ON THE EXTENSION OF HOLOMORPHIC FUNCTIONS WITH GROWTH CONDITIONS ACROSS ANALYTIC SUBVARIETIES

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1. INTRODUCTION AND STATEMENT OF RESULTS

1.1 In this paper we prove two extension theorems for holomorphic functions in the H^p classes, one for the unit ball $B^n \subset \mathbf{C}^n$, and one for the unit polydisk $U^n \subset \mathbf{C}^n$. We also prove an extension theorem for functions in the Nevanlinna class on B^n. The theorems on the ball can be formulated in the context of an arbitrary bounded domain in $\mathbf{C}^n$ with smooth boundary, as indicated in the course of the proofs. The results are as follows:

THEOREM A. *Let V be an analytic subvariety of B^n. Let f be a holomorphic function on $B^n - V$ such that for some $p > 0$, $|f|^p$ has a harmonic majorant u defined on $B^n - V$. Then f extends to a holomorphic function $\hat{f}$ on B^n which belongs to the class $H^p(B^n)$.*

THEOREM B. *Let V be an analytic subvariety of U^n. Let f be a holomorphic function on $U^n - V$ such that for some $p > 0$, $|f|^p$ has an n-harmonic majorant u defined on $U^n - V$. Then f extends to a holomorphic function $\hat{f}$ on U^n which belongs to the class $H^p(U^n)$.*

THEOREM C. *Let V be an analytic subvariety of B^n. Let f be a holomorphic function on $B^n - V$ such that* $\log^+ |f|$ *has a pluriharmonic majorant u on $B^n - V$. Then f extends to a meromorphic function $\hat{f}$ on B^n which belongs to the Nevanlinna class $N(B^n)$.*

1.2 In the case of the first two theorems, the methods involve extending the majorant u to a superharmonic (respectively n-superharmonic) function $\hat{u}$ on B^n (respectively U^n), and applying the Riesz decomposition theorem for superharmonic functions to obtain a growth estimate for $\hat{u}$. This in turn implies that f has a meromorphic extension to B^n (respectively U^n). The argument is completed by showing that a meromorphic function which is not holomorphic cannot have a harmonic majorant. The fact that the extended function $\hat{f}$ belongs to the appropriate H^p class is a consequence of a property of the integral means of a superharmonic (respectively n-superharmonic) function.

Theorem C uses one-variable methods and the Weierstrass preparation theorem.

1.3 Theorems A, B, and C generalize results of Parreau in one variable [7, Theorem 20] which in fact are formulated in the context of an open Riemann surface and (in place of the zero set of a holomorphic function) a compact subset of logarithmic capacity 0. As far as we know, however, our results in the one-variable

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case have not previously been stated when the zero set (a discrete set of points in U) is not compact, i.e., not finite. The first result in several variables of the type considered here is due to Cima [2], who showed that if V is a subvariety of U^n which is bounded away from the distinguished boundary (a Rudin variety), and if f is a holomorphic function on $U^n - V$ such that $|f|$ has a majorant of the form $u = \text{Re}(g)$ with g holomorphic on $U^n - V$, then f extends to a holomorphic function $\hat{f} \in H^1(U^n)$. The method applies without change to the case of pluriharmonic majorants. In [1], P. S. Chee gave a proof of Theorem B for H^p functions on the polydisk with the restriction that V satisfy a condition considered by Zarantanello [10]. Our results do not require any restrictions on the analytic subvariety V.

1.4 Of course there are many other types of extension results in several complex variables. There are also other techniques for proving the existence of harmonic majorants, of which one of the most interesting is a result of Gauthier and Hengartner [4] and Gauthier and Goldstein [3]. These authors show that a sufficient condition for a subharmonic function on (say) the ball to have a harmonic majorant is the existence of such a majorant in a neighborhood of each boundary point.

2. ON VARIOUS GROWTH CONDITIONS AND CLASSES OF MAJORANTS

2.1 *Definition.* Let f be holomorphic on the unit ball B^n in $\mathbf{C}^n$ and let $0 < p < \infty$. We say that $f \in H^p(B^n)$ if

$$\|f\|_p = \sup_{0<r<1} \left\{ \int_{\partial B^n} |f(r\xi)|^p \, d\sigma(\xi) \right\}^{1/p} < \infty.$$

$d\sigma$ denotes normalized Lebesgue measure on ∂B^n.

2.2 *Definition.* Let f be holomorphic on the unit polydisk U^n in $\mathbf{C}^n$ and let $0 < p < \infty$. We say that $f \in H^p(U^n)$ if

$$\|f\|_p = \sup_{0<r_1,\dots,r_n<1} \left\{ \int_{\partial_0 U^n} |f(r_1\xi_1,\dots,r_n\xi_n)|^p \, dm(\xi) \right\}^{1/p} < \infty.$$

Here $\xi = (\xi_1,\dots,\xi_n)$, $\partial_0 U^n$ denotes the distinguished boundary of U^n, and $dm(\xi)$ denotes normalized Lebesgue measure on $\partial_0 U^n$.

2.3 *Definition.* Let f be holomorphic on B^n. We say that f belongs to the Nevanlinna class $N(B^n)$ if

$$\|f\|_0 = \sup_{0<r<1} \int_{\partial B^n} \log^+ |f(r\xi)| \, d\sigma(\xi) < \infty. \tag{2.3.1}$$

As usual $\log^+(t) = \max(0, \log(t))$ for $t > 0$. A meromorphic function which satisfies (2.3.1) will also be said to belong to the Nevanlinna class.

2.4 *Definition.* Let f be holomorphic on B^n. We say that f belongs to the Bloch space $\mathscr{B}(B^n)$ if

$$\|f\|_{\mathscr{B}} = \sup_{z \in B^n} |\nabla f(z)|(1 - \|z\|) < \infty.$$

2.5 A function $f \in H^p(B^n)$ has admissible boundary values a.e. on ∂B^n [9, Theorem 10] and

$$\int_{\partial B^n} |f(r\xi) - f(\xi)|^p \, d\sigma(\xi) \to 0 \qquad \text{as } r \to 1^-.$$

Using this and the fact that $|f|^p$ is subharmonic (in fact plurisubharmonic) it is not hard to show that $f \in H^p(B^n) \Leftrightarrow |f|$ has a harmonic majorant on B^n. It is also true that $f \in N(B^n) \Leftrightarrow \log^+ |f|$ has a harmonic majorant on B^n.

