SMOOTH FREE INVOLUTIONS ON HOMOTOPY 4k-SPHERES

Ronald Fintushel and Ronald J. Stern

1. Introduction. In [7] we constructed a homotopy real projective 4-space Q^4 not s-cobordant to the standard real projective 4-space $\mathbf{R}P^4$ with the property that the 2-fold cover of Q was diffeomorphic to the 4-sphere S^4, thereby constructing a smooth exotic involution on S^4. In order to determine that Q was not s-cobordant to $\mathbf{R}P^4$ we introduced an invariant which was a $\mathbf{Q}/\mathbf{Z}$-linear combination of the μ-invariant of an almost-framed characteristic homology $\mathbf{R}P^3$ and the α-invariant of its 2-fold cover. In this paper we generalize this invariant to study smooth manifolds of the homotopy type of $\mathbf{R}P^{4k}$, $k>1$, or equivalently to study smooth free involutions on homotopy 4k-spheres. This invariant, defined in §2 and called ρ, is again essentially a $\mathbf{Q}/\mathbf{Z}$-linear combination of the Eells-Kuiper μ-invariant of a spin structure on a characteristic homotopy projective space and the Atiyah–Singer α-invariant of its double cover.

In §3 we show that (except for a single exceptional case $\rho=1/4$), every normal cobordism class of homotopy $\mathbf{R}P^{4k-1}$'s which contains a representative whose double cover is the standard sphere S^{4k-1} gives rise to at least two homotopy $\mathbf{R}P^{4k}$'s which are distinguished by ρ. In §4 we construct normal cobordism classes of homotopy $\mathbf{R}P^{4k-1}$'s which contain representatives whose double cover is S^{4k-1}, and we distinguish these normal cobordism classes by a difference invariant, essentially 2ρ, introduced in §2. This then yields twice as many homotopy $\mathbf{R}P^{4k}$'s.

In Appendix I we give an explicit calculation of the μ-invariants of $\mathbf{R}P^{4k-1}$ which is used in §3 and §4.

We obtain no more smooth homotopy $\mathbf{R}P^{4k}$'s than claimed earlier by, for instance, Giffen [8] or Löffler [12]. The strength of our approach lies in the simplicity of the invariant ρ and its utility in distinguishing specific examples. This is exemplified by the explicit construction of all the homotopy $\mathbf{R}P^8$'s and homotopy $\mathbf{R}P^{12}$'s in Appendix II.

Finally, we thank Paul Melvin for asking us if we knew how to construct any exotic smooth involutions on S^8. We also wish to thank Terry Lawson and the referee for their interest and for useful advice.

2. The invariants. Given a closed manifold Q^{4k-1} of the homotopy type of $\mathbf{R}P^{4k-1}$ whose double cover $\tilde{Q}$ is diffeomorphic to S^{4k-1} we construct a homotopy $\mathbf{R}P^{4k}$ as follows. Identify $\tilde{Q}$ with the boundary of the 4k-ball B^{4k} and let

Received December 7, 1981. Revision received November 19, 1982.

The first author was supported in part by NSF grant MCS 7900244A01. The second author was supported in part by NSF grant MCS 8002843.

Michigan Math. J. 30 (1983).

$\Sigma^{4k} = B^{4k} \cup_t B^{4k}$ where t is the free involution on $\tilde{Q}$ given by the covering translation. Let T be the involution on the homotopy sphere Σ^{4k} which sends a point x in one copy of B^{4k} to the point x in the other copy. (This is compatible with the involution t on $\tilde{Q}$.) The pair (Σ^{4k}, T) will be called a *suspension* of $(\tilde{Q}, t)$ with $\tilde{Q}$ a *characteristic sphere* for T. The quotient Σ^{4k}/T is homotopy equivalent to $\mathbf{R}P^{4k}$ and will be called a *suspension* of Q^{4k-1} with $Q^{4k-1} \subset \Sigma^{4k}/T$ a *characteristic homotopy* $\mathbf{R}P^{4k-1}$. Every involution on a homotopy $4k$-sphere, $k > 1$, has characteristic homotopy spheres (see [13]).

Given a (Σ^{4k}, T) as above we shall define an invariant of Σ^{4k}/T which is a $\mathbf{Q}/\mathbf{Z}$-linear combination of the Eells-Kuiper μ-invariant of Q^{4k-1} and the Atiyah-Singer α-invariant (or equivalently the Browder-Livesay invariant) of the involution t on $\tilde{Q}$. We first digress to recall the definitions of these invariants.

Let X^{4k-1} be a spin $4k-1$ manifold with a given spin structure s. The μ-invariant of (X^{4k-1}, s) as defined by Eells and Kuiper [6] and generalized by Milnor [14] is given as follows. The spin cobordism group $\Omega_{4k-1}^{\mathrm{SPIN}} = 0$ since $\Omega_*^{\mathrm{SPIN}} \to \Omega_*^{\mathrm{SO}}$ has kernel only in dimensions $8k+1$ and $8k+2$, and $\Omega_{4k-1}^{\mathrm{SO}} = 0$ (see [17]). So $(X^{4k-1}, s) = \partial(W^{4k}, S)$ where S is a spin structure on W^{4k}. We are interested in the case where X^{4k-1} is a rational homology sphere (e.g. X is a homotopy $\mathbf{R}P^{4k-1}$). Then the inclusion induced homomorphisms

$$j^*: H^{4q}(W, X; \mathbf{Q}) \to H^{4q}(W; \mathbf{Q}) \quad (0 \leqslant q < k) \qquad \text{and}$$
$$j^*: H^{2k}(W, X; \mathbf{Q}) \to H^{2k}(W; \mathbf{Q})$$

are isomorphisms.

Let $L_k(p_1, \ldots, p_k)$ denote the kth polynomial associated with $z^{1/2}/\tanh(z^{1/2})$, $\hat{A}_k(p_1, \ldots, p_k)$ the kth polynomial associated with $\frac{1}{2}z^{1/2}/\sinh(\frac{1}{2}z^{1/2})$, $t_k = -1/(2^{2k+1}(2^{2k-1}-1))$, and

$$N_k(p_1, \ldots, p_{k-1}) = \hat{A}_k(p_1, \ldots, p_{k-1}, 0) - t_k L_k(p_1, \ldots, p_{k-1}, 0).$$

Let $p_j(W)$ be the jth rational Pontrjagin class of W and consider the rational number

$$N_k(W) = \langle N_k(j^{*-1}(p_1(W)), \ldots, j^{*-1}(p_{k-1}(W))), [W, X] \rangle.$$

If $\sigma(W)$ denotes the signature of W and $a_k = 4/(3 + (-1)^k)$, then define $\mu(X, s) \equiv (N_k(W) + t_k \sigma(W))/a_k \pmod 1$, which depends only on (X, s).

The other invariant which we shall need is the α-invariant of Atiyah and Singer [2]. Suppose that (M^{4k-1}, t) is an oriented $(4k-1)$-manifold with an orientation-preserving free involution t such that (M^{4k-1}, t) is the oriented equivariant boundary of (Y^{4k}, T) where T is not necessarily free. Then $\alpha(M, t) = \mathrm{sign}(Y^{4k}, T) - Y^T \cdot Y^T$ where $\mathrm{sign}(Y^{4k}, T)$ is the equivariant signature and $Y^T \cdot Y^T$ is the self-intersection number of the fixed point set. When M^{4k-1} is a homotopy $(4k-1)$-sphere, α is just 8 times the Browder-Livesay invariant of t (see [13]). In this case we usually write $\alpha(M^{4k-1}/t)$ to mean $\alpha(M^{4k-1}, t)$.

We are now prepared to define our first invariant. Let Q^{4k-1} be a smooth homotopy $\mathbf{R}P^{4k-1}$ $(k \geqslant 2)$ with double cover $\tilde{Q}$, and let s_0 and s_1 be the two distinct spin structures of Q^{4k-1}. Define

$$\hat{\rho}(Q^{4k-1},s_i) \equiv \mu(Q^{4k-1},s_i) - \frac{t_k}{2a_k}\alpha(Q) - \frac{1}{2}\mu(\tilde{Q}) \pmod 1.$$

THEOREM 2.1. *If Q and Q' are normally cobordant homotopy* $\mathbf{R}P^{4k-1}$*'s* $(k \geq 2)$ *the pairs of* (mod 1) *rational numbers* $\{2\hat{\rho}(Q,s_0), 2\hat{\rho}(Q,s_1)\}$ *and* $\{2\hat{\rho}(Q',s_0'), 2\hat{\rho}(Q',s_1')\}$ *are equal.*

Proof. The homotopy equivalences $f\colon Q \to \mathbf{R}P^{4k-1}$ and $f'\colon Q' \to \mathbf{R}P^{4k-1}$ extend to a normal map $F\colon W^{4k} \to \mathbf{R}P^{4k-1}$ where W is a normal cobordism from Q to Q'. Performing surgery if necessary, we may assume $\pi_1(W) = \mathbf{Z}_2$. Since $\mathbf{R}P^{4k-1}$ is spin it follows that W is a spin manifold. Further, since F is a normal map the rational Pontryagin classes $p_q(\nu(W))$ of the stable normal bundle of W are pulled back from a bundle over $\mathbf{R}P^{4k-1}$. (This is actually the stable normal bundle of $\mathbf{R}P^{4k-1}$, see [3], p. 215.) Since $H^{4q}(\mathbf{R}P^{4k-1};\mathbf{Q}) = 0$ for $q > 0$ we have $p_q(\nu(W)) = 0$ for $q > 0$, and so $p_q(W) = 0$.

