ON THE SIMULTANEOUS UNIVALENCE OF f AND f'

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1. Introduction. In [1, p. 142, Problem 647] Clunie poses the following problem.

Let $f \in S$. Under the additional assumption that f' is also univalent in $|z|<1$, what can be said about $\max|a_n|$ $(n \geqslant 2)$? The function $z(1-z)^{-1}$ shows that $\max|a_n| \geqslant 1$ for all $n \geqslant 2$.

We solve this problem for the case $n=2$ with the additional assumption, f has real coefficients. We obtain the rather surprising and non-intuitive solution determined by the extremal function given in (1) below. We first solve the problem for the case when f and f' map the unit disk onto convex domains where the expected function is indeed the solution. We then show how this leads into a "natural" candidate (i.e. a variation on the Koëbe function) for the solution to the original problem. In showing why this candidate does not work we show how we were motivated to determine the final solution in the real coefficient case.

Denote the class of functions described in Clunie's problem (i.e. $f \in S$ and f' univalent) by S'. We first show: if $f \in S'$, $f(z)=z+a_2z^2+\cdots$, $f'(z)=1+2a_2g(z)$ and f and g map the disk onto convex domains then $|a_k| \leqslant 4/(3k)$, $k \geqslant 2$ with equality if and only if $a_2=(2/3)e^{i\gamma}$ and $g(z)=z/(1-ze^{i\gamma})$ for some real γ.

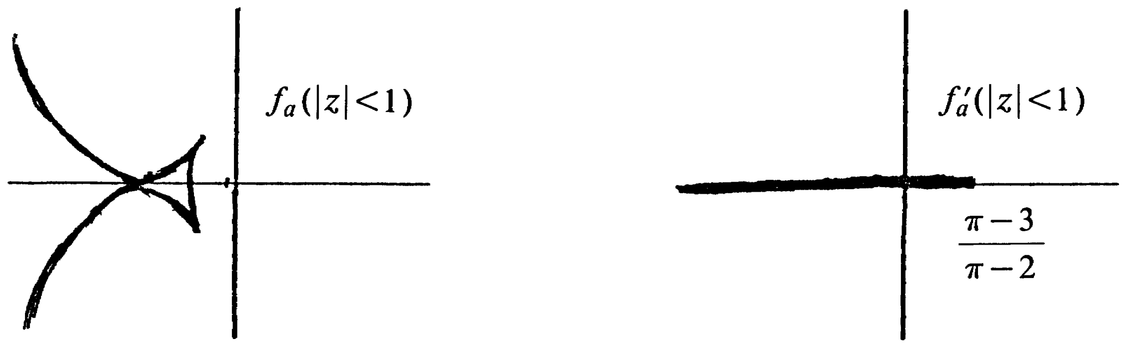
This result is relatively easy to prove as we shall see in Section 2 of this paper. The method is to write $f'(z)=1+2a_2g(z)$, $a_2>0$ when $g(z)=z+b_2z^2+\cdots$ is a convex map of the disk and to show $a_2 \leqslant 2/3$ with equality if and only if $g(z)=z/(1-z)$, the usual extremal function for functionals on the class of normalized convex maps.

By analogy, we are led to suspect that to find $f \in S'$ that maximizes $|a_n|$, we should consider the values of $a>0$ for which the function $f_a(z)=z+az^2+\cdots$ is univalent where $f_a'(z)=1+2az/(1-z)^2$. A detailed analysis of $f_a(|z|=r)$ and the normal to this curve shows that $f_a \in S'$ if and only if $0<a \leqslant 2/(\pi-2) \approx 1.75$. When $a=2/(\pi-2)$, f_a and f_a' map the unit disk to the regions shown roughly in Figure 1. For this value of a, the boundary curve $f_a(|z|=1)$ is tangent to the real axis at the point $f(i)=-4/(\pi-2)(\frac{1}{2}-\log\sqrt{2})$. However, a study of

$$(1-t)f_a(z)+t(z+bz^2), \quad b>0,$$

t small and positive, shows that f_a does not maximize $|a_2|$ in the class S'. Roughly speaking, this variation has the effect of pulling the boundary curve away from the real axis at its point of tangency and the varied function remains in the class S'. The nature of this variation suggests that for the extremal, we should have $\mathrm{Im}[f(e^{i\theta})]=0$ on an arc $(\theta_0<\theta<\theta_1)$. We therefore ask: For what

Received September 9, 1980. Revision received September 10, 1982.
This research was partially supported by the National Science Foundation.
Michigan Math. J. 30 (1983).