Similarly if $f \in H^p(U^n)$ then the radial limit $\lim_{r \to 1^-} f(r\xi)$ exists for almost all $\xi \in \partial_0 U^n$ and

$$\int_{\partial_0 U^n} |f(r\xi) - f(\xi)|^p \, dm(\xi) \to 0 \qquad \text{as } r \to 1^-.$$

Using the fact that $|f|^p$ is n-subharmonic, i.e., subharmonic in each variable separately [8, p. 39], it again follows that $f \in H^p(U^n) \Leftrightarrow |f|^p$ has an n-harmonic majorant on U^n.

2.6 The following lemma, the proof of which is immediate, gives the relations among the various classes of majorants which one might wish to consider in several complex variables:

LEMMA. *Of the following conditions on a real-valued continuous function u as an open set D of C^n, we have* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) *but not conversely:*

(1) *u is the real part of a holomorphic function g on D;*

(2) *u is pluriharmonic,* i.e., *harmonic in all complex directions;*

(3) *u is n-harmonic,* i.e., *harmonic in each variable separately;*

(4) *u is harmonic.*

(On a simply-connected domain we have (1) $\Leftrightarrow$ (2).)

2.7 It is natural to consider harmonic majorants in formulating extension theorems for H^p functions on the ball, while on the polydisk n-harmonic functions are the natural class of majorants. The assumption that u be pluriharmonic permits the application of one-variable results via the Weierstrass preparation theorem, cf., [2]. The assumption (1) coupled with the obvious fact that a majorant must be nonnegative actually suffices to extend the function g across any analytic subvariety in view of

2.8 PROPOSITION. *Let D be a domain in $\mathbf{C}^n$ and let V be an analytic subvariety of D. Suppose that g is holomorphic on $D - V$ and* $\operatorname{Re} g \geq 0$. *Then g has a holomorphic extension $\hat{g}$ to D.*

Proof. Let $T(\lambda) = (\lambda - 1)/(\lambda + 1)$ where $\lambda \in C$. Since T maps the right half plane onto the unit disk the function $T \circ g$ is bounded holomorphic function. It extends across V by the Riemann removable singularities theorem. The extension of $T \circ g$ does not assume the value $+1 = T(\infty)$ for otherwise $T \circ g \equiv 1$, a contradiction. Hence $g = T^{-1} \circ T \circ g$ has a holomorphic extension across V.

2.9 COROLLARY. *Suppose V is an analytic subvariety of B^n and f is holomorphic on $B^n - V$.*

(1) If $|f|^p \leq u = \text{Re}\,(g)$ *where g is holomorphic on $B^n - V$ then f has a holomorphic extension $\hat{f} \in H^p(B^n)$.*

(2) *If* $\log^+ |f| \leq u = \text{Re}\,(g)$ *where g is holomorphic on $B^n - V$ then f has a holomorphic extension $\hat{f} \in N(B^n)$.*

Proof. Since g has a holomorphic extension $\hat{g}$ to B^n it follows that in both cases (1) and (2) the function f is locally bounded near V. Hence f has a holomorphic extension $\hat{f}$ to B^n by the Riemann removable singularities theorem. In case (1) $\hat{f} \in H^p(B^n)$ since $|\hat{f}|^p$ has the harmonic majorant $\hat{u} = \text{Re}\,(\hat{g})$, while in case (2) $\hat{f} \in N(B^n)$ since $\log^+ |\hat{f}|$ has the harmonic majorant $\hat{u} = \text{Re}\,(\hat{g})$.

2.10 Theorem 1 in [2] is also a consequence of Proposition 2.8, but as noted the proof in that paper applies without change to the case of pluriharmonic majorants. Such majorants may have logarithmic singularities along analytic subvarieties. In section 7 we shall prove the following

PROPOSITION. *Let h be a holomorphic function on B^n and let $Z(h)$ denote its zero set. Suppose that u is pluriharmonic on $B^n - Z(h)$ and $u \geq 0$. Then if $z_0 \in Z(h)$, there is a neighborhood Y of z_0 and a constant $c > 0$ such that $u(z) \leq -c \log d(z, Z(h)$ for $z \in Y - Z(h)$.*

3. SUPERHARMONIC AND n-SUPERHARMONIC FUNCTIONS

3.1 Our extension theorem for H^p functions on the ball depends partially on known results for superharmonic functions. A convenient source for this theory is [6]. We recall some known facts concerning superharmonic functions and indicate how analogous results for n-superharmonic functions may be established. These results are needed for the proof of our extension theorem for H^p functions on the polydisk. In some cases it suffices to refer to [8] where properties of n-subharmonic functions are given.

3.2 *Definition.* Let D be an open subset of $\mathbf{R}^k$. An extended real-valued function $u : D \to \mathbf{R} \cup \{\infty\}$ is superharmonic if

(1) u is not identically $+\infty$ on any component of D;

(2) u is lower semi-continuous;

(3) u is super-mean-valued, i.e., whenever the closed ball $\overline{B(x,r)}$ is contained in D, then, letting $M(u,x,r)$ denote the average value of u on $\partial B(x,r)$, we have

$$u(x) \geq M(u,x,r) \tag{3.2.1}$$

The following properties of superharmonic functions are established in [6, Chapter 4]: First, it is sufficient that (3.2.1) should hold for all r satisfying $0 < r < r_0(x)$. The average $M(u,x,r)$ is finite for all r and, for fixed x, is a nonincreasing function of r. If u is superharmonic on D and $x \in D$ then

$$u(x) = \liminf_{y \to x \ y \neq x} u(y). \tag{3.2.2}$$

3.3 *Definition.* A set $X \subset \mathbf{R}^k$ is a polar set if there is an open set D containing X and a function u superharmonic on D such that $u = +\infty$ on X.

An analytic subvariety of a domain in C^n is always a polar set. The basic result about polar sets we shall need is the following:

3.4 THEOREM [6, Theorem 7.7]. *If X is a relatively closed polar subset of an open set $D \subset \mathbf{R}^k$ and if u is superharmonic on $D - X$ and locally bounded below on D then u has a unique superharmonic extension to D.*

For the purposes of obtaining an analogous theorem for n-superharmonic functions (Theorem 3.11), we derive the following lemma. This lemma can also be used in the proof of Theorem 3.4, replacing a procedure indicated in [6, Theorem 7.7].

3.5 LEMMA. *Suppose X is a relatively closed polar subset of an open set $D \subset \mathbf{R}^k$. Let v be a superharmonic function on D such that $v = +\infty$ on X. Let u be a superharmonic function on $D - X$ which is locally bounded below on D and define the extension $\hat{u}$ of u on D by $\hat{u}(x) = \liminf_{y \to x \ y \in D-X} u(y)$. Then if $x \in X$ there exists a sequence of points $\{x_j\}_{j=1}^{\infty} \subset D - X$ such that*

$$\lim_{j\to\infty} x_j = x, \qquad \lim_{j\to\infty} u(x_j) = \hat{u}(x), \qquad \text{and } v(x_j) < \infty$$

for all j.