Now choose a spin structure S_0 for W and let $s_0 = S_0 \mid Q$ and $s_0' = S_0 \mid Q'$. We have

$$\mu(Q',s') - \mu(Q,s) \equiv \mu(\partial W, S \mid \partial W) \equiv \frac{t_k}{a_k}\sigma(W) \pmod 1.$$

Since F lifts to a normal cobordism $\tilde{F}\colon \tilde{W} \to S^{4k-1}$ of double covers $\tilde{Q}$ and $\tilde{Q}'$, a similar argument shows

$$\mu(\tilde{Q}') - \mu(\tilde{Q}) \equiv \frac{t_k}{a_k}\sigma(\tilde{W}) \pmod 1.$$

However $2\sigma(W) - \sigma(\tilde{W}) = \alpha(Q') - \alpha(Q)$ (see [18], p. 198), hence

$$2\mu(Q',s_0') - 2\mu(Q,s_0) - \mu(\tilde{Q}') + \mu(\tilde{Q}) \equiv \frac{t_k}{a_k}(\alpha(Q') - \alpha(Q)) \pmod 1.$$

That is, $2\hat{\rho}(Q_0',s_0') \equiv 2\hat{\rho}(Q_0,s_0) \pmod 1$.

Since $\pi_1(W) = \mathbf{Z}_2$ we have $H^1(W;\mathbf{Z}_2) = \mathbf{Z}_2$ so W has exactly two spin structures, and the diagram

$$\begin{array}{ccc} & H^1(\mathbf{R}P^{4k-1};\mathbf{Z}_2) & \\ {\scriptstyle F^*}\swarrow & & \searrow{\scriptstyle f^*}\,\approx \\ H^1(W;\mathbf{Z}_2) & \xrightarrow{\quad i^* \quad} & H^1(Q;\mathbf{Z}_2) \end{array}$$

shows that i^* is onto. Thus each spin structure on Q is the restriction of some spin structure on W. The same is true for each spin structure on Q'. Since W has exactly two spin structures, say S_0 and S_1, the restriction of S_1 to Q and Q' must give the other spin structures s_1 and s_1' of Q and Q'. So the above argument shows that $2\hat{\rho}(Q',s_1') \equiv 2\hat{\rho}(Q,s_1) \pmod 1$. □

Notice in the above proof that if $\mu(\tilde{Q})$, $\mu(\tilde{Q}')$ and $\sigma(\tilde{W})$ were all equal to zero, the proof would show that $\hat{\rho}(Q',s_i') \equiv \hat{\rho}(Q,s_i) \pmod 1$, $i = 0, 1$.

PROPOSITION 2.2. *Let Σ^{4k} be a homotopy $4k$-sphere $(k \geq 2)$ with a free involution T. Let Q and Q' be characteristic homotopy* $\mathbf{R}P^{4k-1}$*'s for Σ^{4k}/T. Then* $\{\hat{\rho}(Q,s_0), \hat{\rho}(Q,s_1)\} \equiv \{\hat{\rho}(Q',s_0'), \hat{\rho}(Q',s_1')\} \pmod 1$.

Proof. Let W^{4k} be a characteristic cobordism from Q to Q'; so W is a normal cobordism. Consider the double cover $\tilde{W} \subset \Sigma^{4k} \times I$ with $\partial \tilde{W} = \tilde{Q} \cup \tilde{Q}'$. Note that $\tilde{Q} \cong \tilde{Q}' \cong S^{4k-1}$. Now $\tilde{Q}$ bounds a ball U in $\Sigma^{4k} \times 0$ and $\tilde{Q}'$ bounds a ball U' in $\Sigma^{4k} \times 1$ so that the manifold $U \cup \tilde{W} \cup U'$ bounds a codimension 0 submanifold of $\Sigma^{4k} \times I$. In particular $\sigma(\tilde{W}) = \sigma(U \cup \tilde{W} \cup U') = 0$. So the proposition follows from the comment above. □

Proposition 2.2 shows that the pair of (mod 1) rational numbers

$$\{\hat{\rho}(Q, s_0), \hat{\rho}(Q, s_1)\}$$

is an invariant of the involution T of Σ^{4k}. Our goal for the remainder of this section is to prove that $\hat{\rho}(Q, s_1) \equiv -\hat{\rho}(Q, s_0)$ (mod 1) for a characteristic homotopy $\mathbf{R}P^{4k-1}$; so we can reduce the invariant to a single number.

We shall next need the Dold construction. Let Q^{4k-1} be a smooth homotopy $\mathbf{R}P^{4k-1}$ and let $f: Q \to \mathbf{R}P^{4k-1}$ be a homotopy equivalence with $V^{4k-2} = f^{-1}(\mathbf{R}P^{4k-2})$ a transverse preimage. The normal bundle E of V^{4k-2} in Q^{4k-1} is a line bundle and so has a unique fiber-preserving involution J fixing V^{4k-2}. The Dold construction is $\mathfrak{D}(Q) = Q \times I \cup_{J: E \to E} Q \times I$ where E is viewed as a submanifold of $Q \times 1$. Notice that $\partial \mathfrak{D}(Q) = \tilde{Q} - 2Q$. Since f is transverse to $\mathbf{R}P^{4k-1}$, $f \mid E$ maps E onto the normal bundle of $\mathbf{R}P^{4k-2}$ in $\mathbf{R}P^{4k-1}$ via a bundle map, and so $f \mid E$ is equivariant with respect to the involutions used to paste together the Dold constructions. Hence F extends to $\mathfrak{D}(f): D(Q) \to \mathfrak{D}(\mathbf{R}P^{4k-1})$.

Now $f: Q \to \mathbf{R}P^{4k-1}$ is covered by a map of stable tangent bundles

$$\hat{f}: \tau(Q) \to \tau(\mathbf{R}P^{4k-1})$$

([1]). If f is transverse to $\mathbf{R}P^{4k-2}$ then $\hat{f}$ is transverse to $\tau(\mathbf{R}P^{4k-1})|_{\mathbf{R}P^{4k-2}}$. Therefore the Dold construction can be applied to the manifolds $\tau(Q)$ and $\tau(\mathbf{R}P^{4k-1})$ giving $\mathfrak{D}(\hat{f}): \mathfrak{D}(\tau(Q)) \to \mathfrak{D}(\tau(\mathbf{R}P^{4k-1}))$. But $\mathfrak{D}(\tau(Q)) = \tau(\mathfrak{D}(Q))$ and $\mathfrak{D}(\tau(\mathbf{R}P^{4k-1})) = \tau(\mathfrak{D}(\mathbf{R}P^{4k-1}))$ because the gluing map used in the construction of $\mathfrak{D}(\tau(Q))$ from two copies of $\tau(Q) \times I$ can be thought of as the differential of the gluing map used to construct $\mathfrak{D}(Q)$ from two copies of $Q \times I$. Hence $\mathfrak{D}(f): \mathfrak{D}(Q) \to \mathfrak{D}(\mathbf{R}P^{4k-1})$ is covered by $\mathfrak{D}(\hat{f}): \tau(\mathfrak{D}(Q)) \to \tau(\mathfrak{D}(\mathbf{R}P^{4k-1}))$. Notice that $\mathfrak{D}(\hat{f})$ is the same map over the two copies of $Q \times 0$ in $\mathfrak{D}(Q)$.

