A RUDIN-CARLESON THEOREM FOR UNIFORMLY CONVERGENT TAYLOR SERIES

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Let T be the unit circle in the complex plane and let m be normalized Lebesgue measure on T. For a continuous complex-valued function $f(z)$ on T and an integer j, define the Fourier coefficient $\hat{f}(j)$ by $\hat{f}(j) = \int_T f(z) z^{-j} dm(z)$. The letter A will stand for the space of continuous functions $f(z)$ on T such that $\hat{f}(j) = 0$ for $j < 0$, while U will denote the set of all $f \in A$ such that

$$f(z) = \lim_{n \to \infty} \sum_{j=0}^{n} \hat{f}(j) z^j$$

uniformly on T. Recall the classical Rudin-Carleson theorem.

THEOREM 1. ([6], [2]). *Let $K \subseteq T$ be a closed set with $m(K) = 0$ and suppose that g is a continuous function on K. There exists $f \in A$ such that $f(z) = g(z)$ for $z \in K$ and $|f(z)| < \sup\{|g(w)|: w \in K\}$ if $z \notin K$.*

The purpose of this note is to strengthen Theorem 1 as follows:

THEOREM 2. *Let K and g be as in Theorem 1. Then there exists $f \in U$ such that the conclusions of Theorem 1 are valid.*

This theorem answers a question on p. 89 of [5] and extends certain previously known results. (See, e.g., [4].)

We now begin the proof. Let $D_n(z) = \sum_{j=0}^{n} z^j$ so that $\sum_{j=0}^{n} \hat{f}(j) z^j$ is equal to the convolution over the group T $D_n * f(z)$. Let Y be the set $\{0\} \cup \{n^{-1}\}_{n=1}^{\infty}$, and let $\tilde{T}$ be the space $T \times Y$. Then, if $f \in A$, $f \in U$ if and only if the function

$$\tilde{f}(z,y) = \begin{cases} f(z) & \text{if } \ y = 0 \\ D_n * f(z) & \text{if } \ y = n^{-1} \end{cases}$$

is continuous on $\tilde{T}$. Thus U corresponds to a uniformly closed subspace $\tilde{U}$ of the space of continuous functions on $\tilde{T}$. The conclusion of our theorem can be stated as follows: if $K \subseteq T$ is compact and of measure zero, then $\tilde{K} = \{(k,0) : k \in K\}$ is a set of interpolation for the space $\tilde{U}$ of functions on $\tilde{T}$. The generalized Rudin-Carleson theorem [1] now shows that it is enough to establish the following fact.

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If $\tilde{\lambda}$ is a Borel measure on $\tilde{T}$ satisfying

(1) $$\int_{\tilde{T}} \tilde{f}\, d\tilde{\lambda} = 0 \text{ for every } \tilde{f} \in \tilde{U}, \text{ then}$$

$|\tilde{\lambda}|(\tilde{K}) = 0$ for all $\tilde{K}$ as above.

Here $|\tilde{\lambda}|$ denotes the total variation measure associated with $\tilde{\lambda}$.

Each measure $\tilde{\lambda}$ on $\tilde{T}$ can be considered as a sequence $\{\lambda_n\}_{n=0}^{\infty}$ of measures on T such that the series of total variation norms $\sum_{n=0}^{\infty} ||\lambda_n||$ is finite. Thus (1) follows from the next statement.

(2) If $\{\lambda_n\}_{n=0}^{\infty}$ is a sequence of measures on T such that $\sum_{n=0}^{\infty} ||\lambda_n|| < \infty$, and if $\int_T f(z)\, d\lambda_0(\bar{z}) = \sum_{n=1}^{\infty} \int_T D_n * f(z)\, d\lambda_n(\bar{z})$ for all $f \in U$, then λ_0 is absolutely continuous with respect to m.