Proof. If u is superharmonic in a neighborhood of y then the super-mean-value property $u(y) \geq M(u,y,r)$ implies that the set $\{\eta \in \mathbf{R}^k : u(y) \geq u(\eta)\} \cap B(y,r)$ has positive measure for all $r > 0$. Choose a sequence $\{y_j\}_{j=1}^{\infty} \subset D - X$ such that $\lim_{j\to\infty} y_j = x$ and $\lim_{j\to\infty} u(y_j) = \hat{u}(x)$. The given function u is known to be superharmonic in some ball B_j centered at y_j for each j. Since the function v cannot be infinite on a set of positive measure, we may by the first sentence of the proof choose $x_j \in B_j$ such that $u(x_j) \leq u(y_j)$ and $v(x_j) < \infty$. If follow that $\lim_{j\to\infty} x_j = x$ since the radius of B_j tends to 0 as $j \to \infty$. We obtain $\lim_{j\to\infty} u(x_j) = \hat{u}(x)$ in view of the definition of $\hat{u}(x)$ and the inequality $u(x_j) \leq u(y_j)$.

The basic representation theorem for superharmonic functions is due to F. Riesz.

3.6 RIESZ DECOMPOSITION THEOREM [6, Theorem 6.18] *Let D be an open subset of $\mathbf{R}^k$ having a Green's function G_D and let u be superharmonic on D. There is a unique measure μ on D such that if W is an open subset with compact closure in D then $u = G_W\mu|_W + v_W$ where G_W denotes the Green's function of W, $G_W\mu|_W(x) = \int_W G_W(x,y)d\mu(y)$ is the Green's potential of $\mu|_W$, and v_W is the greatest harmonic minorant of u on W. If in addition $u \geq 0$ on D then $u = G_D\mu + v$ where v is the greatest harmonic minorant of u on D. The measure μ assigns* *finite mass to any compact subset of D and is supported on the complement in D of the largest open subset on which u is harmonic.*

Turning now to the case of n-superharmonic functions, we make the following

3.7 *Definition.* Let D be an open subset of C^n. An extended real-valued function $u : D \to \mathbf{R} \cup \{\infty\}$ is n-superharmonic if

(1) u is not identically $+\infty$ on any component of D;

(2) u is lower semi-continuous;

(3) for fixed $a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n \in \mathbf{C}$ the function $u(a_1,\ldots,a_{k-1},z_k,a_{k+1},\ldots,a_n)$ is either $+\infty$ or superharmonic on each component of its domain of definition, $k = 1, \ldots, n$.

Thus an n-superharmonic function has the super-mean-value property in each variable separately. If the polydisk $U^n(z; r_1,\ldots,r_n)$ with center z and radii $r_1, \ldots, r_n$ is contained in D then the average value of u on the distinguished boundary $\partial_0 U^n(z; r_1,\ldots,r_n)$ is

$$M(u; z; r_1,\ldots,r_n) = \int_{\partial_0 U^n} u(z_1 + r_1\xi_1,\ldots,z_n + r_n\xi_n)\,dm(\xi). \tag{3.7.1}$$

3.8 LEMMA. *If* $\overline{U^n(z;r_1,\ldots,r_n)} \subset D$ *then* $M(u;z;r_1,\ldots,r_n)$ *is finite and* $M(u; z; r_1,\ldots,r_n) \leq u(z)$. *For fixed* z, $M(u; z; r_1,\ldots,r_n)$ *is a function which is non-increasing in each of the variables* $r_1, \ldots, r_n$. *That is, if*

$$U^n(z; s_1,\ldots,s_n) \subset D \qquad \textit{and} \qquad 0 < r_j \leq s_j, \qquad j = 1, \ldots, n$$

then $M(u; z; r_1,\ldots,r_n) \geq M(u; z; s_1,\ldots,s_n)$.

Proof. The corresponding results for n-subharmonic functions are proved in [8, p. 40]. Since a function u is n-superharmonic if and only if $-u$ is n-subharmonic we simply apply the results of [8].

3.9 *Remarks.* An n-superharmonic function is easily seen to be superharmonic. It seems reasonable to conjecture that property (3) in Definition 3.7 may be replaced by (3′) whenever $U^n(z; r_1,\ldots,r_n) \subset D$ we have $u(z) \geq M(u; z; r_1,\ldots,r_n)$. A function satisfying properties (1) and (2) of Definition 3.7 as well as property (3′) might be called super-n-harmonic, so that our conjecture can be phrased as follows: a super-n-harmonic function is n-superharmonic. (The converse follows from Lemma 3.8.)

3.10 *Definition.* A set $X \subset C^n$ is a polar set for n-superharmonic functions if there is an open set D containing X and a function u which is n-superharmonic on D such that $u = +\infty$ on X.

We need to know that the analogue of Theorem 3.4 holds for n-superharmonic functions.

3.11 THEOREM *If X is a relatively closed polar subset for n-superharmonic functions in an open set* $D \subset C^n$ *and if u is n-superharmonic on* $D - X$ *and locally bounded below on D then u has a unique n-superharmonic extension to D.*

Proof. We proceed as in [6, Theorem 7.7], defining the extension $\hat{u}$ of u by $\hat{u}(x) = \lim_{y \to x} \inf_{y \in D - X} u(y)$ for $x \in D$. We use Lemma 3.5 to show that if v is a superharmonic function which assumes the value $+\infty$ on X and if $x \in X$ then there exists a sequence of points $\{x_j\} \subset D - X$ such that

$$\lim_{j \to \infty} x_j = x, \qquad \lim_{j \to \infty} u(x_j) = \hat{u}(x), \qquad \text{and } v(x_j) < \infty$$

for all j. The rest of the proof is the same as in [6, Theorem 7.7] except that we must check the super-mean-value property of $\hat{u}$ in each variable separately. This involves an obvious modification of the argument there.

4. THE NON-EXISTENCE OF A HARMONIC MAJORANT OF $|z_1|^{-p}$ ON $B^n - \{z_1 - 0\}$.

Let $Z = \{z \in C^n : z_1 = 0\}$ and let B_r denote the ball in C^n of centre 0 and radius r. (Thus $B_1 = B^n$.)