$a=2/(\pi-2)$

Figure 1

values of θ can $f \in S'$ satisfy $\operatorname{Im} f(e^{i\theta})=0$? The following argument shows that if $f \in S'$ is analytic at $e^{i\theta_0}$ $(0<\theta_0<\pi/2)$ and f has real coefficients with $a_2>0$ then $\operatorname{Im} f(e^{i\theta_0})>0$. Assume otherwise, i.e. assume $f(e^{i\theta})=u(\theta)+iv(\theta)$, $v(\theta_0)=0$. Then $v'(\theta_0)=0$ because f is typically real. Further, $f'(e^{i\theta})=e^{-i\theta}(-iu'(\theta)+v'(\theta))$ so that $\operatorname{Im}[f'(e^{i\theta_0})]=-(\cos\theta_0)u'(\theta_0)$. Thus we conclude $u'(\theta_0)\leqslant 0$ because f' is typically real. Since f is conformal, it follows that the sector $\theta_0-\epsilon<\arg z<\theta_0+\epsilon$, $0<|z|<1$, maps to a region that has nonempty intersection with the lower half plane when $\epsilon>0$ is sufficiently small. This contradicts the fact that f is typically real.

The above remarks suggest that we should consider a function $f(z)=z+a_2z^2+\cdots \in S'$ with the properties: $\operatorname{Im} f(e^{i\theta})=0$, $\pi/2<\theta<\pi$ and $\operatorname{Im} f'(e^{i\theta})=0$, $0<\theta<\pi/2$. Thus f and f' would map the unit disk to regions similar to those shown in Figure 2. We can construct such a function inferring the properties of $\operatorname{Re}[zf''(z)/f'(z)]$ on $|z|=1$. The result is that

$$f'(z)=\frac{(1+z)}{(1-z)^2}\exp\left[\int_0^z \frac{1}{\pi i w}\log\frac{1+iw}{1-iw}\,dw\right] \tag{1}$$
$$=1+(3+2/\pi)z+\cdots.$$

In this paper, we show that the function $f\in S'$ given by (1) maximizes $|a_2|$ over the subclass of functions in S' that have real coefficients. We also show that this function maximizes $|a_2|$ over the class of normalized f $(f(0)=0, f'(0)=1)$ such that f and f' are typically real.

2. The convex case. In the convex case, we have the following result.

THEOREM 1. *If $f(z)=z+a_2z^2+\cdots$ and $f'(z)=1+2a_2g(z)$ are univalent functions that map the unit disk onto convex regions then $|a_2|\leqslant \frac{2}{3}$ with equality if and only if $g(z)=z/(1-ze^{i\gamma})$ and $a_2=\frac{2}{3}e^{i\gamma}$.*

Proof. Without loss of generality, we may assume $a_2=a>0$. We assume g and f are convex and write