PROPOSITION 2.3. *Let Q^{4k-1} be a homotopy* $\mathbf{R}P^{4k-1}$ *($k \geq 2$) with $\alpha(Q) = 0$. Then $\hat{\rho}(Q, s_1) \equiv -\hat{\rho}(Q, s_0)$ (mod 1).*

Proof. Since $\alpha(Q) = 0$ there is a homotopy equivalence $f: Q \to \mathbf{R}P^{4k-1}$ transverse to $\mathbf{R}P^{4k-2}$ such that $f \mid f^{-1}(\mathbf{R}P^{4k-2}) = V \to \mathbf{R}P^{4k-2}$ is also a homotopy equivalence. Let $\mathfrak{D} = \mathfrak{D}(Q)$ and $\mathfrak{D}_R = \mathfrak{D}(\mathbf{R}P^{4k-1})$. There is an extension $F = \mathfrak{D}(f): \mathfrak{D} \to \mathfrak{D}_R$ covered by $b = \mathfrak{D}(\hat{f}): \tau \to \tau_R$. Because Q and $\mathbf{R}P^{4k-1}$ are spin manifolds a Mayer–Vietoris sequence argument shows that $\mathfrak{D}$ and $\mathfrak{D}_R$ are also spin manifolds. Choose a spin structure $\mathcal{S}$ for $\mathfrak{D}$. If we can show that $\mathcal{S} \mid Q \cup Q = s_0 \cup s_1$ where $Q \cup Q$ is the union of the two copies of Q in $\partial \mathfrak{D}$ then

$$\mu(\partial \mathfrak{D}, \mathcal{S} \mid \partial \mathfrak{D}) \equiv \mu(\tilde{Q}) - \mu(\tilde{Q}, s_0) - \mu(Q, s_1) \equiv \frac{t_k}{a_k} \sigma(\mathfrak{D}) \text{ (mod 1)}$$

because $H^{4q}(\mathfrak{D};\mathbf{Q})=0$ for $q\geq 1$, so $p_q(\mathfrak{D})=0$ for $q\geq 1$. But $\mathfrak{D}$ has the property that $\sigma(\mathfrak{D})=-\alpha(Q)$ (see [13]). So

$$\mu(\tilde{Q})-\mu(Q,s_0)-\mu(Q,s_1)\equiv\frac{-t_k}{a_k}\alpha(Q)\ (\mathrm{mod}\ 1)$$

and the result follows.

We must now show that $\mathcal{S}\mid Q\cup Q=s_0\cup s_1$ for some spin structure $\mathcal{S}$ on $\mathfrak{D}$. Spin structures on $\mathfrak{D}$ and $\mathfrak{D}_R$ correspond uniquely to spin structures on τ and τ_R (see [14], Lemma 3). Let $P\xi$ denote the total space of the principal SO(n)-bundle associated with an orientable n-plane bundle ξ. Fix a spin structure $\mathcal{R}$ on τ_R, i.e. $\mathcal{R}\in H^1(P\tau_R;\mathbf{Z}_2)$. Then $b:\tau\to\tau_R$ induces $\bar{b}:P\tau\to P\tau_R$; so

$$\mathcal{S}=\bar{b}^*\mathcal{R}\in H^1(P\tau;\mathbf{Z}_2)$$

is an induced spin structure on τ.

Now $\mathcal{R}\mid \mathbf{R}P^{4k-1}\cup\mathbf{R}P^{4k-1}\subseteq\partial\mathfrak{D}_R$ is $r_0\cup r_1$ with $r_0\neq r_1$; for if $r_0=r_1$ we would have

$$2\mu(\mathbf{R}P^{4k-1},r_0)\equiv\frac{-t_k}{a_k}\alpha(\mathbf{R}P^{4k-1})\equiv 0\ (\mathrm{mod}\ 1)$$

which is not the case. (See Appendix I for a computation of $\mu(\mathbf{R}P^{4k-1},r_0)$.)

Consider the commutative diagram

$$\begin{array}{ccccc}
H^1(P\tau(\mathbf{R}P^{4k-1});\mathbf{Z}_2) & \longleftarrow & H^1(P\tau_R;\mathbf{Z}_2) & \longrightarrow & H^1(P\tau(\mathbf{R}P^{4k-1}),\mathbf{Z}_2)\\
\bar{b}_0^*\downarrow & & \downarrow\bar{b}^* & & \downarrow\bar{b}_1^*\\
H^1(P\tau(Q);\mathbf{Z}_2) & \longleftarrow & H^1(P\tau;\mathbf{Z}_2) & \longrightarrow & H^1(P\tau(Q);\mathbf{Z}_2).
\end{array}$$

By construction $b_0=b_1$; so $\bar{b}_0^*=b_1^*$. Since $r_0\neq r_1$ it follows easily that $\mathcal{S}\mid Q\cup Q=s_0\cup s_1$. □

COROLLARY 2.4. *Let Q^{4k-1} be any homotopy* $\mathbf{R}P^{4k-1}$ $(k\geq 2)$. *Then* $2\hat{\rho}(Q,s_0)\equiv-2\hat{\rho}(Q,s_1)$ (mod 1).

Proof. Q is normally cobordant to another homotopy $\mathbf{R}P^{4k-1}$, Q', satisfying $\alpha(Q')=0$ ([13], V.2.1). So the result follows from (2.3) and (2.1). □

THEOREM 2.5. *Let Q^{4k-1} be a homotopy $\mathbf{R}P^{4k-1}$ with double cover $\tilde{Q}\cong S^{4k-1}$. Then $\hat{\rho}(Q,s_1)\equiv-\hat{\rho}(Q,s_0)$* (mod 1).

Proof. As in the proof of (2.3) it suffices to show that for the Dold construction $\mathfrak{D}(Q)$ all $p_q(\mathfrak{D}(Q))=0$, $q>0$, and that there is a spin structure S for $\mathfrak{D}(Q)$ restricting to the two different spin structures s_0 and s_1 on the two copies of Q in $\partial\mathfrak{D}(Q)$.

To prove that the rational Pontryagin classes of $\mathfrak{D}(Q)$ are trivial let P^{4k} be a suspension of Q. (P^{4k} exists because $\tilde{Q}\cong S^{4k-1}$). Let $f:P^{4k}\to\mathbf{R}P^{4k}$ be a homotopy equivalence transverse to $\mathbf{R}P^{4k-1}$ which restricts to a homotopy equivalence $Q\to\mathbf{R}P^{4k-1}$. Now let $\mathbf{R}P_0^{4k-1}$ be a cross section of the normal bundle of $\mathbf{R}P^{4k-1}$ in $\mathbf{R}P^{4k}$ such that $\mathbf{R}P_0^{4k-1}$ meets $\mathbf{R}P^{4k-1}$ transversely in $\mathbf{R}P^{4k-2}$. Then f is transverse to $\mathbf{R}P_0^{4k-1}$ and $f^{-1}(\mathbf{R}P_0^{4k-1})=Q_0$ is a cross section of the normal bundle of Q in P (so $Q_0\cong Q$) and $f^{-1}(\mathbf{R}P^{4k-2})=Q_0\cap Q=V^{4k-2}$. Let $\mathfrak{D}(Q)$ be the Dold construction corresponding to the homotopy equivalence $f:Q\to\mathbf{R}P^{4k-1}$.

Even though the involution on $\tilde{P}$ is orientation-reversing, we can form the nonorientable $(4k+1)$-manifold

$$\mathfrak{D}(P)=P\times I \bigcup_{J_0:E_0\to E_0} P\times I$$

where E_0 is the normal bundle of Q_0 in $P\times 1$ and the gluing map J_0 is a fiber-preserving involution of E_0 fixing Q_0. Since Q and Q_0 intersect transversely in V we have $E=E_0\cap Q$ and $J=J_0\,|\,E$; so $\mathfrak{D}(Q)$ is a codimension 1 submanifold of $\mathfrak{D}(P)$. A Mayer–Vietoris argument shows that $H^{4q}(\mathfrak{D}(P);\mathbf{Q})=0$ for $0<q<k$; so $p_q(\mathfrak{D}(P))=0$ for $0<q<k$. Thus $0=i^*p_q(\mathfrak{D}(P))=p_q(\mathfrak{D}(Q))$.

