AN EXAMPLE RELATED TO THE AFFINE THEOREM OF CASTELNUOVO

Jeffrey Lang

In a recent article, Peter Russell [13] proved the following theorem.

(Affine theorem of Castelnuovo): *Suppose k is perfect. Let $A \subset k^{[2]}$ be a finitely generated, regular k-algebra of dimension 2 such that $\bar{k} \otimes_k A$ is factorial ($\bar{k}$ an algebraic closure of k) and $qt(k^{[2]})/qt(A)$ is a separable extension. Then $A \simeq k^{[2]}$.*

The question that naturally arises is: Does the conclusion of the theorem still hold if we drop the condition that $qt(k^{[2]})/qt(A)$ is a separable extension?

We answer in the negative, by considering the surface $X_1 : z^2 = x(xy+1)^2 + y^3$ over an algebraically closed field k of characteristic two.

The coordinate ring of X_1 is isomorphic to $A = k[x^2, y^2, x(xy+1)^2 + y^3]$. Using methods of P. Samuel we show that A is factorial. All of the other conditions of the theorem are easily seen to be met except that $qt(k^{[2]})/qt(A)$ is a purely inseparable extension. Let $\tilde{X}$ be a smooth projective model of $qt(A)$. We show that the geometric genus $p_g(\tilde{X}) > 0$ which implies that $\tilde{X}$ is not rational, hence A is not isomorphic to $k^{[2]}$.

1. PRELIMINARIES

Let k be an algebraically closed field of characteristic $p > 0$ and $F \subseteq A_k^3$ be a normal affine surface defined by an equation of the form $z^p = G(x,y)$ where $G(x,y) \in k[x,y] \setminus k[x^p, y^p]$. Then the coordinate ring of F is isomorphic to $B = k[x^p, y^p, G]$. (In general this is not a k-isomorphism unless the coefficients of G are in the prime subfield of k).

Let $D: k(x,y) \to k(x,y)$ be the k-derivation defined by $D(x) = \partial G/\partial y$, $D(y) = -\partial G/\partial x$. Let $R = D^{-1}(0) \cap k[x,y]$.

LEMMA 1.1. $R = B$.

Proof. We have that $Dx \neq 0$ or $Dy \neq 0$ since $G(x,y) \notin k[x^p, y^p]$. Thus we have that $k(x^p, y^p) \subsetneqq qt(B) \subsetneqq qt(R) \subsetneqq k(x,y)$. Since $[k(x,y) : k(x^p, y^p)] = p^2$ we see that $qt(B) = qt(R)$. But R is integral over B and B is integrally closed. We conclude that $R = B$.

The following are results of P. Samuel given in his Tata notes (see [14], pp. 61–65).

1.2. Let Cl(B) denote the divisor class group of B. Then Cl(B) is isomorphic to $\mathscr{L}$, the additive group of logarithmic derivatives of D, where

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$$\mathscr{L} = \left\{ \frac{Df}{f} \,\middle|\, f \in k(x,y) \quad \text{and} \quad \frac{Df}{f} \in k[x,y] \right\}.$$

1.3 There exists $a \in B$ such that $D^p = aD$.

1.4. An element $t \in k[x,y]$ is in $\mathscr{L} \Leftrightarrow D^{p-1}(t) - at = -t^p$.

LEMMA 1.5. *If* $t \in \mathscr{L}$, *then* $\deg(t) \leq \deg(G) - 2$.

Proof. Suppose $t \in \mathscr{L}$. Then there exists $f \in k(x,y)$ such that $Df/f = t$. Since $k(x,y)$ is a purely inseparable extension of $qt(B)$ of degree p, there exists $\alpha_i \in qt(B)$, $i = 0, 1, \ldots, p-1$, such that $f = \alpha_0 + \alpha_1 x + \ldots + \alpha_{p-1}x^{p-1}$. Combining the α_i under a common denominator we see that we can write $f = h/g$ for some $h \in k[x,y]$, $g \in B$. Since $Dg = 0$ we obtain $Dh/h = t$. We have that

$$\deg(Dh) \leq \deg(h) + \deg(G) - 2.$$

From this we conclude that $\deg(t) \leq \deg(G) - 2$.

Remark. By bounding the degree of any logarithmic derivative we obtain a method for computing the divisor class group of any normal surface of the form $z^p = G(x,y)$ over an algebraically closed field of characteristic $p > 0$. We write t as a polynomial in x and y of degree $= \deg(G) - 2$ with undetermined coefficients. We then substitute t into our differential equation (1.4) and compare coefficients. The number of solutions will be the order of the class group. Since $\mathscr{L} \subseteq k[x,y]$, each element in the class group will have p-torsion. From this it can be shown that the class group is a finite p-group of type $(p,p,\ldots,p)$ (see [7] and [10]).

