=l(x)+l(y)-l(xy^{-1})$, giving $l(xy)+l(xy^{-1})=2l(x)$.

By Proposition 3.1 of [4], two of

$$l(yx)=l(xy),\quad l(y^{-1}x)=l(xy^{-1}),\quad l(x)+l(y^2)-l(y)$$

are equal with the third no greater. Thus if $y\in N$ and $l(xy)\neq l(xy^{-1})$, then

$\max(l(xy), l(xy^{-1})) = l(x) + l(y^2) - l(y) \leqslant l(x)$, contradicting the fact that $l(xy) + l(xy^{-1}) = 2l(x)$. Hence $l(xy) = l(xy^{-1}) = l(x)$.

If $y \notin N$ and $l(xy) = l(xy^{-1})$, then $l(xy) = l(xy^{-1}) \geqslant l(x) + l(y^2) - l(y) > l(x)$, contradicting the fact that $l(xy) + l(xy^{-1}) = 2l(x)$. Hence $l(xy) \neq l(xy^{-1})$ with $\max(l(xy), l(xy^{-1})) = l(x) + l(y^2) - l(y)$. Now $l(xy^{-1}) = l(y^{-1}x) = l(x^{-1}y)$ by Axiom A2. Thus $l(xy) \neq l(x^{-1}y)$, whence by Proposition 3.1 of [4],

$$\max(l(xy), l(x^{-1}y) = l(xy^{-1})) = l(y) + l(x^2) - l(x).$$

Now $l(x) + l(y^2) - l(y) = l(y) + l(x^2) - l(x)$, and since $y \notin N$,

$$l(x) + l(y^2) - l(y) > l(x).$$

Thus, as $l(x) > l(y)$, it follows that $l(x^2) > l(x)$ and $x \notin N$, completing the proof of the lemma. □

The following proposition has been proved by Nancy Harrison in [1], in the special case where $N = \{1\}$.

PROPOSITION 2.6. *Let $x, y, z \notin N$. If $xy = yx$ and $xz = zx$ then $yzy^{-1}z^{-1} \in N$.*

Proof. Since $x \notin N$, the lengths $l(x^n)$ are unbounded. We may therefore assume, without loss of generality, that $l(x) > 2l(y), 2l(z)$, since if x does not satisfy this condition then it could be replaced by a suitable power x^n.

Since $y, z \notin N$ and $xy = yx$, $xz = zx$ it follows by Lemma 2.5 that $l(xy) \neq l(xy^{-1})$ and $l(xz) \neq l(xz^{-1})$. We suppose that $l(xy) > l(xy^{-1})$ and $l(xz) > l(xz^{-1})$, and consider the other possibilities later. By Lemma 2.5,

$$l(xy) = l(x) + l(y^2) - l(y) = l(y) + l(x^2) - l(x),$$

$$l(xz) = l(x) + l(z^2) - l(z) = l(z) + l(x^2) - l(x),$$

$$l(xy^{-1}) = l(x) + l(y) - l(y^2),$$

$$l(xz^{-1}) = l(x) + l(z) - l(z^2).$$

Thus $2l(x) - l(x^2) = 2l(y) - l(y^2) = 2l(z) - l(z^2)$.

Consider

$$2d(x, y) = l(x) + l(y) - l(xy^{-1}) = l(y^2),$$

$$2d(x, z^{-1}) = l(x) + l(z) - l(xz) = 2l(x) - l(x^2).$$

Now $2l(x) - l(x^2) = 2l(y) - l(y^2)$, and so, since $y \notin N$, $2d(x, y) > 2d(x, z^{-1})$. Hence by A4

$$2d(y, z^{-1}) = 2d(x, z^{-1}) = 2l(x) - l(x^2).$$

Interchanging the roles of y and z above gives

$$2d(z, y^{-1}) = 2d(x, y^{-1}) = 2l(x) - l(x^2).$$

Thus $2d(y, z^{-1}) = 2d(z, y^{-1}) = 2l(x) - l(x^2)$, showing that $l(yz) = l(zy)$.

Consider

$$2d(xy, y) = l(xy) + l(y) - l(x) = l(y^2).$$

Now $2d(y,z^{-1})=2l(y)-l(y^2)<l(y^2)=2d(xy,y)$, and so by A4, $2d(xy,z^{-1})=2d(y,z^{-1})$. By interchanging the roles of y and z, $2d(xz,y^{-1})=2d(z,y^{-1})$. From above, $2d(y,z^{-1}) = 2d(z,y^{-1}) = 2l(x)-l(x^2)$, and so $2d(xy,z^{-1}) = 2d(xz,y^{-1})$. Since

$$l(xy)+l(z)=l(xz)+l(y)=l(y)+l(z)+l(x^2)-l(x),$$

it follows that $l(xyz) = l(xzy)$. Moreover

$$2d(xy,z^{-1})=2l(x)-l(x^2)=2l(z)-l(z^2),$$

and since $z\notin N$, $2l(z)-l(z^2)<l(z)$. Thus $l(xy)+l(z)-l(xyz)<l(z)$, giving $l(xyz)>l(xy)$. Now $l(xy)>l(x)$, and so $l(xyz)>l(x)$.