Next let Q' be a homotopy $\mathbf{R}P^{4k-1}$ normally cobordant to Q with $\alpha(Q')=0$, and let $F\colon W^{4k}\to \mathbf{R}P^{4k-1}$ be a normal cobordism between the homotopy equivalences $f\colon Q\to \mathbf{R}P^{4k-1}$ and $f'\colon Q'\to \mathbf{R}P^{4k-1}$. We may assume that F is transverse to $\mathbf{R}P^{4k-2}$ and that $\hat{V}=F^{-1}(\mathbf{R}P^{4k-2})$ has $\partial\hat{V}=V\cup V'$ where $V'\subset Q'$ is homotopy equivalent to $\mathbf{R}P^{4k-2}$. Let $\hat{E}$ be a tubular neighborhood of $\hat{V}$ and form the Dold construction $\mathfrak{D}(W)=W\times I\cup_{\hat{J}:\hat{E}\to\hat{E}} W\times I$. Then $\partial\mathfrak{D}(W)=\tilde{W}-2W+\mathfrak{D}(Q)-\mathfrak{D}(Q')$. After performing surgery on W, $\hat{V}$, and V we may suppose that $H^1(W;\mathbf{Z}_2)=\mathbf{Z}_2$ and that

$$H^1(V;\mathbf{Z}_2)\xleftarrow{i_V^*} H^1(\hat{V};\mathbf{Z}_2)\xrightarrow{i_{V'}^*} H^1(V';\mathbf{Z}_2)=\mathbf{Z}_2$$

are isomorphisms. Hence each (inclusion induced) map in the commutative diagram

$$\begin{array}{ccc} H^1(\mathfrak{D}(W);\mathbf{Z}_2) & \longrightarrow & H^1(\hat{V};\mathbf{Z}_2) \\ \downarrow & & \downarrow \\ H^1(\mathfrak{D}(Q');\mathbf{Z}_2) & \longrightarrow & H^1(V';\mathbf{Z}_2) \end{array}$$

is an isomorphism.

It follows that each spin structure of $\mathfrak{D}(Q')$ is induced from some spin structure on $\mathfrak{D}(W)$. Let S' be a spin structure on $\mathfrak{D}(Q')$ and let $\mathcal{S}$ be an extension over $\mathfrak{D}(W)$ with $\mathcal{S}\,|\,\mathfrak{D}(Q)=S$. Since $H^1(W;\mathbf{Z}_2)=\mathbf{Z}_2$, W has exactly two spin structures. We know from the proof of (2.3) that the restriction of S' to the two copies of Q' in $\partial\mathfrak{D}(Q')$ gives $s_0'\cup s_1'$ where $s_0'\neq s_1'$. Hence $\mathcal{S}$ restricts to the two distinct spin structures $\sigma_0\neq\sigma_1$ of W on the two copies of W in $\partial\mathfrak{D}(W)$. Since $H^1(W;\mathbf{Z}_2)\to H^1(Q;\mathbf{Z}_2)=\mathbf{Z}_2$ is an isomorphism, each spin structure on Q extends over W; so $s_0=\sigma_0\,|\,Q\neq s_1=\sigma_1\,|\,Q$. So the spin structure S on $\mathfrak{D}(Q)$ restricts to $s_0\cup s_1$ on $Q\cup Q\subseteq\partial\mathfrak{D}(Q)$ as required. □

Theorem 2.5 can also be proved for *any* homotopy $\mathbf{R}P^{4k-1}$ with k *odd*. For we can ambiently surger a characteristic submanifold V^{4k-2} until all $\tilde{H}^i(V^{4k-2})=0$ except for $i=2k-1, 4k-2$. But $H^{i+1}(\mathfrak{D}(Q);\mathbf{Q})\approx H^i(V;\mathbf{Q})$. So if $2k\not\equiv 0 \pmod 4$ i.e. if k is odd then $p_q(\mathfrak{D}(Q))=0$ for $q>0$.

DEFINITION 2.6. If Σ^{4k} is a homotopy $4k$-sphere with a free involution T define $\rho(\Sigma^{4k}/T)\equiv|\hat{\rho}(Q^{4k-1},s)| \pmod 1$ for Q a characteristic homotopy $\mathbf{R}P^{4k-1}$ and spin structure s on Q.

Theorem 2.5 and Proposition 2.2 show that ρ is well defined. However we wish to emphasize that even though we write ρ as a rational number (mod 1), *ρ is really only defined up to the involution of* $\mathbf{Q}/\mathbf{Z}$ *given by absolute value.*

We shall need another invariant which is related to ρ. Consider the portion of the surgery exact sequence:

$$\mathcal{S}(\mathbf{R}P^{4k-1}) \xrightarrow{\eta} [\mathbf{R}P^{4k-1}, G/0] \xrightarrow{\sigma} L_{4k-1}(\mathbf{Z}_2, +) = \mathbf{Z}_2.$$

Let M^{4k-1} be a smooth closed manifold and $f: M^{4k-1} \to \mathbf{R}P^{4k-1}$ a normal map. Then f is covered by a bundle map $\nu(M^{4k-1}) \to \nu(\mathbf{R}P^{4k-1})$ of stable normal bundles ([1]). Let r_0 and r_1 denote the two distinct spin structures on $\mathbf{R}P^{4k-1}$. If f is degree 1 it induces an injection on cohomology; so r_0 and r_1 induce distinct spin structures s_0 and s_1 on M.

DEFINITION 2.7. The *difference invariant* of a degree 1 normal map $f: M \to \mathbf{R}P^{4k-1}$ is $d(M) \equiv |\mu(M, s_0) - \mu(M, s_1)|$ (mod 1).

Of course $d(M)$ is also defined only up to the involution of $\mathbf{Q}/\mathbf{Z}$ by absolute value. More apt notation for $d(M)$ would include the normal map f; however this will usually be understood. In any case if f is a homotopy equivalence $M \to \mathbf{R}P^{4k-1}$ it is unique up to homotopy ([3], (5.1)).

PROPOSITION 2.8. *Let Σ^{4k} be a homotopy 4k-sphere ($k \geqslant 2$) with a free involution T and let Q^{4k-1} be any characteristic homotopy $\mathbf{R}P^{4k-1}$ for Σ/T. Then $2\rho(\Sigma/T) \equiv d(Q)$ (mod 1).*

Proof. This follows immediately from the definitions and Theorem 2.5. □

PROPOSITION 2.9. *Let $f_i: P_i^{4k-1} \to \mathbf{R}P^{4k-1}$, $i = 0, 1$, be normally cobordant degree 1 normal maps (and $k \geqslant 2$). Then $d(P_0) \equiv d(P_1)$ (mod 1).*

Proof. Let $F: W^{4k} \to \mathbf{R}P^{4k-1}$ be a normal cobordism between f_0 and f_1. We have seen above that F is covered by a bundle monomorphism $\nu(W) \to \nu(\mathbf{R}P^{4k-1})$, hence W is a spin manifold. Let S_0, S_1 be the spin structures on W induced by r_0 and r_1. So by definition, $S_j \mid P_i^{4k-1} = S_j^{(i)}$, the spin structure on P_i induced from r_j. Using W to compute μ-invariants we have

$$\mu(\partial W, S_0 \mid \partial W) \equiv \mu(\partial W, S_1 \mid \partial W) \text{ (mod 1)}.$$

So $\mu(P_1, s_0^{(1)}) - \mu(P_0, s_0^{(0)}) \equiv \mu(P_1, s_1^{(1)}) - \mu(P_0, s_1^{(0)})$ (mod 1) and the result follows. □

3. Suspending normal invariants. In §2 we defined an invariant for smooth involutions on homotopy 4k-spheres and then showed in Proposition 2.9 that twice this invariant is an invariant of the normal cobordism class of the orbit space of any characteristic sphere. Our goal, then, is to realize normal cobordism classes of homotopy $\mathbf{R}P^{4k-1}$'s whose double cover is S^{4k-1}, distinguish them by the invariant d of §2, and for each such normal cobordism class construct two smooth free involutions on a homotopy 4k-sphere distinguished by the invariant ρ. In this section we make this process precise.

Consider the surgery exact sequences

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{S}(\mathbf{R}P^{4k}) & \xrightarrow{\ \eta\ } & [\mathbf{R}P^{4k}, G/0] & \xrightarrow{\ \sigma\ } & \mathbf{Z}_2 \\
 & & & \searrow^{\delta} & \downarrow i_* & & \\
\mathbf{Z}\oplus\mathbf{Z} & \longrightarrow & \mathcal{S}(\mathbf{R}P^{4k-1}) & \xrightarrow{\ \eta\ } & [\mathbf{R}P^{4k-1}, G/0] & \xrightarrow{\ \sigma\ } & \mathbf{Z}_2
\end{array}$$

where i_* is induced by the inclusion $i\colon \mathbf{R}P^{4k-1} \to \mathbf{R}P^{4k}$ and where $\delta = i_*\eta$.

DEFINITION 3.1. Let $N_{4k-1} \subset [\mathbf{R}P^{4k-1}, G/0]$ be the subset consisting of those elements which are represented by a homotopy $\mathbf{R}P^{4k-1}$ double covered by S^{4k-1}.

LEMMA 3.2. *The image of* $\delta = i_*\eta$ *is exactly* N_{4k-1}.