In this section we prove

4.1 THEOREM. Let $0 < p < \infty$. *The function* $f(z) = |z_1|^{-p}$ *has no harmonic majorant on* $B^n - Z$.

This theorem is an important step in the proof of Theorem A which is carried out in the next section. Theorem 4.1 is a consequence of

4.2 PROPOSITION. *There exists a positive constant* $\beta = \beta(p)$ *such that if* u *is a harmonic majorant of* $|z_1|^{-p}$ *on* $B_r - Z$*, then for* $z \in B_{r/2} - Z$*,*

$$u(z) \geq 4^{-\beta} r^{\beta} |z_1|^{-(p+\beta)}. \tag{4.2.1}$$

Furthermore $\beta(p)$ *is an increasing function of* p.

4.3 *Proof of Theorem 4.1 Using Proposition 4.2.* Proposition 4.2 implies that given $q > 0$, there exists $\epsilon > 0$ such that $u(z) \geq \text{const.}\ |z_1|^{-q}$ on $B_\epsilon - Z$. But this gives a contradiction as soon as $q \geq 2$, for $|z_1|^{-2}$ is not integrable on spheres transverse to Z, whereas u is integrable on such spheres. (u has a superharmonic extension $\hat{u}$ on B^n by Theorem 3.4, and a superharmonic function is integrable over any sphere.)

Proposition 4.2 is proved by means of a sequence of lemmas which are based on a consideration of the averages of $|z_1|^{-p}$ on spheres which are tangent to Z.

Let Σ_r denote the sphere of centre $(r,0,\ldots,0)$ and radius r. Let $M(p,r)$ denote the average value of $|z_1|^{-p}$ on Σ_r, a finite quantity for $0 < p < n$.

4.4 LEMMA. (1) $M(p,r) = M(p,1)r^{-p} = \text{const.}\ r^{-p}$

(2) $M(p,1) > 1$ *and is an increasing function of* p, $0 < p < n$.

Proof. (1) We first translate the centre of the sphere Σ_r to the origin and then make a change of scale. Letting $d\sigma_r$ denote normalized Lebesgue measure on ∂B_r, we have

$$M(p,r) = \int_{\partial B_r} |r - z_1|^{-p}\, d\sigma_r(z) = \int_{\partial B_1} |r - rw_1|^{-p} \frac{d\sigma_r(rw)}{d\sigma_1(w)} \cdot d\sigma_1(w)$$

$$= r^{-p} \int_{\partial B_1} |1 - w_1|^{-p}\, d\sigma_1(w) = r^{-p} M(p,1).$$

($d\sigma_r(rw)/d\sigma_1(w) = 1$ since the measures are normalized.)

(2) Since

$$\int_{\partial B_r} |1 - z_1|^{-p}\, d\sigma_r(z) \to \int_{\partial B_1} |1 - z_1|^{-p}\, d\sigma_1(z) = M(p,1)$$

it is sufficient to show that $\int_{\partial B_r} |1 - z_1|^{-p}\, d\sigma_r(z) \geq c > 1$ independently of r in order to establish that $M(p,1) > 1$. Write $\phi(z) = |1 - z_1|^{-p}$. By Green's formula

$$(4.4.1) \qquad 1 = \phi(0) = \int_{\partial B_r} \phi(z)\, d\sigma_r(z) - c_n \int_{B_r} G_{B_r}(0,z)\, \Delta\phi(z) dz$$

where c_n is a positive constant, G_{B_r} denotes the Green's function for the ball B_r, and $d\lambda$ denotes Lebesgue measure. We have $\Delta\phi \geq 0$ and it is not hard to see that the last term in (4.4.1) is bounded away from 0 independently of r. Hence the first term is $\geq c > 1$.

To show that $M(p,1)$ is an increasing function of p it suffices to apply Holder's inequality. Thus if $p < q < n$ and $P = q/p$, $Q = P/(P - 1)$,

$$\int_{\Sigma_1} |z_1|^{-p}\, d\sigma_1(z) \leq \left(\int_{\Sigma_1} |z_1|^{-pP}\, d\sigma_1(z)\right)^{1/P} \left(\int_{\Sigma_1} 1^Q d\sigma_1(z)\right)^{1/Q}$$

$$\leq \left(\int_{\Sigma_1} |z_1|^{-q}\, d\sigma_1(z)\right)^{1/P} \leq \int_{\Sigma_1} |z_1|^{-q}\, d\sigma_1(z).$$

The last inequality follows since $M(q,1) > 1$. This completes the proof of Lemma 4.4.

For the next lemma we define a family of subsets of B^n inductively. Euclidean distance is denoted by d, so that $d(z,Z) = |z_1|$.

4.5 *Definition.* Let $E_1 = \{z \in B^n | \, d(z,Z) < d(z,\partial B^n)\}$. For $k = 2, 3, \ldots$ let $E_k = \{z \in E^{k-1} | d(z,Z) < d(z,\partial E_{k-1})\}$. (See Figure 1.)

4.6 LEMMA. *Suppose u is harmonic and $u(z) \geq |z_1|^{-p}$ for $z \in B^n - Z$. Then $u(z) \geq (M(p,1))^k\, |z_1|^{-p}$ for $z \in E_k - Z$.*

Proof. The proof is by induction. For any point $z \in E_1 - Z$ the sphere of radius $|z_1|$ and centre z is contained in B^n. Since u has a superharmonic extension $\hat{u}$ to B^n which has the super-mean-value property, it follows that $u(z) \geq M(p,|z_1|)$. By Lemma 4.4 this latter quantity is $M(p,1)\, |z_1|^{-p}$. (A rotation of the z_1 coordinate preserves the harmonicity of u so we may replace z_1 by $|z_1|$.)