The idea behind (2) is that λ_0 is the limit in the dual space of U of a sequence of polynomials, and such a limit should be absolutely continuous. To make this precise, we need to introduce some more notation. For $f \in U$, define $||f||$ to be $\sup\{|D_n * f(z)| : z \in T,\ n = 1, 2, \ldots\}$. For a measure λ on T, define $||\lambda||^*$ to be $\sup\left\{\left|\int_T f(z)\, d\lambda(\bar{z})\right| : f \in U,\ ||f|| \leq 1\right\}$. Then it follows from the hypotheses of (2) that for each $\varepsilon > 0$ there is a polynomial $p(z)$ such that, identifying p and the measure $p(\bar{z})\, dm(z)$, we have $||\lambda_0 - p||^* < \varepsilon$. Now (2) will follow when we prove assertions (3) and (4) below.

(3) If λ_0 is a measure which is the limit in $||\cdot||^*$ of polynomials, then so is $\lambda_0|_K$, the restriction of λ_0 to any compact $K \subseteq T$ with $m(K) = 0$.

(4) Suppose that ν is a measure on T supported on a closed set E with $m(E) = 0$. If ν is the limit in $||\cdot||^*$ of polynomials, then $\nu = 0$.

The proof of (3) is easy. Fix $\varepsilon > 0$. Let $g(z)$ be a polynomial such that $g\lambda_0$ approximates $\lambda_0|_K$ well in the total variation norm: $||\lambda_0|_K - g\lambda_0|| < \varepsilon/2$. (The existence of g follows from Theorem 1.) The operator $f \to gf$ is a bounded linear operator on U. Since the adjoint operator is bounded on U^* and since λ_0 is the limit in $||\cdot||^*$ of polynomials, there is a polynomial p with $||g\lambda_0 - gp||^* < \varepsilon/2$. Now $||\lambda_0|_K - gp||^* < \varepsilon$ follows from the inequality $||\eta||^* \leq ||\eta||$ for measures η on T.

The proof of (4) is somewhat longer. Suppose $\{\eta_n\}_{n=0}^{\infty}$ is a sequence of measures on T satisfying $\sum_{n=0}^{\infty} \|\eta_n\| < \infty$ and define an analytic function $A(z)$ on the open unit disc by $A(z) = \sum_{j=0}^{\infty} a_j z^j$, $a_j = \sum_{n=0}^{j} \hat{\eta}_n(j)$, and $\hat{\eta}_n(j) = \int_T z^{-j} d\eta_n(z)$. It follows from the proof of Theorem 1 of [7] that $A(z) \in H^p (0 < p < 1)$ and that if $A^*(e^{i\theta}) = \lim_{r\to 1} A(re^{i\theta})$, then

$$m\{e^{i\theta}: |A^*(e^{i\theta})| > s\} \leq \frac{M}{s} \sum_{n=0}^{\infty} \|\eta_n\| \quad (s > 0)$$

for some absolute constant M. It is easy to see that if

$$B(z) = \sum_{j=0}^{\infty} b_j z^j, \; b_j = \hat{\eta}_0(j) + \sum_{n \geq j} \hat{\eta}_n(j),$$

then $B(z) \in H^p \; (0 < p < 1)$ and

$$m\{e^{i\theta}: |B^*(e^{i\theta})| > s\} \leq \frac{M}{s} \sum_{n=0}^{\infty} \|\eta_n\| \quad (s > 0), \tag{5}$$

but with a possibly larger absolute constant M.

Now suppose that μ is a measure on T and define $C_\mu(z) = \int_T (1 - z\bar{w})^{-1} d_\mu(w)$. Then $C_\mu(z)$ is the Cauchy transform of μ, so $C_\mu^*(e^{i\theta}) = \lim_{r\to 1} C_\mu(re^{i\theta})$ exists for almost all $e^{i\theta} \in T$. We will need the following fact.