Proof. Let P^{4k} be a homotopy $\mathbf{R}P^{4k}$. Then P^{4k} has a characteristic homotopy $\mathbf{R}P^{4k-1}$, say Q^{4k-1}, unique up to characteristic cobordism (see [13]), hence up to normal cobordism. Since the double cover $\tilde{Q}$ of Q is characteristic in $\tilde{P}$, the double cover of P, $\tilde{Q}$ bounds a contractible submanifold of $\tilde{P}$; so $\tilde{Q} \cong S^{4k-1}$. But $i_*\eta(P) = \eta(Q)$.

Conversely, if $\eta(Q) \in [\mathbf{R}P^{4k-1}, G/0]$ where $\tilde{Q} \cong S^{4k-1}$ then any suspension P^{4k} of Q has the property that $i_*\eta(P) = \eta(Q)$. □

LEMMA 3.3. *Let* Σ^{4k-1} *be the nontrivial element of* bP_{4k} *with* $2\Sigma = 0$. *Then* $\mu(\Sigma) = 1/2$.

Proof. The group bP_{4k} of homotopy $(4k-1)$-spheres which bound parallelizable manifolds is a cyclic group of order $a_k 2^{2k-2}(2^{2k-1}-1)N$ where $N =$ numerator$(B_k/4k)$, B_k the kth Bernoulli number, and $N \equiv 1 \pmod 2$. (See [11].) A generator $\Sigma_0 \in bP_{4k}$ has $\mu(\Sigma_0) \equiv 1/(2^{2k-2}(2^{2k-1}-1)a_k) \pmod 1$. Since $2\Sigma = 0$, $\Sigma = a_k 2^{2k-3}(2^{2k-1}-1)N\Sigma_0$; so $\mu(\Sigma) \equiv N/2 \equiv 1/2 \pmod 1$. □

Now consider an element of N_{4k-1} represented by Q^{4k-1}, a homotopy $\mathbf{R}P^{4k-1}$ with $\tilde{Q}$ diffeomorphic to S^{4k-1}. We would like to find two distinct homotopy $\mathbf{R}P^{4k}$'s, say P and P', such that $\delta(P) = \delta(P') = \eta(Q)$. To do this we let $Q' = Q \# \Sigma$ where Σ is the homotopy sphere of order 2 in bP_{4k}. Then $\tilde{Q}' \cong S^{4k-1}$ and Q' is normally cobordant to Q. Let P^{4k} and P'^{4k} be suspensions of Q and Q'. Then $\delta(P) = \eta(Q) = \eta(Q') = \delta(P')$. But by Lemma 3.3 $\rho(P') \equiv \rho(P) + 1/2 \pmod 1$. Since ρ is well defined up to absolute value (mod 1) this means that P and P' are distinct provided $\rho(P)$ is not congruent to $1/4$ (or $3/4$) (mod 1). It is easily seen that $\rho(P) = 1/4$ if and only if $d(Q) = 1/2$. Hence we have:

THEOREM 3.4. *Let* $k \geqslant 2$ *and let* $\eta(Q^{4k-1}) \in N_{4k-1}$. *If* $d(Q) \neq 1/2$ *then* $\delta^{-1}(\eta(Q))$ *contains at least two distinct* $\mathbf{R}P^{4k}$*'s distinguished by their* ρ*-invariants.* □

We know of no example of a homotopy $\mathbf{R}P^{4k-1}$ with difference invariant $1/2$.

LEMMA 3.5. *If* P_0^{4k} *and* P_1^{4k} *are homotopy equivalent to* $\mathbf{R}P^{4k}$ *and if* P_0 *and* P_1 *have characteristic homotopy* $\mathbf{R}P^{4k-1}$*'s which are diffeomorphic, then* $P_1 \cong P_0 \# \Sigma$ *for some* $\Sigma \in \theta^{4k}$.

Proof. Let Q^{4k-1} be the common characteristic homotopy $\mathbf{R}P^{4k-1}$. Then $P_i = D^{4k} \cup_{g_i}$ Mapping cylinder$(S^{4k-1} \to Q^{4k-1})$, $i=0,1$; so $P_1 \cong P_0 \# \Sigma$ where Σ corresponds to $g_1 g_0^{-1} \in \Gamma_{4k}$. □

Let P^{4k} be a homotopy $\mathbf{R}P^{4k}$ and let $I(P^{4k})$ denote the subset of θ^{4k} consisting of those homotopy 4k-spheres Σ with $P^{4k} \# \Sigma \cong P^{4k}$; i.e. $I(P^{4k})$ is the *inertia group* of P^{4k}.

THEOREM 3.6. *Let* $k \geqslant 2$ *and* $P_0^{4k} \in \mathcal{S}(\mathbf{R}P^{4k})$ *with* $\rho(P_0) \not\equiv 1/4$. *Let* $\Pi = \{P \in \mathcal{S}(\mathbf{R}P^{4k}) \mid 2\rho(P) \equiv 2\rho(P_0) \pmod 1\}$. *Then* $\delta^{-1}(\delta(P))$ *contains exactly two elements for each* $P \in \Pi$ *if and only if* $I(P) = \theta^{4k}$ *for each* $P \in \Pi$.

Proof. Suppose that $I(P) = \theta^{4k}$ for every $P \in \mathcal{S}(\mathbf{R}P^{4k})$ satisfying $2\rho(P) \equiv 2\rho(P_0) \pmod 1$. Let Q_0 be a characteristic $\mathbf{R}P^{4k-1}$ for P_0. In the proof of Theorem 3.4 we constructed $P_1 \in \mathcal{S}(\mathbf{R}P^{4k})$ with $\rho(P_1) \equiv \rho(P_0) + 1/2 \pmod 1$ and $\delta(P_1) = \delta(P_0) = \eta(Q_0)$. Let P_2 be another homotopy $\mathbf{R}P^{4k}$ with $\delta(P_2) = \eta(Q_0)$. Then $\rho(P_2) = \rho(P_0)$ or $\rho(P_1)$ since $2\rho(P_2)$ is a normal cobordism invariant of the characteristic homotopy $\mathbf{R}P^{4k-1}$ by (2.8) and (2.9). Say $\rho(P_2) = \rho(P_0)$.

Let Q_2 be a characteristic homotopy $\mathbf{R}P^{4k-1}$ for P_2; then $\eta(Q_2) = \eta(Q_0)$. By ([13], V.3) we have $\tilde{Q}_2 \cong \tilde{Q}_0 \# m\Sigma_0$ where Σ_0 is a generator of bP_{4k} and

$$m \equiv 1/8(\alpha(Q_2) - \alpha(Q_0)) \pmod 2.$$

But both $\tilde{Q}_2$ and $\tilde{Q}_0$ are diffeomorphic to S^{4k-1}; so $m \equiv 0 \pmod{|bP_{4k}|}$ and thus $m \equiv 0 \pmod 2$. So $\alpha(Q_2)/8 \equiv \alpha(Q_0)/8 \pmod 2$, and thus by ([13], IV.4.2) we can find another characteristic homotopy $\mathbf{R}P^{4k-1}, Q_2'$, for P_2 with $\alpha(Q_2') = \alpha(Q_0)$. Since Q_2' and Q_0 are normally cobordant and have the same α-invariant, $Q_2' \cong Q_0 \# \Sigma$ for some $\Sigma \in bP_{4k}$ (see [13], IV.3, Theorem 2′).

Since $\rho(P_2) \equiv \rho(P_0) \pmod 1$ and $\alpha(Q_2') = \alpha(Q_0)$, we have $\mu(Q_2', s_2) \equiv \mu(Q_0, s_0) \pmod 1$ for some spin structures s_0 on Q_0 and s_2 on Q_2'. Now the double covers of Q_0 and $Q_2' \cong Q_0 \# \Sigma$ are both diffeomorphic to S^{4k-1}; so 2Σ is also diffeomorphic to S^{4k-1}. But $\mu(Q_2', s_2) \equiv \mu(Q_0, s_0') + \mu(\Sigma) \pmod 1$. If $s_0' = s_0$ then $\mu(\Sigma) \equiv 0 \pmod 1$ so $\Sigma \cong S^{4k-1}$ by (3.3). If $s_0' \neq s_0$ then $d(Q_0) \equiv |\mu(Q_0, s_0') - \mu(Q_0, s_0)| \equiv \mu(\Sigma) \pmod 1$. But our hypothesis is that $\rho(P_0) \not\equiv 1/4$ or equivalently that $d(Q_0) \not\equiv 1/2$. So (3.3) again implies that $\Sigma \cong S^{4k-1}$. Thus $Q_2' \cong Q_0$; so Lemma 3.5 implies that $P_2 \cong P_0 \# M^{4k}$ for some $M^{4k} \in \theta^{4k}$. By hypothesis $I(P_0) = \theta^{4k}$ so we have $P_2 \cong P_0$. This shows that $\delta^{-1}(\delta(P_0)) = \{P_0, P_1\}$ as desired.