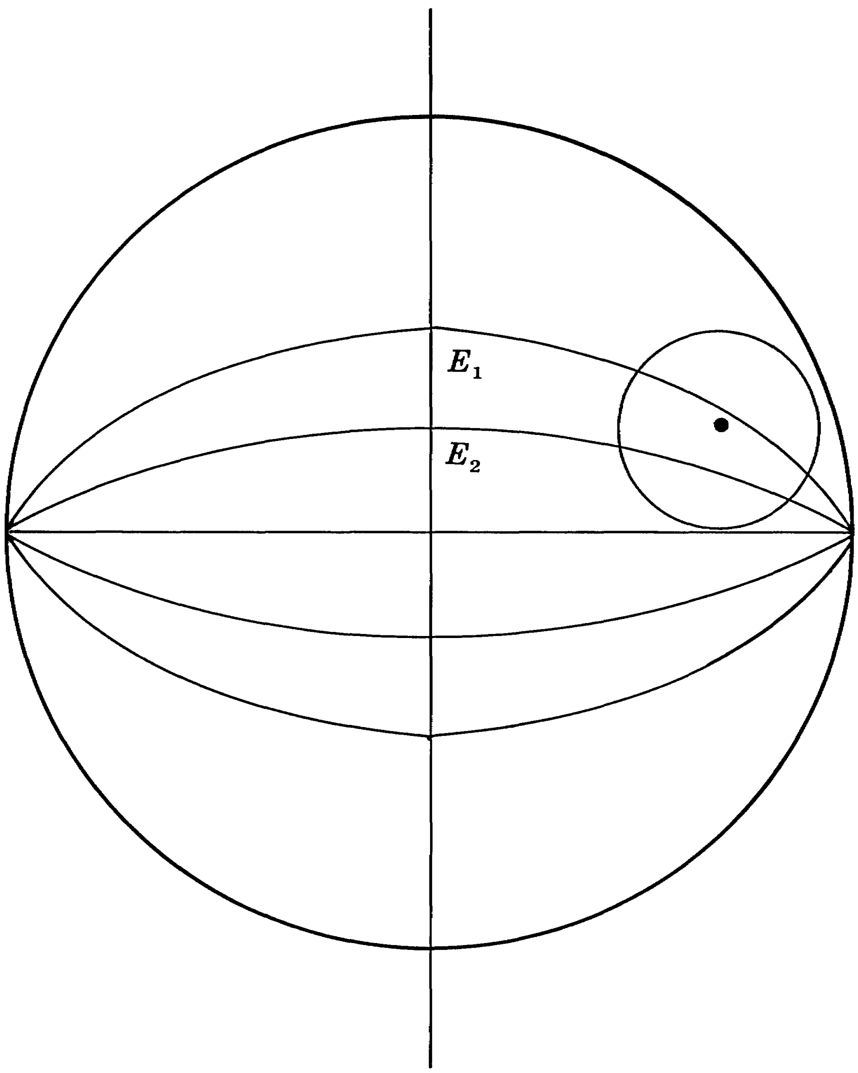

Figure 1

For the inductive step assume $u(z) \geq M(p,1)^{k-1} |z_1|^{-p}$ for $z \in E_{k-1} - Z$. If $z \in E_k$ the sphere of centre z and radius $|z_1|$ lies in E_{k-1}. The harmonic function $u(z)\, M(p,1)^{-k+1}$ majorizes $|z_1|^{-p}$ in $E_{k-1} - Z$, so by Lemma 4.4,

$$u(z)M(p,1)^{-k+1} \geq M(p,1)\, |z_1|^{-p}.$$

(Again we may replace z_1 by $|z_1|$ in order to apply Lemma 4.4.) Hence $u(z) \geq (M(p,1))^k\, |z_1|^{-p}$ for $z \in E_k - Z$.

4.7 LEMMA. *If the coordinates of the point $z = (z_1,\ldots,z_n)$ satisfy*

$$|z_2|^2 + \ldots + |z_n|^2 < 1/4 \qquad \textit{and} \qquad 2^{-(k+1)} \leq |z_1| < 2^{-k}$$

then $z \in E_{k-1}$.

Proof. The set ∂E_k intersects the z_1 axis in the circle $|z_1| = 2^{-k}$. Hence if

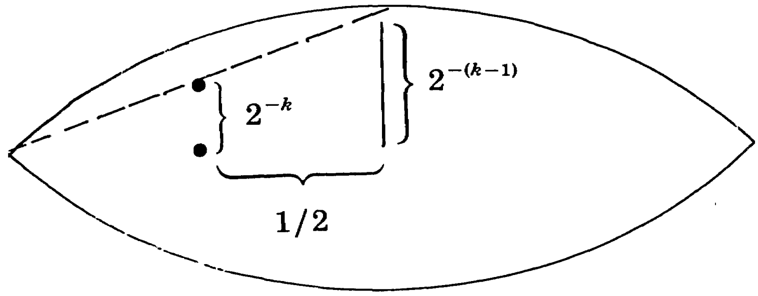

Figure 2

$2^{-k} \le |z_1| < 2^{-(k-1)}$ then $(z_1,0,\dots,0) \in E_{k-1}$. It is clear that the set E_{k-1} is convex, hence it contains the convex hull of $(Z \cap B^n) \cup (E_{k-1} \cap z_1\text{-axis})$. From this the assertion of the lemma follows. (See Figure 2.)

4.8 LEMMA. *Suppose* $u(z) \ge M(p,1)^{k-1}|z_1|^{-p}$ *when* $2^{-(k+1)} \le |z_1| < 2^{-k}$ and $|z_2|^2 + \dots + |z_n|^2 < 1/4$. *Choose* β *such that* $2^\beta = M(p,1)$. Then $u(z) \ge 4^{-\beta}|z_1|^{-(p+\beta)}$ *when* $z \in B_{1/2} - Z$.

Proof. For $2^{-(k+1)} \le |z_1| < 2^{-k}$, we have

$$(4.8.1)\quad 4^{-\beta}|z_1|^{-\beta}|z_1|^{-p} \le 4^{-\beta}2^{(k+1)\beta}|z_1|^{-p} = 2^{(k-1)\beta}|z_1|^{-p} = (M(p,1))^{k-1}|z_1|^{-p}.$$

Hence if u majorizes the term on the right of (4.8.1) then it majorizes the term on the left.

4.9 *End of Proof of Proposition 4.2.* If $u(z) \ge |z_1|^{-p}$ on $B_r - Z$ then $v(w) = u(rw) \ge |rw_1|^{-p}$ on $B_1 - Z$. Of course v is harmonic. Hence by Lemma 4.8, $r^p v(w) \ge 4^{-\beta}|w_1|^{-(p+\beta)}$ when $w \in B_{1/2} - Z$. Equivalently,

$$r^p u(z) \ge 4^{-\beta}|z_1/r|^{-(p+\beta)}$$

when $z \in B_{r/2} - Z$. This reduces to $u(z) \ge 4^{-\beta}r^\beta|z_1|^{-(p+\beta)}$ when $z \in B_{r/2} - Z$. Since $M(p,1) > 1$ we have $\beta > 0$. Also since $M(p,1)$ is an increasing function of p, so is $\beta(p)$.

Professor Norberto Kerzman has indicated another proof that $|z_1|^{-1}$ has no harmonic majorant on $B_n - Z$. The proof does not cover the case of $|z_1|^{-p}$ for $p < 1$. One considers a smooth cylinder in $B_n - Z$ which meets the surface Z on a side. By using the representation of the purported harmonic majorant in terms of its Poisson integral over the boundary of the cylinder one obtains a contradiction.