The element x commutes with yz and zy, and $l(yz)=l(zy)\leqslant l(y)+l(z)<l(x)$. Lemma 2.5 may therefore be applied to x and yz, and to x and zy. Thus, since $l(xyz)=l(xzy)>l(x)$ we have $yz, zy\notin N$ with

$$2l(x)-l(x^2)=2l(yz)-l((yz)^2)=2l(zy)-l((zy)^2).$$

Since $l(yz)=l(zy)$, we therefore have $l((yz)^2)=l((zy)^2)$. Also by Lemma 2.5,

$$l(xyz)=l(xzy)=l(x)+l((yz)^2)-l(yz),$$
$$l(x(yz)^{-1})=l(x(zy)^{-1})=l(x)+l(yz)-l((yz)^2).$$

Now,

$$2d(x,yz)=2d(x,zy)=l(x)+l(yz)-l(x(yz)^{-1})=l((yz)^2),$$
$$2d(x,(yz)^{-1})=2d(x,(zy)^{-1})=l(x)+l(yz)-l(xyz)=2l(yz)-l((yz)^2).$$

Thus by A4, $2d(yz,zy)\geqslant l((yz)^2)$, and $2d((yz)^{-1},(zy)^{-1})\geqslant 2l(yz)-l((yz)^2)$. So $d(yz,zy)+d((yz)^{-1},(zy)^{-1})\geqslant l(yz)=l(zy)$, and it follows by Proposition 2.4 of [2] that $yz(zy)^{-1}=yzy^{-1}z^{-1}\in N$. □

In the above argument we assumed that $l(xy)>l(xy^{-1})$ and $l(xz)>l(xz^{-1})$. If one or the other of these inequalities were reversed, then the argument would proceed as before with y replaced by y^{-1} or z replaced by z^{-1}. Thus the conclusion would be that one of $y^{-1}zyz^{-1}$, $yz^{-1}y^{-1}z$, $y^{-1}z^{-1}yz$ is in N. Since inverses and conjugates of elements of N are again in N, it follows that $yzy^{-1}z^{-1}\in N$, since

$$yzy^{-1}z^{-1}=y(y^{-1}zyz^{-1})^{-1}y^{-1}=z(yz^{-1}y^{-1}z)^{-1}z^{-1}=(zy)(y^{-1}z^{-1}yz)(zy)^{-1}.$$

Proof of Theorem 2.1. In (i) $Z\not\subset N$, and so by Lemma 2.4 the lengths of elements of N are bounded. By Theorem 5.3 of [4], $T=N$ and l is an extension of a non-Archimedean length function l_1 on N, by an Archimedean length function l_2 on $H=G/N$. For two elements $y,z\notin N$, there exists $x\in Z\backslash N$, and so by Proposition 2.6, the element $yzy^{-1}z^{-1}\in N$. The projections of y and z therefore commute in $H=G/N$, which is thus an abelian group.

In (ii), $N=T$ and so by Theorem 5.3 of [4], l is an extension of a non-Archimedean length function l_1 on N, by an Archimedean length function l_2 on $H=G/N$. If H has non-trivial centre Z_2, then Proposition 2.6 may be applied to non-trivial elements $x \in Z_2$, $y, z \in H$, giving $yzy^{-1}z^{-1}=1$, the single non-Archimedean element of H. Thus $yz=zy$ and H is an abelian group.

In (iii), $Z \subset N \neq T$, and so T is a proper subgroup of N. By Theorem 1.8, l is equivalent to an extension of a non-Archimedean length function l_1 on C, the core of T, by a length function l_2 on $H=G/C$. Let Z_2 be the centre, N_2 the set of non-Archimedean elements, and T_2 the maximal l_2-trivializable subgroup of H. Since $C \subset T \neq N$, we have $f(T)=T_2 \neq N_2=f(N)$, where $f\colon G \to H$ is the projection homomorphism. If $Z_2 \not\subset N_2$ then by Lemma 2.4 the lengths of elements of N_2 are bounded, and so by Theorem 5.3 of [4], $N_2=T_2$, which is false. Thus $Z_2 \subset N_2$ and by Lemma 2.3, $Z_2 \subset C_2$, the core of T_2. But by Theorem 1.8, T_2 contains no non-trivial normal subgroup of H, and so $Z_2=C_2=\{1\}$, completing the proof of Theorem 2.1. □

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