Conversely, suppose that $\delta^{-1}(\delta(P))$ contains exactly two elements for each $P \in \mathcal{S}(\mathbf{R}P^{4k})$ such that $2\rho(P) \equiv 2\rho(P_0) \pmod 1$. Consider such a P. If $M^{4k} \in \theta^{4k}$ then $\rho(P) = \rho(P \# M)$. Our construction in Theorem 3.4 gives a $P' \in \mathcal{S}(\mathbf{R}P^{4k})$ with $\rho(P') \equiv \rho(P) + 1/2 \pmod 2$ and $\delta(P') = \delta(P)$. So $\delta^{-1}(\delta(P)) = \{P, P'\}$. Since a characteristic homotopy $\mathbf{R}P^{4k-1}$ for $P \# M$ is also one for P we have $\delta(P \# M) = \delta(P)$. Hence $P \# M$ is diffeomorphic to P or P'. But $\rho(P) \equiv \rho(P \# M) \not\equiv \rho(P')$. So $P \# M \cong P$ and $M \in I(P)$. □

4. Realizing elements of N_{4k-1}. In Appendix I we show that $d(\mathbf{R}P^{4k-1}) = 1/(a_k 2^{2k}) \pmod 1$ where $a_k = 4/(3 + (-1)^k)$.

THEOREM 4.1. *There are at least* $a_k 2^{2k-3}$ *distinct elements of* N_{4k-1} (*each with difference invariant* $\neq 1/2$).

Proof. The indicated number of elements will be obtained by taking multiples of $\mathbf{R}P^{4k-1}$ and then determining which multiples are normally cobordant to homotopy equivalences using [3].

Following Browder [3], let $n=2s+1$ and let $\mathbf{R}P^{4k-1}(n)=n\mathbf{R}P^{4k-1}\cup -sS^{4k-1}$ and let $f_n: \mathbf{R}P^{4k-1}(n)\to \mathbf{R}P^{4k-1}$ be the degree 1 normal map which is the identity on each copy of $\mathbf{R}P^{4k-1}$ and the double covering map on each copy of S^{4k-1}. It is shown in ([3], Theorem 4.8) that f_n is normally cobordant to a homotopy equivalence $g_n: M_n \to \mathbf{R}P^{4k-1}$ if and only if $n\equiv \pm 1 \pmod 8$. Furthermore in the proof of ([3], Prop. 4.18) it is shown that the double cover $\tilde{M}_n \in bP_{4k}$. Now $d(M_n)=d(\mathbf{R}P^{4k-1}(n))\equiv |n/(a_k 2^{2k})| \pmod 1$; so (taking into account the involution of $\mathbf{Q}/\mathbf{Z}$ by the absolute value function) there are $a_k 2^{2k-3}$ such M_n with distinct difference invariants $d(M_n)$, and $d(M_n)\neq 1/2$ for each such n. It remains to show that we can find homotopy projective spaces Q_n^{4k-1} normally cobordant to M_n with double covers $\tilde{Q}_n$ diffeomorphic to S^{4k-1}.

If $\tilde{M}_n\in 2bP_{4k}$ let $M_n'=M_n$. Otherwise let M_n' be a homotopy $\mathbf{R}P^{4k-1}$ normally cobordant to M_n with $\alpha(M_n')=\alpha(M_n)+8$ (Theorem V.2.1 of [13]). Then by ([13], V.3) $\tilde{M}_n'\cong \tilde{M}_n \# r\Sigma_0$ where Σ_0 generates bP_{4k} and r is odd. So in either case $\tilde{M}_n'\cong 2m\Sigma_0$ for some integer m. Let $Q_n^{4k-1}=M_n'\# -m\Sigma_0$. Then Q_n is normally cobordant to M_n and $\tilde{Q}_n\cong S^{4k-1}$. The Q_n^{4k-1}, $n\equiv \pm 1 \pmod 8$ are $a_k 2^{2k-3}$ distinct elements of N_{4k-1} distinguished by their difference invariants (none of which is 1/2). □

THEOREM 4.2. *Suppose that there exists a degree* 1 *normal map*

$$f: M^{4k-2}\to S^{4k-2}$$

with nonzero surgery obstruction, e.g. for $k=2,4,8,$ *or* 16. *Then there are at least* $a_k 2^{2k-2}$ *distinct elements of* N_{4k-1} *distinguished by their difference invariants* (*all* $\neq 1/2$).

Proof. Consider

$$id\# f: \mathbf{C}P^{2k-1}\# M^{4k-2}\to \mathbf{C}P^{2k-1}\# S^{4k-2}=\mathbf{C}P^{2k-1}.$$

Let N^{4k-1} be the total space of the pull back of the Euler class 2 S^1-bundle over $\mathbf{C}P^{2k-1}$. We then obtain a degree 1 normal map $g: N\to \mathbf{R}P^{4k-1}$ since $\mathbf{R}P^{4k-1}$ is the total space of the Euler class 2 S^1-bundle over $\mathbf{C}P^{2k-1}$. It is easy to see that a transverse preimage of $\mathbf{R}P^{4k-2}$ is $id\# f: \mathbf{R}P^{4k-2}\# M\to \mathbf{R}P^{4k-2}$; so g has nonzero surgery obstruction. We claim that N admits an orientation-reversing diffeomorphism. For $N=S^{4k-3}\times_{\mathbf{Z}_2} D^2\cup (M-D^{4k-2})\times S^1$ with gluing map

$$\partial(S^{4k-3}\times_{\mathbf{Z}_2} D^2)=S^{4k-3}\times S^1 \xrightarrow{r\times id} S^{4k-3}\times S^1$$

where r is reflection in S^{4k-4} and $\mathbf{Z}_2$ acts on $S^{4k-3}\times D^2$ by antipodal $\times$ antipodal. (Note that $S^{4k-3}\times_{\mathbf{Z}_2} D^2$ is the tubular neighborhood of $\mathbf{R}P^{4k-3}$ in $\mathbf{R}P^{4k-1}$.) The orientation-reversing diffeomorphism of N is given by $id\times_{\mathbf{Z}_2}$ reflection on $S^{4k-3}\times_{\mathbf{Z}_2} D^2$ and $id\times$ reflection on $(M-D^{4k-2})\times S^1$.

Now let $E_{\pm 2}$ be the total spaces of the Euler class ± 2 D^2-bundles over $\mathbf{C}P^{2k-1}$ and let $W_{\pm 2}$ be the total spaces of their pull backs over $\mathbf{C}P^{2k-1}\#M$. Then $N=\partial W_2$ and $N\cong -N=\partial W_{-2}$. Since $id\#f$ is a degree 1 stable tangential map, it is covered by degree 1 stable tangential maps $W_{\pm 2}\to E_{\pm 2}$. A Mayer–Vietoris argument shows that $\sigma(W_{\pm 2})=\sigma(E_{\pm 2})$. Since Pontryagin classes pull back, $N_k(W_{\pm 2})=N_k(E_{\pm 2})$. The two μ-invariants of $\mathbf{R}P^{4k-1}$ are computed from $E_{\pm 2}$; so it follows that $d(N)=d(\mathbf{R}P^{4k-1})$.

Let $n=2s+1$ and let $N(n)=nN\cup -s\tilde{N}$ and $g_n: N(n)\to \mathbf{R}P^{4k-1}$ the degree 1 normal map which is g on each copy of N and on each copy of $\tilde{N}$ is the composition

$$\tilde{N}\to N\xrightarrow{g}\mathbf{R}P^{4k-1}.$$

Again Browder ([3], Theorem 4.8) shows that g_n is normally cobordant to a homotopy equivalence $h_n: P_n^{4k-1}\to \mathbf{R}P^{4k-1}$ if and only if $n\equiv \pm 3 \pmod 8$. Also $d(P_n)\equiv d(N(n))\equiv n/(a_k 2^{2k}) \pmod 1$. As in the proof of (4.1) we can assume that the double covers of P_n, $n\equiv \pm 3 \pmod 8$, are diffeomorphic to S^{4k-1}. The P_n yield $a_k 2^{2k-3}$ distinct elements of N_{4k-1} which the difference invariant shows are different from those obtained in (4.1). □

Combining (3.4), (4.1), and (4.2) (and [7] or [5] for $k=1$) we have:

COROLLARY 4.3. (Compare [8] and [12].) *There are at least $a_k 2^{2k-2}$ distinct smooth homotopy $\mathbf{R}P^{4k}$'s where $a_k=4/(3+(-1)^k)$. If $k=2,4,8,$ or 16 there are at least $a_k 2^{2k-1}$ distinct smooth homotopy $\mathbf{R}P^{4k}$'s.* □

Appendix I. Computation of $d(\mathbf{R}P^{4k-1})$. In this appendix we shall show that $d(\mathbf{R}P^{4k-1})\equiv 1/(a_k 2^{2k}) \pmod 1$ where $a_k=4/(3+(-1)^k)$.