5. PROOF OF THEOREM A

5.1 THEOREM A. *Let V be an analytic subvariety of B^n. Let f be a holomorphic function on $B^n - V$ such that for some $p > 0$, $|f|^p$ has a harmonic majorant u defined on $B^n - V$. Then f extends to a holomorphic function $\hat{f}$ on B^n which belongs to $H^p(B^n)$.*

5.2 It suffices to prove Theorem A under the hypothesis that V is the zero set of single holomorphic function h on V. We write $V = Z(h)$. To obtain the extension $\hat{f}$, we first show that f has a meromorphic extension to B^n, and subsequently that this meromorphic extension is actually holomorphic. To show that $\hat{f} \in H^p(B^n)$ we consider the averages of the superharmonic extension $\hat{u}$ of the majorant u on spheres centered at 0.

5.3 PROPOSITION. *Let f be holomorphic on $B^n - Z(h)$ where h is holomorphic on B^n. Suppose that for some $p > 0$, $|f|^p$ has a harmonic majorant u on $B^n - V$. Then f extends to a meromorphic function $\hat{f}$ on B^n.*

Proof. It suffices to show that f has a meromorphic extension to a neighborhood of each point $z_0 \in Z(h)$. Now the majorant u is a harmonic function on the complement of a relatively closed polar set, and u is bounded below by 0. By [6, Theorem 7.7] u has a unique superharmonic extension $\hat{u}$ to B^n. We apply the Riesz decomposition theorem (Theorem 3.6) to $\hat{u}$. We conclude that there exists a unique measure μ on B^n which is supported on $Z(h)$ and assigns finite mass to any compact subset of B^n, such that if W is an open set with closure contained in B^n and Green's function G_W then $\hat{u} = G_W\mu|_W + v_W$ on W. v_W is the greatest harmonic minorant of $\hat{u}$ on W.

Now given $z_0 \in Z(h)$ choose the open set W to be a neighborhood of z_0. Since $G_W(z,w) \asymp \|z - w\|^{-2n+2}$ in any compact subset of W and μ is supported on $Z(h)$ we conclude that there is a neighborhood Y of z_0 such that $u(z) \leq \text{const.}\ d(z,Z(h))^{-2n+2}$ when $z \in Y - Z(h)$. We also have an estimate $|h(z)| \leq \text{const.}\ d(z,Z(h))$ on Y since $\bar{Y} \subset B^n$. We shrink the open set Y if necessary so that $|h(z)| < 1$ on Y. Then with p as in the statement of the Proposition, we may choose a positive integer k such that $|h(z)|^{kp} \leq \text{const.}\ d(z,Z(h))^{2n-2}$ on Y. It follows that $|(h(z))^k f(z)|^p \leq \text{const.}$ on $Y - Z(h)$. By the Riemann removable singularities theorem we can write $h^k f = g$ where g is holomorphic on Y. Hence $f = gh^{-k}$ on Y so that f has a meromorphic extension to Y.

5.4 To show that the meromorphic extension of f is actually holomorphic we need to show that we are now essentially in the situation in which Theorem 4.1 applies. Let $z_0 \in Z(h)$ be a smooth point of $Z(h)$. By removing common factors from g and h if necessary and then avoiding points where both g and h vanish we may assume $g(z_0) \neq 0$. Let P be a neighborhood of z_0 such that $Z(h) \cap P$ is smooth and connected, and $|g(z)|$ is bounded below on P. Then $|h(z)|^{-kp}$ has a harmonic majorant which we shall call u (a constant times the original u) on $P - Z(h)$. We shall show that this is a contradiction.

Theorem 4.1 cannot be applied directly, for the local change of co-ordinates which flattens $Z(h)$ need not preserve harmonic functions. We shall re-examine some of the steps in the proof of Theorem 4.1 in order to demonstrate that $|h(z)|^{-p}$ can have no harmonic majorant for any $p > 0$. We may rotate and translate co-ordinates, perform changes of scale, and multiply functions by constants. By considering the restriction of h to a small ball centered at z_0 and then changing scale so that this ball has radius 1 we can achieve a situation in which the following proposition applies:

5.5 PROPOSITION. *Let $p > 0$. There exists $\epsilon = \epsilon(p) > 0$ such that if h is a holomorphic function in a neighborhood of $\overline{B_n}$ such that $h(0) = 0$,*

$$\nabla h(0) = (1,0,\ldots,0), \qquad \textit{and} \qquad \|\nabla h(z) - (1,0,\ldots,0)\| < \epsilon$$

on $\overline{B^n}$ then $|h|^{-p}$ has no harmonic majorant on $B^n - Z(h)$. We may take $\epsilon(p)$ to be an increasing function of p.

Proof. Because of the assumption on ∇h the following is true uniformly for points $z_0 \in Z(h) \cap B^n$: there exists $r_0 = r_0(\epsilon)$ such that if the coordinates are subjected to a unitary transformation to ensure that $z_1 = 0$ defines the complex tangent space to $Z(h)$ at z_0, then

$$(5.5.1) \qquad 1 - \epsilon < \frac{|h(z)|}{|z_1|} < 1 + \epsilon$$

for all points $z \in \bar{B} - \{z_0\}$, where B is any ball which is (1) tangent to $Z(h)$ at z_0; (2) of radius less than r_0; (3) contained together with its closure in B^n. Such a ball B satisfies $\bar{B} \cap Z(h) = \{z_0\}$.

If $u \geq |h|^{-p}$ on $B^n - Z(h)$ and u is harmonic then on any of the balls B (in fact on $\bar{B} - \{z_0\}$), we have $u(z) \geq (1+\epsilon)^{-p}\,|z_1|^{-p}$ using a variable co-ordinate system with origin at z_0 as already indicated. Recalling Lemma 4.4, it follows that if z is the center of one of the balls B, then

$$(5.5.2) \qquad u(z) \geq (1+\epsilon)^{-p} M(p,1)\,|z_1|^{-p} \geq (1+\epsilon)^{-p} M(p,1)(1-\epsilon)^p\,|h(z)|^{-p}.$$

(The second inequality uses (5.5.1).) We now choose ϵ such that

$$(5.5.3) \qquad (1+\epsilon)^{-p}(1-\epsilon)^p M(p,1) = \phi(M(p,1))$$

where $\phi\colon [0,\infty) \to [1,\infty)$ is a strictly increasing function such that $\phi(1) = 1$, but which increases slowly enough that ϵ is an increasing function of p.

The proof of Proposition 5.5 will be completed in Lemmas 5.6 and 5.7 which are analogues of Lemmas 4.6 and 4.8 respectively. For Lemma 5.6 we define the sets $E_k = E_k(h)$ as in Lemma 4.6 except that we replace $Z = Z(z_1)$ by $Z(h)$. It is clear that the sets $E_k(h)$ are convex if ϵ is sufficiently small.