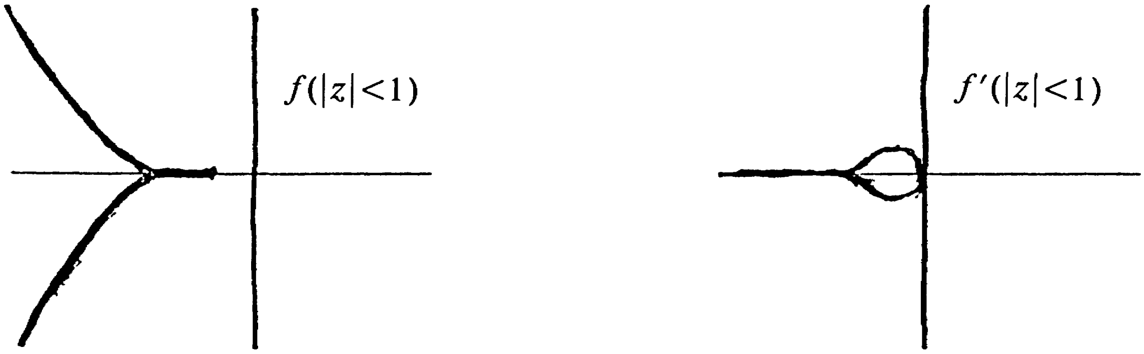


Figure 2

$$0 \leqslant \operatorname{Re}[zf''(z)/f'(z)+1] = \operatorname{Re}[2azg'(z)/(1+2ag(z))+1].$$

This is equivalent to

$$\left|\frac{2azg'(z)}{1+2ag(z)}+2\right| \geqslant \left|\frac{2azg'(z)}{1+2ag(z)}\right|$$

or $|zg'(z)+2g(z)+1/a| \geqslant |zg'(z)|$. Since $zg'(z)+2g(z)=0$ when $z=0$, and $g'(z)\neq 0$, given r, $0<r<1$, there exists z such that $|z|=r$, $zg'(z)+2g(z)<0$ and $1/a-|zg'(z)+2g(z)| \geqslant |zg'(z)|$. Thus $1/a-|g(z)||2+zg'(z)/g(z)| \geqslant |zg'(z)|$. Since

$$|g(z)| \geqslant r/(1+r), \qquad |zg'(z)| \geqslant r/(1+r)^2$$

and

$$\operatorname{Re}(zg'(z)/g(z)) \geqslant 1/(1+r),$$

we conclude

$$1/a - r(3+2r)/(1+r)^2 \geqslant r/(1+r)^2,$$

$$1/a \geqslant (4r+2r^2)/(1+r)^2, \qquad a \leqslant (1+r)^2/(4r+2r^2).$$

Letting $r \to 1$, we conclude $a \leqslant \frac{2}{3}$. Further, it is clear that if $g(z) \neq z/(1-z)$ then $|zg'(z)| \geqslant r(1+\delta)/(1+r)^2$ for some $\delta>0$ for the z such that $zg'(z)+2g(z)<0$, $|z|=r$, hence $a<\frac{2}{3}$. The proof is now completed by observing that

$$\operatorname{Re}\left[\frac{\frac{4}{3}z/(1-z)^2}{1+\frac{4}{3}z/(1-z)}+1\right] = \operatorname{Re}\left[\frac{(3-z)(1+z)}{(3+z)(1-z)}\right] > 0$$

when $|z|<1$ (i.e. $a=\frac{2}{3}$ is possible). □

COROLLARY 1. *If $f\in S'$ and f and f' are convex then $|a_k| \leqslant 4/(3k)$ (where $f(z)=z+a_2z^2+\cdots$). This inequality is sharp.*

Proof. Write $f'(z)=1+2a_2z+3a_3z^2+\cdots=1+2a_2g(z)$. Since the coefficients of g are bounded by 1, $|ka_k| \leqslant |2a_2| \leqslant \frac{4}{3}$ and $|a_k| \leqslant 4/(3k)$ follows. Sharpness is clearly shown by taking $a_2=\frac{2}{3}$ and $g(z)=z/(1-z)$. □

Actually, in Theorem 1 and Corollary 1, one need only assume f is convex and g is starlike of order $\frac{1}{2}$.

COROLLARY 2. *If $f(z)=z+a_2z^2+\cdots$ and $f'(z)=1+2a_2g(z)$ are univalent in the unit disk, $f(|z|<1)$ is convex and g is starlike of order $\frac{1}{2}$ then $|a_k|\leqslant 4/(3k)$ with equality if and only if $a_2=\frac{2}{3}e^{i\gamma}$ and $g(z)=z/(1-ze^{i\gamma})$ for some real γ.*

Proof. We shall show $|g(z)|\geqslant|z|/(1+|z|)$ and $|zg'(z)|\geqslant|z|/(1+|z|)^2$. Since it is also true that $\operatorname{Re}[zg'(z)/g(z)]\geqslant 1/(1+|z|)$, the proof given for Theorem 1 is valid here as well.