2. FACTORIALITY

We now restrict our attention to the case where the characteristic of k is two.

We then are considering normal surfaces of the form $z^2 = G(x,y)$. We now compute a in (1.3).

LEMMA 2.1. $a = G_{xy}$.

Proof. As usual we can assume $G_x \neq 0$. Then $D(y) = G_x$ and $D^2(y) = G_{xy}G_x$. Thus $a = D^2(y)/D(y) = G_{xy}$.

Remark. This implies that in characteristic two that $G_{xy} \in \mathscr{L}$. From this we conclude that for a generic choice of G, the surface $z^2 = G(x,y)$ has nontrivial divisor class group. For if $G_{xy} \neq 0$, then $\mathscr{L} \neq 0$.

Thus when the characteristic is two our differential equation (1.4) becomes

2.2. $Dt + at = t^2$.

We can rewrite this equation as

2.3. $t_x G_y + t_y G_x + G_{xy} t = t^2$.

Differentiating both sides of (2.3) with respect to x(resp., y) we obtain $t_{xy}G_x = 0$ (resp., $t_{xy}G_y = 0$).

Therefore, we have that

2.4. $t \in \mathscr{L} \Rightarrow t_{xy} = 0$.

Let us now calculate Cl(A) where $A = k[x^2, y^2, x(xy+1)^2 + y^3]$. By (1.2) Cl($A$) $\cong \mathscr{L}$. Let $t \in \mathscr{L}$. By (1.5) $\deg(t) \leq 3$ and by (2.4) $t_{xy} = 0$. Thus t must be of the form

2.5 $t = (\alpha_{00} + \alpha_{20}x^2 + \alpha_{02}y^2) + (\alpha_{10} + \alpha_{30}x^2 + \alpha_{12}y^2)x + (\alpha_{01} + \alpha_{03}y^2 + \alpha_{21}x^2)y$

for some $\alpha_{ij} \in k$.

Since $\partial^2/\partial x \partial y\ (x(xy+1)^2 + y^3) = 0$ our equation (2.3) becomes

2.6 $t_x y^2 + t_y(xy+1)^2 = t^2$.

Substituting the expression for t in (2.5) into equation (2.6) we obtain

2.7. $(\alpha_{10} + \alpha_{30}x^2 + \alpha_{12}y^2)y^2 + (\alpha_{01} + \alpha_{03}y^2 + \alpha_{22}x^2)(xy+1)^2$
$= \alpha_{00}^2 + \alpha_{20}^2x^4 + \alpha_{02}^2y^4 + \alpha_{10}^2x^2 + \alpha_{30}^2x^6 + \alpha_{12}^2x^2y^4 + \alpha_{01}^2y^2 + \alpha_{03}^2y^6 + \alpha_{21}^2x^4y^2$.

Comparing coefficients in (2.7) we see that each $\alpha_{ij} = 0$. This implies that $t = 0$. Hence $\mathscr{L} = 0$. Therefore A is factorial.

3. NON RATIONALITY

It remains to show that A is not isomorphic to $k^{[2]}$. We will accomplish this by showing that X_1 is not rational. We do this in the following steps.

Step 1. We make X_1 an affine piece of a projective k-scheme X.

Step 2. We define a double differential σ on X.

Step 3. We resolve X to obtain a smooth projective surface $\tilde{X}$, birational to X_1, and show that σ lifts to a nonzero regular differential $\tilde{\sigma}$ on $\tilde{X}$.

It then follows that $p_g(\tilde{X}) > 0$ and that X_1 is not rational.

Step 1. Let X_2 be the surface in A_k^3 defined by the equation $w^2 = u^3v + uv^5 + v^3$ and X_3 be the surface defined by the equation $t^2 = r^2s + s^5 + r^3s^3$. We then glue X_1, X_2, and X_3 together in the following way.

Let U_{12}(resp., U_{13}) be the open subset of X_1 defined by $y = 0$ (resp., $x \neq 0$),
V_{12}(resp., V_{23}) be the open subset of X_2 defined by $v \neq 0$ (resp., $u \neq 0$),
W_{13}(resp., W_{23}) be the open subset of X_3 defined by $s \neq 0$ (resp., $r \neq 0$).