5.6 LEMMA. *Suppose u is harmonic and $u(z) \geq |h(z)|^{-p}$ on $B^n - Z(h)$ where h satisfies the hypotheses of Proposition 5.5. Then $u(z) \geq (\phi(M(p,1)))^{k-k_0+1}\,|h(z)|^{-p}$ for $z \in E_k(h) - Z(h)$ and $k \geq k_0$. Here $k_0 = k_0(\epsilon)$ is the smallest integer such that each point in $E_{k_0}(h)$ is the center of one of the balls B which occur in the proof of Proposition* 5.5.

One can formulate a geometrical result for the set $E_k(h)$ similar to Lemma 4.7 and this together with Lemma 5.6 leads to

5.7 PROPOSITION. *There exists a positive constant $\gamma = \gamma(p,\epsilon)$ such that if h is a holomorphic function on $\overline{B_r} - Z(h)$ which satisfies the hypothesis of Proposition* 5.5 *with B^n replaced by B_r, and if u is a harmonic majorant of $|h(z)|^{-p}$ on $B_r - Z(h)$, then, for $z \in B_{r/2} - Z(h)$, $u(z) \geq$ const. $r^{\gamma}\,|h(z)|^{-(p+\gamma)}$. For fixed ϵ, γ is an increasing function of p.*

5.8 The proof of Proposition 5.5 is completed by arguing as in Section 4.3. Hence we now know that the meromorphic extension $\hat{f}$ of f given by Proposition 5.3 is actually holomorphic.

5.9 *Proof that* $\hat{f} \in H^p(B^n)$. If $\hat{u}$ denotes the superharmonic extension of u to B^n then

$$\int_{\partial B^n} |f(r\xi)|^p \, d\sigma(\xi) \leq \int_{\partial B^n} \hat{u}(r\xi) d\sigma(\xi) \tag{5.9.1}$$

The right-hand side of (5.9.1) is a finite nonincreasing function of r for $0 < r < 1$. Hence the left hand side of (5.9.1) which is a nondecreasing function of r is uniformly bounded for $0 < r < 1$. This completes the proof of Theorem A.

5.10 The case of arbitrary bounded domains with smooth boundary.

Let $D \subset \mathbf{C}^n$ be a bounded domain with C^2 boundary. Let ϕ be a real-valued C^2 function defined in a neighborhood of $\bar{D}$ such that $\phi = 0$ on ∂D, $d\phi \neq 0$ on ∂D, $\phi < 0$ in D and $\phi > 0$ on D^C. Let $D_\epsilon = \{z \in D : \phi(z) < -\epsilon\}$, $0 < \epsilon < \epsilon_0$. We define

$$H^p(D) = \{f \text{ holomorphic on } D \mid \sup_{0<\epsilon<\epsilon_0} \int_{\partial D_\epsilon} |f(\xi)|^p \, d\sigma_\epsilon(\xi) < \infty\}$$

where $d\sigma_\epsilon$ denotes normalized Lebesgue measure on ∂D_ϵ. The definition is independent of ϕ [9]. Suppose V is a subvariety of D, f is holomorphic on $D - V$, $|f|^p \leq u$ where u is harmonic on $D - V$. To extend f holomorphically to D we apply the foregoing results locally, for any point $z_0 \in V$ is contained in a ball which lies in D. To show that the extension $\hat{f} \in H^p(D)$ we need to know that the superharmonic extension $\hat{u}$ of u is uniformly integrable on ∂D_ϵ, $0 < \epsilon < \epsilon_0$. This is easily shown, for if $z_0 \in D$ is such that $\hat{u}(z_0) < \infty$, then for all ϵ such that $z_0 \in D_\epsilon$ we have $\hat{u}(z_0) \geq \int_{\partial D_\epsilon} P_\epsilon(z_0,\xi)\, \hat{u}(\xi) d\sigma(\xi)$ where P_ϵ denotes the Poisson kernel on D_ϵ. For ϵ sufficiently small we have $P_\epsilon(z_0,\xi) \geq c$ where c is independent of $\xi \in \partial D_\epsilon$ and of ϵ, hence $\hat{u}(z_0) \geq c \int_{\partial D_\epsilon} \hat{u}(\xi) d\sigma(\xi)$.

6. PROOF OF THEOREM B

6.1 THEOREM B. *Let V be an analytic subvariety of U^n. Let f be a holomorphic function on $U^n - V$ such that for some $p > 0$, $|f|^p$ has an n-harmonic majorant u defined on $U^n - V$. Then f extends to a holomorphic function $\hat{f}$ on U^n which belongs to the class $H^p(U^n)$.*

Proof. We first extend f to $\hat{f}$ and then show that $\hat{f} \in H^p(U^n)$. The extension of $\hat{f}$ is a local question, i.e., it suffices to show that each point $z_0 \in V$ is contained in a ball $B(z_0)$ such that f has a holomorphic extension to $B(z_0)$. But this follows from Theorem A, in fact we can take $B(z_0)$ to be any ball contained in U^n and containing z_0.

To show that $\hat{f} \in H^p(U^n)$, we use Theorem 3.11 to extend the majorant u to an n-superharmonic function $\hat{u}$ on U^n. Now if $0 < r_j < 1$, $j = 1, \ldots, n$, then

$$\int_{\partial_0 U^n} |f(r_1\xi_1,\ldots,r_n\xi_n)|^p \, dm(\xi) \leq \int_{\partial_0 U^n} \hat{u}(r_1\xi_1,\ldots,r_n\xi_n)\, dm(\xi). \tag{6.1.1}$$

The right-hand side of (6.1.1) is finite and gives a non-increasing function of each of $r_1, \ldots, r_n$ by Lemma 3.8. Since the left-hand side of (6.1.1) is a non-decreasing function of each of $r_1, \ldots, r_n$ it must be uniformly bounded.

7. PROOF OF THEOREM C

7.1 THEOREM C. *Let V be an analytic subvariety of* B^n. *Let f be a holomorphic function on* $B^n - V$ *such that* $\log^+ |f|$ *has a pluriharmonic majorant u on* $B^n - V$. *Then f extends to a meromorphic function* $\hat{f}$ *on* B^n *which belongs to the Nevanlinna class* $N(B^n)$.

Theorem C will follow easily from

7.2 PROPOSITION. *Let h be a holomorphic function on* B^n *and let* $Z(h)$ *denote its zero set. Suppose that u is a pluriharmonic function on* $B^n - Z(h)$ *and* $u \geq 0$. *Then if* $z_0 \in Z(h)$, *there is a neighborhood Y of* z_0 *and a constant* $c > 0$ *such that* $u(z) \leq -c \log d(z,Z,(h))$ *for* $z \in Y - Z(h)$.