By the Herglotz representation, $zg'(z)/g(z)=\int_0^{2\pi}1/(1-ze^{it})\,d\mu(t)$ for some probability measure μ on $[0,2\pi]$. Hence

$$\log[g(z)/z]=\int_0^{2\pi}-\log(1-ze^{it})\,d\mu(t).$$

Since $k(z)=-\log(1-z)$ is a convex map (i.e. $k(|z|<1)$ is convex), $\log(g(z)/z)$ lies in the region $k(|z|<1)$. This clearly implies $\operatorname{Re}(g(z)/z)>\frac{1}{2}$ so $|g(z)|\geqslant|z|/(1+|z|)$. Now $\operatorname{Re}[zg'(z)/g(z)]>\frac{1}{2}$ and hence

$$\frac{1}{1+|z|}\leqslant\operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right)\leqslant\left|\frac{zg'(z)}{g(z)}\right|\leqslant\frac{|zg'(z)|}{|z|/(1+|z|)}.$$

Therefore $|zg'(z)|\geqslant|z|/(1+|z|)^2$ and the proof is complete. □

3. The real coefficient case. We now consider the functions $f(z)=z+a_2z^2+\cdots$, $a_2\neq 0$ such that f is typically real and $f'(z)=1+2a_2g(z)$ where g is typically real. Since f is typically real if and only if the function $-f(-z)$ is typically real, we assume throughout this section that $a_2>0$. Thus, we consider $f(z)=z+a_2z^2+\cdots$ such that $a_2,a_3,\ldots$ are real, $(1/\sin\theta)[\sin\theta+a_2r\sin 2\theta+\cdots]\geqslant 0$ and $(1/\sin\theta)[2a_2\sin\theta+3a_3r\sin 2\theta+\cdots]\geqslant 0$. We shall prove the following theorem.

THEOREM 2. *If f and f' are typically real, $a_2>0$, then $a_2\leqslant\frac{3}{2}+1/\pi$ with equality if and only if $f'(z)$ is given by* (1).

We note that if $f(z)=z+a_2z^2+\cdots$ has real coefficients and is univalent then it is typically real. We shall show that f given by (1) (and the condition $f(0)=0$) is in the class S'. Hence we have the following result.

THEOREM 3. *If $f\in S'$ and all coefficients of f are real, $a_2>0$, then $a_2\leqslant\frac{3}{2}+1/\pi$ with equality if and only if f is given by* (1).

We first assume that f is given by (1) with $f(0)=0$ and show that $f\in S'$ with $f(|z|<1)$ and $f'(|z|<1)$ as shown in Figure 2. Set

$$F(z)=\frac{zf''(z)}{f'(z)}=\frac{z}{1+z}+\frac{2z}{1-z}+\frac{1}{\pi i}\log\left(\frac{1+iz}{1-iz}\right)$$

and note that

$$\frac{\partial \arg f'(z)}{\partial \theta} = \mathrm{Re}(F(z)),$$

$$\frac{\partial \log|f'(z)|}{\partial \theta} = -\mathrm{Im}(F(z)) \quad \text{and} \quad r\frac{\partial \log|f'(z)|}{\partial r} = \mathrm{Re}(F(z)).$$

We will use the principle of the argument to show that f' is univalent in the region $|z|<1$, $\mathrm{Im}\, z>0$ and maps that region into the upper half plane (the intersection of the upper half plane with the region given in Figure 2). It will then follow that f' is univalent in $|z|<1$ and also typically real. From (1), we see that $f'(x)$ is the product of increasing functions for real x and takes $-1<z<1$ onto $0<w<\infty$. When $z=e^{i\theta}$, $0<\theta<\pi$, $\mathrm{Im}(F(z))>0$ so

$$\frac{\partial \log|f'(z)|}{\partial \theta} < 0.$$

Further,

$$\mathrm{Re}(F(e^{i\theta})) = \begin{cases} 0, & 0<\theta<\pi/2 \\ -1, & \pi/2<\theta<\pi \end{cases}$$