Let W be the total space of the Euler class 2 D^2-bundle ξ over $\mathbf{C}P^{2k-1}$. Then $\partial W=\mathbf{R}P^{4k-1}$. Let s be the spin structure on $\mathbf{R}P^{4k-1}$ induced by the unique spin structure on W, and let j: $(W,\phi)\to (W,\partial W)$ be the inclusion. Since $\sigma(W)=1$ we have $\mu(\mathbf{R}P^{4k-1},s)\equiv (N_k(W)+t_k)/a_k \pmod 1$ (see §2). It is easy to check (e.g. in [9]) that the coefficient of p_k in $(\hat{A}_k-t_kL_k)(p_1,\ldots,p_{k-1},p_k)$ is 0. Hence for any p_k,

$$(\hat{A}_k-t_kL_k)(p_1,\ldots,p_{k-1},p_k)=(\hat{A}_k-t_kL_k)(p_1,\ldots,p_{k-1},0)=N_k(p_1,\ldots,p_{k-1}).$$

Now $p(W)=p(\mathbf{C}P^{2k-1})p(\xi)=(1+a^2)^{2k}(1+4a^2)$ where a is the image in $H^2(W;\mathbf{Q})$ of a generator of $H^2(W;\mathbf{Z})\approx H^2(\mathbf{C}P^{2k-1};\mathbf{Z})$. So $j^{*-1}p(W)=(1+(b/2)^2)^{2k}(1+b^2)$ where b is a generator of $H^2(W,\partial W;\mathbf{Q})$ such that $j^*(b)=2a$. Since $\hat{A}$ is the multiplicative sequence associated with the function

$$f(z)=\tfrac{1}{2}z^{1/2}/\sinh(\tfrac{1}{2}z^{1/2})$$

we obtain

$$\hat{A}_k(j^{*-1}p_1(W),\ldots,j^{*-1}p_{k-1}(W),p_k)=\left(\frac{b/4}{\sinh(b/4)}\right)^{2k}\left(\frac{b/2}{\sinh(b/2)}\right)$$

where p_k is the $4k$-component of $(1+(b/2)^2)^{2k}(1+b^2)$.

But if $x \in H^p(W, \partial W; \mathbf{Q})$ and $y \in H^{4k-p}(W, \partial W; \mathbf{Q})$, by definition

$$\langle x \cup y, [W, \partial W] \rangle = \langle x \cup j^*(y), [W, \partial W] \rangle.$$

It follows that $\langle \hat{A}_k(j^{*-1}p_1(W), \ldots, j^{*-1}p_{k-1}(W), p_k), [W, \partial W] \rangle$ is the coefficient of z^{2k} in $(z/(2\sinh(z/2)))^{2k}(z/2\sinh z)$. By the Cauchy integral formula this is obtained by dividing by $2\pi i z^{2k+1}$ and integrating around the origin. Hence

$$\begin{aligned}
&\langle \hat{A}_k(j^{*-1}p_1(W), \ldots, j^{*-1}p_{k-1}(W), p_k), [W, \partial W] \rangle \\
&= \frac{1}{2^{2k+1}} \frac{1}{2\pi i} \oint \frac{dz}{\sinh^{2k}(z/2)\sinh z} \\
&= \frac{1}{2^{2k+2}} \frac{1}{2\pi i} \oint \frac{dz}{\sinh^{2k+1}(z/2)\cosh(z/2)} \\
&= \frac{1}{2^{2k+1}} \frac{1}{2\pi i} \oint \frac{du}{u^{2k+1}(1+u^2)} \quad \text{(substituting } u = \sinh(z/2)) \\
&= \frac{1}{2^{2k+1}} \frac{1}{2\pi i} \oint \frac{1 - u^2 + u^4 - u^6 + - \cdots}{u^{2k+1}} du \\
&= (-1)^k / 2^{2k+1}.
\end{aligned}$$

Since L is the multiplicative sequence associated with $z^{1/2}/\tanh(z^{1/2})$ we obtain

$$L_k(j^{*-1}p_1(W), \ldots, j^{*-1}p_{k-1}(W), p_k) = \left(\frac{b/2}{\tanh(b/2)}\right)^{2k} \left(\frac{b}{\tanh b}\right),$$

and as above $\langle L_k(j^{*-1}p_1(W), \ldots, j^{*-1}p_{k-1}(W), p_k), [W, \partial W] \rangle$ is the coefficient of z^{2k} in $(z/\tanh z)^{2k}(z/\tanh(2z))$. Thus

$$\begin{aligned}
&\langle L_k(j^{*-1}p_1(W), \ldots, j^{*-1}p_{k-1}(W), p_k), [W, \partial W] \rangle \\
&= \frac{1}{2\pi i} \oint \frac{dz}{\tanh^{2k} z \tanh(2z)} = \frac{1}{2\pi i} \oint \frac{1 + \tanh^2 z}{2\tanh^{2k+1} z} \\
&= \frac{1}{2\pi i} \oint \frac{1+u^2}{2u^{2k+1}(1-u^2)} du \quad \text{(substituting } u = \tanh z) \\
&= \frac{1}{2\pi i} \oint \frac{-1 + 2/(1-u^2)}{2u^{2k+1}} du \\
&= \frac{1}{2\pi i} \oint \frac{-du}{2u^{2k+1}} + \frac{1}{2\pi i} \oint \frac{du}{u^{2k+1}(1-u^2)} \\
&= 0 + \frac{1}{2\pi i} \oint \frac{1 + u^2 + u^4 + \cdots}{u^{2k+1}} du = 1.
\end{aligned}$$

So $N_k(W) = (-1)^k/2^{2k+1} - t_k$; hence $\mu(\mathbf{R}P^{4k-1}, s) \equiv (-1)^k/(a_k 2^{2k+1}) \pmod 1$. That is

$$\mu(\mathbf{R}P^{4k-1},s) \equiv \begin{cases} 1/2^{2k+1}, & k \text{ even} \\ -1/2^{2k+2}, & k \text{ odd} \end{cases} \pmod 1$$

Since $\mathbf{R}P^{4k-1}$ has an orientation-reversing diffeomorphism its other μ-invariant is $-\mu(\mathbf{R}P^{4k-1},s)$. Thus

$$d(\mathbf{R}P^{4k-1}) \equiv 2|\mu(\mathbf{R}P^{4k-1},s)| \equiv \begin{cases} 1/2^{2k}, & k \text{ even} \\ 1/2^{2k+1}, & k \text{ odd} \end{cases} \pmod 1$$

Appendix II. Explicit computations. In this appendix we shall exhibit explicit constructions of the homotopy $\mathbf{R}P^8$'s and $\mathbf{R}P^{12}$'s. The constructions go as follows. Start with a homotopy $(4k-1)$-sphere Σ^{4k-1} admitting a free circle action; thus we have an S^1-bundle of Euler class 1 $\Sigma^{4k-1} \to X^{4k-2}$ where X^{4k-2} is a homotopy $\mathbf{C}P^{2k-1}$. We consider only those homotopy spheres which are divisible by 2 in θ^{4k-1}; i.e. $\Sigma^{4k-1} = 2\bar{\Sigma}^{4k-1}$. Let $\bar{Q}^{4k-1}$ be the total space of the Euler class 2 S^1-bundle over X^{4k-2}, and let $Q^{4k-1} = \bar{Q} \# -\bar{\Sigma}$. Then $\tilde{Q} \cong S^{4k-1}$; $\mu(Q,s) \equiv \mu(\bar{Q},\bar{s}) - \mu(\bar{\Sigma}) \pmod 1$ and these μ-invariants can be computed if the rational Pontryagin classes of X^{4k-2} are known. To compute $\hat{\rho}(Q,s)$ it remains to compute $\alpha(Q) = \alpha(\bar{Q})$, and this is accomplished by using a theorem of Montgomery and Yang.

