ON THE BOUND FOR THE DEFRANCHIS-SEVERI THEOREM

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The classical DeFranchis-Severi Theorem (cf. [2], [3], [4]) states: Given a function field of one variable L over an algebraically closed field K, then the intermediary extensions $K \subsetneqq L' \subset L$ such that genus $(L') \geq 2$ and L/L' is separable, are finitely many in number. A. Weitsman and I raise independently the question whether the number in the above theorem is bounded by some number depending only on the genus of the field L. The purpose of the article is to settle the hyperelliptic case. Indeed we have

THEOREM. *Given an integer g, there is a number m_g such that for any hyperelliptic field L of genus g over an algebraically closed field of characteristic $\neq 2$, the number m_g is bigger than the number of intermediate field L' with L/L' separable and the genus of $L' \geq 2$.*

The above theorem establishes the credibility of the following conjecture.

CONJECTURE. *The number of subfields in the original DeFranchis-Severi theorem is bounded by some number which depends only on the genus.*

1. THE PROOF

Let the genus of L' be g' and $[L:L'] = n$. Then it follows from the Hurwitz formula $2g - 2 = n(2g' - 2) + \delta$ that there are finitely many choices for g' and n. Note that $n \leq g - 1$. Thus as usual we may assume that both g' and n are given. Note that L' must be hyperelliptic (cf. [1]).

The canonical map will send L to a rational field $K(x)$ with $[L:K(x)] = 2$. The field $K(x)$ is thus uniquely determined. Let $K(y)$ be the corresponding field for L'. Then we have the following diagram

$$\begin{array}{cccc} & L & \overset{n}{\supset} & L' \\ 2 & \cup & & \cup\, 2 \\ & K(x) & \overset{n}{\supset} & K(y). \end{array}$$

Let $y = f(x)/g(x)$ and a defining equation of L' over $k(y)$ be $v^2 = \psi(y) = \prod (y - \beta_i)$ where $\psi(y)$ is of degree $2g' + 2$. Then $v \notin k(x)$ and v will generate L over $k(x)$. The above equation can be rewritten as

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$$(g(x)^{g+1}v)^2 = \prod (f(x) - \beta_i g(x)).$$

It is clear that the existence of the above diagram is equivalent to the existence of a polynomial $h(x)$ satisfying the following equation

$$\prod (f(x) - \beta_i g(x)) = h(x)^2 \prod (x - \alpha_j) \tag{1}$$

where the β_i are the ramified points of L' over $k(y)$ and the α_j are the ramified points of L over $K(x)$.

Our theorem means given arbitrary $2g+2$ distinct points $\alpha_1, \ldots, \alpha_{2g+2}$ there are a bounded number of $2g'+4$ tuples $(\beta_1,\ldots,\beta_{2g'+2'}, f(x), g(x))$ such that β_i's are all distinct, $(f(x),g(x)) = 1$, max (deg $f(x)$,deg $g(x)) = n$ and the equation (1) is satisfied for suitable $h(x)$. Note that in case the β_i's are not all distinct then the genus of L' defined by $v^2 = \prod (f(x)/g(x) - \beta_i)$ is determined by the number of β_i's with odd multiplicities. Furthermore the conditions that

$$(f(x),g(x)) = 1 \qquad \text{and} \qquad \max(\deg f(x), \deg g(x)) = n$$

simply specify the field degrees of L over L' and $k(x)$ over $k(y)$.

Let all coefficients of $f(x)$, $g(x)$, $h(x)$, α_j, β_i be replaced by indeterminates. Then we have a formal equation of the following form

$$\prod_i (F(x) - B_i G(x)) = H(x)^2 \prod_j (x - A_j) \tag{2}$$

where F and G are of degree n each. Equating the coefficients of x in the above equation produces an algebraic variety $V \subset A^m$ where m is the number of indeterminates. Let $\bar{V}$ be the completion of V in the corresponding projective space P^m.

Let $L_{\alpha_1,\ldots,\alpha_{2g+2}}$ be the linear space defined by $A_j - \alpha_j = 0$ for $j = 1, \ldots, 2g+2$. Let p be a point in $V \cap L_{\alpha_1,\ldots,\alpha_{2g+2}}$. Let

$$\prod_i (\bar{f}(x) - \bar{\beta}_i \bar{g}(x)) = \bar{h}(x)^2 \prod_j (x - \alpha_j) \tag{1*}$$

be the corresponding equation. suppose all α_j's are distinct. Then clearly not all $\bar{\beta}_i$'s are with even multiplicities. Otherwise the left-hand side of the equation (1*) will be the square of a polynomial while the right-hand side of the equation (1*) is not a square. Furthermore the number of $\bar{\beta}_i$'s with odd multiplicities must be strictly greater than 2. Otherwise the number of roots with odd multiplicities of the left-hand side of the equation (1*) is at most $2n$ while the number of roots with odd multiplicities of the right-hand side of the equation (1*) is at least $2g+2$ which is strictly greater than $2n$.

The subfield $\bar{L}'$ defined by

$$v^2 = \prod (\bar{f}(x)/\bar{g}(x) - \bar{\beta}_i) = \prod (\bar{y} - \bar{\beta}_i)$$

with $\bar{y} = \bar{f}(x)/\bar{g}(x)$ will be of genus $= 1/2$ (number of $\bar{\beta}_i$ with odd multiplicities $- 2) \geq 1$ and $[L:\bar{L}'] = \max(\deg \bar{f}(x), \deg \bar{g}(x)) - \deg(\bar{f}(x), \bar{g}(x)) \leq n$. It follows from Satz 1′ of [4] that the cardinality of $V \cap L_{\alpha_1,\ldots,\alpha_{2g+2}}$ is finite. Let $V^* = U$ all irreducible components of V which meet $L_{\alpha_1,\ldots,\alpha_{2g+2}}$ for some distinct $\alpha_1, \ldots, \alpha_{2g+2} = UV_i^*$. Let $m_g = \sum \operatorname{ord} v_i^*$. Note that it follows from the purity of intersections that $2g + 2 + \dim V^* \leq m$. Hence the cardinality of

$$V \cap L_{\alpha_1,\ldots,\alpha_{2g+2}} = \text{the cardinality of } V^* \cap L_{\alpha_1,\ldots,\alpha_{2g+2}} \leq m_g.$$

Our theorem is thus established.

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