hence $\arg f'(e^{i\theta}) = \text{constant}$, $0<\theta<\pi/2$ and $\arg f'(e^{i\theta})$ decreases by $\pi/2$ as θ varies from $\pi/2$ to π. By conformality of f' at -1, we conclude $f'(e^{i\theta})$, $\pi/2<\theta<\pi$ is a curve in the upper half plane tangent to the imaginary axis at $0=f'(-1)$ and that the image of $e^{i\theta}$, $0<\theta<\pi/2$ is the part of the negative real axis from $f'(i)$ (<0) to $-\infty$. Note that $f'(z)$ behaves like $A/(1-z)^2$ near $z=1$ so that $\arg f'(z)$ jumps by π as $z\to 1^-$ along the positive reals and then starts around the semicircle. Thus f' is univalent along the boundary $\{-1\leqslant z\leqslant 1\}\cup\{z=e^{i\theta}: 0<\theta<\pi\}$. It is now easy to see by the argument principle that f' is univalent and typically real in $|z|<1$ and that $f'(|z|<1)$ is as shown in Figure 2.

We now apply a similar argument for $f(z)$. Since $f'(x)>0$, $-1<x<1$, $f(x)=\int_0^x f'(t)\,dt$ maps the segment $-1\leqslant x\leqslant 1$ univalently onto the half line $f(-1)\leqslant w$ (here $-\infty<f(-1)<0$ and $f(1)=\infty$ because f' behaves like $A/(1-z)^2$ near 1). Write $f(e^{i\theta})=u(\theta)+iv(\theta)$ so that $f'(e^{i\theta})=e^{-i\theta}(v'(\theta)-iu'(\theta))$. We know $f'(e^{i\theta})\neq 0$, $0<\theta<\pi$. Further $0>f'(i)=iv'(\pi/2)-u'(\pi/2)$ and hence $v'(\pi/2)=0$. From the mapping $f'(e^{i\theta})$ we conclude $v'(\theta)=0$, $\pi/2<\theta<\pi$ (because $\mathrm{Re}\, f(e^{i\theta})=-1$) so $v(\theta)=v(\pi)=0$ when $\pi/2<\theta<\pi$, $v'(\theta)<0$, $0<\theta<\pi/2$ and $u'(\theta)>0$, $0<\theta<\pi$. Using this information, reasoning as before for f', we conclude f is univalent and typically real.

We now proceed to prove Theorem 2. We require two lemmas.

LEMMA 1. *Suppose $f(z)=z+a_2z^2+\cdots$ maximizes a_2 within the class of normalized functions that are typically real and have typically real first derivative. Then f is analytic on the* arc $z=e^{i\theta}$, $\pi/2<\theta<3\pi/2$ *and* $\mathrm{Im}\, f(e^{i\theta})=0$ *on that arc.*

Proof. By Robertson's representation formula for typically real functions [2], we have

$$f(z)=\int_0^\pi \frac{z}{1-2z\cos t+z^2}\,d\mu(t)$$

and hence

$$f'(z)=\int_0^\pi \frac{1-z^2}{(1-2z\cos t+z^2)^2}\,d\mu(t)$$

where μ is a nondecreasing function on $[0,\pi]$ such that $\mu(\pi)-\mu(0)=1$.

Set

$$\mu_1(t)=\begin{cases}\mu(t), & 0\leqslant t\leqslant \pi/2\\ \mu(\pi/2), & \pi/2\leqslant t\leqslant \pi\end{cases}$$

$$\mu_2(t)=\begin{cases}\mu(\pi/2), & 0\leqslant t\leqslant \pi/2\\ \mu(t), & \pi/2\leqslant t\leqslant \pi\end{cases}.$$