Proof. By the Weierstrass preparation theorem there is a nonsingular linear change of co-ordinates in $\mathbf{C}^n$ such that h becomes regular of order k in z_n at z_0. That is, we may write $h(z) = \phi(z)q(z)$ where ϕ is holomorphic and nonvanishing in a neighborhood of z_0 and q is a Weierstrass polynomial of degree k in $z_n - (z_0)_n$. It follows that there exist two positive numbers p_1 and p_2 such that if $z^* = (z_1,\ldots,z_{n-1}) \in \overline{U^{n-1}(z_0^*;p_1,\ldots,p_1)}$ then there are precisely k points $z_n^1(z^*), \ldots, z_n^k(z^*)$ which satisfy $|z_n^j(z^*) - (z_0)_n| < p_2$ and $h(z^*,z_n^j(z^*)) = 0, j = 1, \ldots, k$. We may also assume that $h(z^*,z_n) \neq 0$ if

$$z^* \in \overline{U^{n-1}(z_0^*;p_1,\ldots,p_1)} \qquad \text{and} \qquad |z_n - (z_0)_n| = p_2.$$

We now consider $u(z^*,z_n)$ with z^* fixed. $u(z^*,z_n)$ extends to a superharmonic function on the disk $U((z_0)_n,p_2) \subset C$ which we denote by $\hat{u}(z^*,z_n)$. We apply the Riesz decomposition theorem (Theorem 3.6) to $\hat{u}(z^*,z_n)$, writing

$$\hat{u}(z^*,z_n) = G_{\mu_{z^*}} + v_{z^*} \tag{7.1.1}$$

where G denotes the Green's function on $U((z_0)_n,p_2)$ and μ_{z^*} consists of point masses $m_j(z^*)$ located at the points $z_n^j(z^*)$, $j = 1, \ldots, k$. The function v_{z^*} is the greatest harmonic minorant of $\hat{u}(z^*,z_n)$ on $U((z_0)_n,p_2)$. Since the Green's potential of a finite sum of point masses clearly tends to 0 at $\partial U((z_0)_n,p_2)$, v_{z^*} is simply the Poisson integral of the boundary values of $\hat{u}(z^*,\cdot)$ on $\partial U((z_0)_n,p_2)$. Hence v_{z^*} depends continuously on z^*, or, more precisely, the function $(z^*,z_n) \to v_{z^*}(z_n)$ is continuous on $\overline{U^n(z_0;p_1,\ldots,p_1,p_2)}$.

We now choose a number $p_3 < p_2$ which has the same properties as p_2, i.e., if $z^* \in U^{n-1}(z_0^*,p_1,\ldots,p_1)$ then the k points $z_n^1(z^*), \ldots, z_n^k(z^*)$ satisfy

$$|z_n^j(z^*) - (z_0)_n| < p_3.$$

We may also assume that the k points in question are bounded away from the circle $\partial U((z_0)_n,p_3)$ uniformly in z^*. Of course $h(z^*,z_n) \neq 0$ if

$$z^* \in U^{n-1}(z_0^*;p_1,\ldots,p_1) \qquad \text{and} \qquad |z_n - (z_0)_n| = p_3.$$

Now $\int_0^{2\pi} \hat{u}(z^*,(z_0)_n + p_3 e^{i\theta})d\theta$ is a continuous function of $z^* \in \overline{U^{n-1}(z_0^*;p_1,\ldots,p_1)}$ and so is $\int_0^{2\pi} v_{z^*}((z_0)_n + p_3 e^{i\theta})d\theta$. Hence the same is true of $\int_0^{2\pi} G\mu_{z^*}((z_0)_n + p_3 e^{i\theta})d\theta$. But this can only be true if the sum $\sum_{j=1}^{k} m_j(z^*)$ is bounded above on $\overline{U^{n-1}(z_0^*;p_1,\ldots,p_1)}$ say by the constant c. Since

$$G\mu_{z^*}(z_n) \leq -\sum m_j(z^*) \log |z_n - z_n^j(z^*)|$$

on $U((z_0)_n,p_2)$ it is clear that $G\mu_{z^*}(z_n) \leq -c \log \min |z_n - z_n^j(z^*)|$. By changing the constant c we obtain a similar majorization for $\hat{u}(z^*,z_n)$ and hence

$$u(z) \leq -c \log (d(z,Z(h)).$$

7.3 *Proof of Theorem C.* We may assume that V is the zero set of a single function h. It suffices to show that f has a meromorphic extension to a neighborhood of any point $z_0 \in Z(h)$.

Since $|h(z)| \leq$ const. $d(z,Z(h))$ in a neighborhood of z_0 it follows from Proposition 7.2 that $u(z) \leq -c \log |h(z)|$ (c may have a different value) in a nieghborhood of z_0. We may assume that c is an integer. We have $\log^+ |f(z)| + c \log |h(z)| \leq 0$ which implies $\log^+ |f(z)(h(z))^c| \leq 0$ for $z \in Y - Z(h)$, Y some neighborhood of z_0. From this it follows that $f(z)(h(z))^c$ is bounded in $Y - Z(h)$, hence has a holomorphic extension to Y which we call g. Thus $f = g/h^c$ on Y which gives the desired meromorphic extension of f to Y.

Let $\hat{f}$ denote the meromorphic extension of f to B^n. To show that $\hat{f}$ belongs to the Nevanlinna class we argue as in Theorems A and B: Extend the majorant u to a superharmonic function $\hat{u}$ on B^n. Then

$$\text{(7.3.1)} \qquad \int_{\partial B^n} \log^+ |\hat{f}(r\xi)| d\sigma(\xi) \leq \int_{\partial B^n} \hat{u}(r\xi) d\sigma(\xi).$$

Since the left-hand side of (7.3.1) is a non-decreasing function of r and the right-hand side is a nonincreasing function of r, the left-hand side is bounded independently of r.

7.4 *Remarks.* (1) An open question is whether the majorant u in Theorem C can be taken to be harmonic.

(2) This theorem can also be formulated for an arbitrary bounded domain in $\mathbf{C}^n$ with smooth boundary. Referring to Section 5.10, one defines

$$N(D) = \left\{ f \text{ holomorphic on } D \mid \sup_{0<\epsilon<\epsilon_0} \int_{\partial D_\epsilon} \log^+ |f(\xi)| d\sigma_\epsilon(\xi) < \infty \right\}$$

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