THEOREM ([16]). *Suppose S^1 acts freely on the homotopy sphere Σ^{4k-1}. Define $I(\Sigma^{4k-1},S^1) = \sigma(f^{-1}(\mathbf{C}P^{2k-2}))-1$ where $f\colon \Sigma/S^1 \to \mathbf{C}P^{2k-1}$ is an orientation-preserving homotopy equivalence transverse to $\mathbf{C}P^{2k-2}$. Let $Q = \Sigma/\mathbf{Z}_2$, the quotient of Σ by the involution in the S^1-action. Then $\alpha(Q) = I(\Sigma, S^1)$.* □

We pointed out in §4 that there are exactly 8 distinct free involutions on homotopy 8-spheres. We shall now realize all of these as involutions on S^8 with distinct ρ-invariants by using the above program. We use the computation of free S^1-actions on homotopy 7-spheres given in [15]. Homotopy $\mathbf{C}P^3$'s are distinguished by their first Pontryagin classes $p_1(X_i^6) = (24i+4)a^2$ where a is a generator of $H^2(X_i;\mathbf{Z})$. The corresponding $\Sigma_i^7 = (18i^2+4)\Sigma_0 \in \theta^7$ where Σ_0 is the generator of θ^7 with $\mu(\Sigma_0) \equiv 1/28 \pmod 1$.

Let W_i^8 be the total space of the Euler class 2 D^2-bundle over X_i^6, and let Q_i^7 be the homotopy projective space ∂W_i^8. We have $p_1(W) = (24i+8)a^2$. Let $\bar{s}_i$ be the spin structure on $\bar{Q}_i$ inherited from the unique spin structure on W_i. We have

$$\mu(\bar{Q}_i,\bar{s}_i) \equiv \frac{p_1^2[W_i]-\sigma(W_i)}{2^7\cdot 7} \equiv \frac{72i^2+48i+7}{2^7\cdot 7} \pmod 1.$$

Let $Q_i = \bar{Q}_i - (9i^2+2i)\Sigma_0$; so $\tilde{Q}_i \cong S^7$ and

$$\mu(Q_i,s_i) \equiv \frac{32i+7}{224} \pmod 1.$$

An easy computation using the quoted theorem of Montgomery and Yang (see [16], p. 191) yields $\alpha(Q_i) = -8i$. Thus

$$\hat{\rho}(Q_i, s_i) \equiv \mu(Q_i, s_i) + \frac{1}{448}\alpha(Q_i) \equiv \frac{28i+7}{224} \pmod 1;$$

and so $Q_0, \ldots, Q_7$ are homotopy $\mathbf{R}P^7$'s with $\hat{\rho}$-invariants which are distinct in $\mathbf{Q}/\mathbf{Z}$ modulo absolute value. Since there are exactly 8 free involutions on homotopy 8-spheres, we realize them all on S^8 (with distinct ρ-invariants) by suspending these examples.

The construction of homotopy $\mathbf{R}P^{12}$'s starts with the description of Brumfiel of homotopy $\mathbf{C}P^5$'s which are distinguished by their rational Pontryagin classes ([4]; see also [10]):

$$p_1(X^{10}_{m,n}) = (6+24m)a^2 \qquad p_2(X^{10}_{m,n}) = (15+228m^2-456m-1440n)a^4$$

(where m is even). The corresponding homotopy sphere is

$$\Sigma^{11}_{m,n} = \left(\frac{m(32m^2+301)}{3} - 84m^2 - 224mn + 348n\right)\Sigma_0 \in \theta^{11}$$

where $\mu(\Sigma_0) = 1/992 \pmod 1$.

Again we let $W^{12}_{m,n}$ be the Euler class 2 D^2-bundle over $X^{10}_{m,n}$ and $\bar{Q}^{11}_{m,n} = \partial W_{m,n}$. Then

$$p_1(X_{m,n}) = (10+24m)a^2 \quad \text{and} \quad p_2(X_{m,n}) = (288m^2-360m-1440n+39)a^4;$$

so

$$\begin{aligned}\mu(\bar{Q}_{m,n}, \bar{s}) &\equiv \frac{1}{2^{11}\cdot 3\cdot 31}(4p_1p_2[W] - 3p_1^3[W] - 24\sigma(W)) \\ &\equiv -\frac{1}{2^8\cdot 31}(288m^3+1560m^2+672m+2880mn+1200n+31) \pmod 1.\end{aligned}$$

We let

$$Q^{11}_{m,n} = \bar{Q}^{11}_{m,n} \# -\left[\frac{m}{6}(32m^2+301) - 42m^2 - 112mn + 174n\right]\Sigma_0;$$

so $\tilde{Q}_{m,n} \cong S^{11}$ and

$$\begin{aligned}&\mu(Q^{11}_{m,n}, s) \\ &\equiv \frac{1}{2^8\cdot 31}\left[\frac{-992}{3}m^3 - 1224m^2 - \frac{3220}{3}m + 31 - 1984mn - 2592n\right] \pmod 1.\end{aligned}$$

Again an easy computation as in [16] shows $\alpha(Q_{m,n}) = 32m^2 - 80m - 224n$. So

$$\begin{aligned}\hat{\rho}(Q_{m,n}, s) &\equiv \mu(Q_{m,n}, s) - \frac{1}{2^9\cdot 31}\alpha(Q_{m,n}) \\ &\equiv \frac{-1}{256}\left(\frac{32}{3}m^3 + 40m^2 + \frac{100}{3}m - 1 + 64mn + 80n\right) \pmod 1.\end{aligned}$$

Then $Q^{11}_{m,n}$, $m = 0, 2$, $n = 0, \ldots, 15$ are homotopy $\mathbf{R}P^{11}$'s which have $\hat{\rho}$-invariants which are distinct in $\mathbf{Q}/\mathbf{Z}$ modulo absolute value. Their suspensions give 32 free involutions of S^{12} distinguished by their ρ-invariants. Since there are exactly 32 free involutions on homotopy 12-spheres (see §4), this realizes all of them.

REFERENCES

1. J. F. Adams, *On the groups $J(X)$, II,* Topology 3 (1965), 137–171.
2. M. F. Atiyah and I. M. Singer, *The index of elliptic operators, III,* Ann. of Math. (2) 87 (1968), 546–604.
3. W. Browder, *Cobordism invariants, the Kervaire invariant and fixed point free involutions,* Trans. Amer. Math. Soc. 178 (1973), 193–225.
4. G. Brumfiel, *Differentiable S^1-actions on homotopy spheres,* Mimeographed Notes, Berkeley, 1970.
5. S. Cappell and J. Shaneson, *Some new four-manifolds,* Ann. of Math. 104 (1976), 61–72.
6. J. Eells and N. H. Kuiper, *An invariant for certain smooth manifolds,* Ann. Mat. Pura Appl. (4) 60 (1962), 93–110.
7. R. Fintushel and R. J. Stern, *An exotic free involution on S^4,* Ann. of Math. 113 (1981), 357–365.
8. C. H. Giffen, *Smooth homotopy projective spaces,* Bull. Amer. Math. Soc. 75 (1969), 509–513.
9. F. Hirzebruch, *Topological methods in algebraic geometry,* 3rd. ed., Springer, New York, 1966.
10. H. T. Ku and M. C. Ku, *Characteristic spheres of free differentiable actions of S^1 and S^3 on homotopy spheres,* Trans. Amer. Math. Soc. 156 (1971), 493–504.
11. M. Kervaire and J. Milnor, *Groups of homotopy spheres, I,* Ann. of Math. (2) 77 (1963), 504–537.
12. P. Löffler, *Äquivariante Homotopie, äquivarianter Bordismus und frei differenzierbare Involutionen auf Sphären,* Math. Z. 170 (1980), 233–246.
13. S. López de Medrano, *Involutions on Manifolds,* Ergebnisse der Math., 59, Springer, New York, 1971.
14. J. Milnor, *Remarks concerning spin manifolds,* Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse), pp. 55–62, Princeton Univ. Press, Princeton, N.J., 1965.
15. D. Montgomery and C. T. Yang, *Differentiable actions on homotopy seven spheres, II,* Proc. Conf. on Transformation Groups (New Orleans, La., 1967), pp. 125–134, Springer, New York, 1968.
16. ———, *Free differentiable actions on homotopy spheres,* Proc. Conf. on Transformation Groups (New Orleans, La., 1967), pp. 175–192, Springer, New York, 1968.
17. R. E. Stong, *Notes on cobordism theory,* Princeton Math. Notes, Princeton Univ. Press, Princeton, N.J., 1958.
18. C. T. C. Wall, *Surgery on compact manifolds,* London Math. Soc. Monographs, No. 1, Academic Press, London, 1970.

Department of Mathematics
Tulane University
New Orleans, Louisiana 70118

Department of Mathematics
The University of Utah
Salt Lake City, Utah 84112