Let $\phi_{12}: U_{12} \to V_{12}$ be the isomorphism defined by $x \to u/v, y \to 1/v, z \to w/v^3$,
$\phi_{13}: U_{13} \to W_{13}$ be the isomorphism defined by $x \to 1/s, y \to r/s, z \to t/s^3$,
$\phi_{23}: V_{23} \to W_{23}$ be the isomorphism defined by $u \to 1/r, v \to s/r, w \to t/r^3$.

We glue X_1, X_2, and X_3 together via these isomorphisms to obtain a scheme X. We note that the coordinate ring of X_1 (resp., X_2,X_3) is the integral closure of $k[x,y]$ (resp., $k[u,v]$, $k[r,s]$) in its quotient field. Thus we have a finite morphism $X \to \mathbf{P}^2$. Since a finite morphism is projective (see [6, p. 113]) and a composition of projective morphisms is projective, it follows that X is a projective k-scheme.

Step 2. For each $i = 1, 2, 3$ we define σ_i, a differential on X_i as follows:

$$\text{On } X_1, \quad \sigma_1 = \frac{dxdz}{y^2} = \frac{dydz}{x^2y^2+1}.$$

$$\text{On } X_2, \quad \sigma_2 = \frac{dudw}{u^3+uv^4+v^2} = \frac{dydw}{u^2v+v^5}.$$

$$\text{On } X_3, \quad \sigma_3 = \frac{drdt}{r^2+s^4+r^3s^2} = \frac{dsdt}{r^2s^3}.$$

We check that these differentials agree on the above overlaps.

Under ϕ_{12}, σ_1 becomes

$$\frac{d(1/v)d(w/v^3)}{(u/v)^2(1/v)^2+1} = \frac{(1/v^2)dv(1/v^2)((vdw+wdv)/v^2)}{(u^2/v^4)+1}$$

$$= \frac{dvdw}{vu^2+v^5} = \sigma_2.$$

Similarly σ_2 maps to σ_3 under ϕ_{23} and σ_1 maps to σ_3 under ϕ_{13}.

Thus these differentials glue together to give a differential σ on X. We now resolve X to obtain a smooth projective scheme $\tilde{X}$ and show that σ lifts to a regular differential $\tilde{\sigma}$ on $\tilde{X}$.

Step 3. $\tilde{\sigma}$, the lifting of σ to $\tilde{X}$, will be a regular differential on $\tilde{X}$ if we show that X has only rational singularities (see [8], page 153).

Since X_1 is smooth, X can only have singularities on $X_2 \cup X_3$.

On $X_2: w^2 = u^3v + uv^5 + v^3$ singularities can only occur when $v = 0$. Otherwise we would be considering points on $X_1 \cap X_2$, which we know is smooth. So we see that X_2 has only an isolated singularity at $(u,v,w) = (0,0,0)$.

Similarly, we see that $X_3: t^2 = r^2s + s^5 + r^3s^3$ has only an isolated singularity at $(r,s,t) = (0,0,0)$.

These double point singularities will be rational if we show that they can be resolved by quadratic transformations alone (see [9, p. 255]).

J. Lipman has shown that if an isolated singularity on a normal affine surface has local equation of the form

3.1. $z^2 = xy^2 + x^2 g(x,y),\ g(x,y) \in k[x,y]$

then the singularity is rational (see [9, p. 266]).

Thus we see immediately that the singularity on X_3 is rational. This leaves only the singularity on X_2.

We begin by blowing up the origin on X_2. Since w is integrally dependent on the ideal generated by u and v, the blow up of (0,0,0) is covered by two charts (see [1] page 96). Namely,

$$F_1:\quad w^2 = u^2 v + u^4 v^5 + uv^3 \quad \text{and}$$
$$F_2:\quad w^2 = u^3 v^2 + uv^4 + v.$$

F_1 has only an isolated singularity at the origin which is a rational singularity by (3.1).

F_2 is smooth since $\partial/\partial v = 1$. Thus the singularity on X_2 can be resolved by quadratic transformations alone and is a rational singularity.

Therefore $\tilde{\sigma}$ is a regular differential on $\tilde{X}$, which shows that X_1 is not rational.

CONCLUDING REMARKS

After the circulation of a preliminary version of this paper, M. Miyanishi and P. Russell have shown that the ring $A = k[x^p, y^p, x(xy+1)^p + y^{p+1}]$ gives an example of a regular, factorial, nonrational ring for all primes $p > 0$ (see [10]).

Also Miyanishi and Russell have observed that a theorem of Ganong [4] yields a more concise proof that A is not isomorphic to $k^{[2]}$.

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Department of Mathematics
Purdue University
West Lafayette, IN 47907.