The lemma will clearly follow if $\mu_2(t)$ is constant. Since $a_2=2\int_0^\pi \cos t\,d\mu(t)>0$, μ_1 is not constant (i.e. $\mu_1(\pi/2)>\mu_1(0)$). If $\mu_1(\pi/2)-\mu_1(0)<1$ so that μ_2 is not constant, then clearly

$$g(z)=\int_0^{\pi/2} \frac{z}{1-2z\cos t+z^2}\,d\nu(t)=z+b_2z^2+\cdots$$

is typically real where

$$\nu(t)=\frac{1}{\mu_1(\pi/2)-\mu_1(0)}\mu(t).$$

Further,

$$b_2=2\int_0^{\pi/2}\cos t\,d\nu(t)>2\int_0^{\pi/2}\cos t\,d\mu(t)\geqslant 2\int_0^{\pi}\cos d\mu(t)=a_2.$$

If we can show $g'(z)$ is typically real, we will contradict the maximality of a_2, and the proof of the lemma will be complete. Set

$$k(z,t)=\frac{1-z^2}{(1-2z\cos t+z^2)^2}\quad\text{so that}\quad g'(z)=\int_0^{\pi/2}k(z,t)\,d\nu(t).$$

We wish to show that $\operatorname{Im} k(z,t)\geqslant 0$ when $z=re^{i\theta}$, $0\leqslant r<1$, $\pi/2\leqslant\theta\leqslant\pi$ and $0\leqslant t\leqslant \pi/2$. To that end, consider $k(z,t)$ on the arcs $\Gamma_1=\{e^{i\theta}:\pi/2\leqslant\theta\leqslant\pi\}$, $\Gamma_2=\{-1\leqslant z\leqslant 0\}$ and $\Gamma_3=\{i\rho:0\leqslant\rho\leqslant 1\}$ with t fixed, $0<t<\pi/2$.

On Γ_1,

$$\operatorname{Im} k(e^{i\theta},t)=-\frac{\sin 2\theta}{4(\cos\theta-\cos t)^2}\geqslant 0,$$

on Γ_2, $k(-r,t)\geqslant 0$ and on Γ_3,

$$\operatorname{Im} k(i\rho,t)=\operatorname{Im}\frac{1+\rho^2}{(1-2i\rho\cos t-\rho^2)^2}\geqslant 0$$

since $1-\rho^2-2i\rho\cos t$ is a point in quadrant 4 of the plane. It follows that $\operatorname{Im} k(z,t)\geqslant 0$ when z is a point of quadrant 2 inside the unit disk provided $0\leqslant t\leqslant \pi/2$.

Since $k(-\bar{z},\pi-t)=k(\bar{z},t)=\overline{k(z,t)}$, we see that $\operatorname{Im} k(z,t)\leqslant 0$ when z is a point of quadrant 1 inside the unit disk and $\pi/2\leqslant t\leqslant\pi$. Hence if $0\leqslant\theta\leqslant\pi/2$ we have

$$\begin{aligned} 0\leqslant \operatorname{Im} f'(re^{i\theta}) &= \int_0^{\pi/2}\operatorname{Im} k(re^{i\theta},t)\,d\mu_1(t)+\int_{\pi/2}^{\pi}\operatorname{Im} k(re^{i\theta},t)\,d\mu_2(t)\\ &\leqslant \int_0^{\pi/2}\operatorname{Im} k(re^{i\theta},t)\,d\mu_1(t)\leqslant\int_0^{\pi/2}\operatorname{Im} k(re^{i\theta},t)\,d\nu(t)=\operatorname{Im} g'(re^{i\theta}).\end{aligned}$$

Thus we have proved g' is typically real and the lemma follows. □

LEMMA 2. *If $f(z)=z+a_2z^2+\cdots$ and f' are typically real and if f is analytic on the* arc $z=e^{i\theta}$, $\pi/2<\theta<3\pi/2$ *and* $\operatorname{Im}[f(e^{i\theta})]=0$ *on that arc then the function* $(1+z)\left[\exp\left(-\int_0^z\frac{1}{\pi iw}\log\left(\frac{1+iw}{1-iw}\right)dw\right)\right]\cdot f'(z)$ *is typically real.*

Proof. Set $h(z)=(1+z)\exp\left(-\int_0^z\frac{1}{\pi iw}\log\left(\frac{1+iw}{1-iw}\right)dw\right)$ and $g(z)=h(z)f'(z)$. From the mapping properties of (1), we see that

$$\arg h(e^{i\theta})=\begin{cases}0, & 0<\theta<\pi/2\\ \theta-\pi/2, & \pi/2<\theta<\pi.\end{cases}$$

Thus, $h(e^{i\theta})>0$, $0\leqslant\theta\leqslant\pi/2$ and $0\leqslant\arg h(z)\leqslant\pi/2$ when $0\leqslant\arg z\leqslant\pi$. Since $f(e^{i\theta})=u(\theta)<0$, $\pi/2<\theta<\pi$, $f'(e^{i\theta})=e^{-i\theta}(-iu'(\theta))$, $\arg f'(e^{i\theta})=3\pi/2-\theta$, $\pi/2<\theta<\pi$ so that $\arg g(e^{i\theta})=\pi$, $\pi/2<\theta<\pi$. It is now clear that all boundary points of $g(\{|z|<1\}\cap\{\operatorname{Im} z\geqslant 0\})$ lie in the region $\operatorname{Im} w\geqslant 0$ for if $g(z_n)$ converges to a boundary point with $z_n\to e^{i\theta}$, $0\leqslant\theta\leqslant\pi$ either $g(z_n)\to g(e^{i\theta})$ on the negative real axis (if $\pi/2\leqslant\theta\leqslant\pi$) or $h(z_n)\to h(e^{i\theta})$ on the positive real axis and $f'(z_n)$ converges to a boundary point of $f'(\{|z|<1\}\cap\{\operatorname{Im} z>0\})$. This means that g assumes every value in the lower half-plane the same number of times as z varies in the upper half disk. However

$$0\leqslant\arg g(z)=\arg h(z)+\arg f'(z)\leqslant\pi/2+\pi=3\pi/2$$

and it follows that the image of the upper half disk under the map g is contained in the upper half plane. This completes the proof of the lemma. □

Proof of Theorem 2. Now assume f and f' are typically real, $f(z)=z+a_2z^2+\cdots$ with a_2 maximal. By the lemmas

$$g(z)=(1+z)\left[\exp\left(-\int_0^z\frac{1}{\pi iw}\log\left(\frac{1+iw}{1-iw}\right)dw\right)\right]f'(z)$$

is typically real. We have $g(0)=1$ and $g(x)\geqslant 0$, $-1<x<1$ (otherwise $f'(x)=0$ for some x and f would not be typically real). It follows that $g(z)$ is subordinate to $\left(\frac{1+z}{1-z}\right)^2=1+4z+\cdots$. Therefore, $2a_2+1-2/\pi\leqslant 4$ and Theorem 2 follows.

Since the extremal function for Theorem 2 is in the class S', Theorem 3 is also proved. □

4. Open problems. Our work suggests the following conjectures.

CONJECTURE 1. If $f(z)=z+a_2z^2+\cdots\in S'$, $a_2>0$, then $a_2\leqslant\frac{3}{2}+1/\pi$ with equality if and only if f is given by (1).

CONJECTURE 2. If $f(z)=z+a_2z^2+\cdots$ and $f'(z)$ are close-to-convex, $a_2>0$, and f maximizes a_2 then f has real coefficients and there exists θ_0, $\pi/2<\theta_0<\pi$ such that $\operatorname{Im} f(e^{i\theta})=0$, $\theta_0<\theta<\pi$ and $\operatorname{Im} f'(e^{i\theta})=\beta>0$, $0<\theta<\theta_0$.

CONJECTURE 3. If $f(z)=z+a_2z^2+\cdots$ and $f'(z)=1+2a_2g(z)$ where f and g are starlike (with respect to 0), $a_2>0$ and f maximizes a_2, then f has real coefficients and there exists θ_0, $\pi/2<\theta_0<\pi$ such that $\operatorname{Im} f(e^{i\theta})=0$, $\theta_0<\theta<\pi$ and $\arg g(e^{i\theta})=\text{constant}$, $0<\theta<\theta_0$.

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