(6) If μ is the limit in $\|\cdot\|^*$ of polynomials, then
$$m\{e^{i\theta}: |C_\mu^*(e^{i\theta})| > s\} = o(s^{-1}) \quad \text{as} \quad s \to \infty.$$

To prove (6), fix $\varepsilon > 0$ and let p be a polynomial such that $\|\mu - p\|^* < \varepsilon$. It follows from the Hahn-Banach theorem that there exists a sequence $\{\eta_n\}_{n=0}^{\infty}$ of measures on T such that

$$\int_T f(z)\,d\mu(\bar{z}) - \int_T f(z)p(\bar{z})\,dm(z) = \int_T f(z)\,d\eta_0(\bar{z}) + \sum_{n=1}^{\infty} \int_T D_n * f(z)\,d\eta_n(\bar{z}) \qquad (f \in U),$$

$$\sum_{n=0}^{\infty} \|\eta_n\| < \varepsilon. \tag{7}$$

In particular, $\hat{\mu}(j) = \hat{p}(j) + \hat{\eta}_0(j) + \sum_{n\geq j} \hat{\eta}_n(j) \doteq \hat{p}(j) + b_j \; (j \geq 0)$. Putting

$$B(z) = \sum_{j=0}^{\infty} b_j z^j,$$

we see that $C_\mu(z) = p(z) + B(z)$. Let $b = \sup \{|p(z)|: z \in T\}$. Then for $s > 2b$,

$$m\{e^{i\theta}: |C_\mu^*(e^{i\theta})| > s\} \leq m\{e^{i\theta}: |B^*(e^{i\theta})| > s/2\} \leq 2\varepsilon M/s$$

by (5) and (7). Since M is fixed, this establishes (6).

Next consider the following assertion,

(8) If λ is a probability measure on T supported by a closed set K with $m(K) = 0$, then $m\{e^{i\theta}: |\mathrm{Im} C_\lambda^*(e^{i\theta})| \geq s\} = 2 \arctan(2/s)/\pi$.

This statement, together with (3) and (6), implies (4): Write $\nu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$ where the positive measures μ_1 and μ_2 (respectively, μ_3 and μ_4) are supported on disjoint sets K_1 and K_2 (respectively, K_3 and K_4) partitioning E. It is enough to show that if K is a compact subset of one of the intersections

$$K_i \cap K_j \ (i = 1, 2; j = 3, 4),$$

then $\nu|_K = 0$. But, by (3), $\nu|_K$ is a limit in the norm $\|\cdot\|^*$ of polynomials, so (6) holds with $\mu = \nu|_K$. Together with (8) (applied to $\lambda = \mu_i|_K$ and $\lambda = \mu_j|_K$) and the fact that $\mathrm{Re}(1 - z\bar{w})^{-1} = 1/2$, for $z\bar{w} \in T$, $z\bar{w} \neq 1$, this shows that $\mu_i|_K = \mu_j|_K = 0$.

Thus it remains to establish (8). A computation shows that (8) holds if λ is the unit mass at 1, so it is

(9) $m\{e^{i\theta}: |C_\lambda^*(e^{i\theta})| \geq s\}$ is independent of λ so long as λ satisfies the hypotheses of (8).

For such λ, $C_\lambda(z) = \int_T (1 - z\bar{w})^{-1} d\lambda(w)$ is continuous on $D \sim K$, the complement of K in the closed unit disc. For $z \in D \sim K$,

(10) $\mathrm{Re} C_\lambda(z) \geq 1/2$, $\mathrm{Re} C_\lambda(z) = 1/2$ if and only if $z \in T$.

Now let $C(z) = (1 - z)^{-1}$ and fix λ. Since $C_\lambda(0) = 1$, $C_\lambda(z)$ is subordinate to $C(z)$. Thus there exists an analytic function $g(z)$ on the open unit disc such that $g(0) = 0$, $|g(z)| \leq |z|$ if $|z| < 1$, and $C_\lambda(z) = C(g(z))$. Because of (10), $g(z)$ is an inner function. Thus (9) follows from the following assertion.

(11) If $g(z)$ is an inner function with $g(0) = 0$ and if $g^*(e^{i\theta}) = \lim_{r \to 1} g(re^{i\theta})$, then for any Borel subset B of T, $m\{e^{i\theta}: g^*(e^{i\theta}) \in B\} = m(B)$.

To prove (11), define a measure m_1 on T by the rule

$$\int_T f(z)\,dm_1(z) = \int_T f(g^*(z))\,dm(z)$$

for continuous functions $f(z)$ on T. Then $\int_T f(z)\,dm_1(z) = \int_T f(z)\,dm(z)$ whenever $f(z) = z^n$ for some integer n, so $m_1 = m$.

We remark that the proof of Theorem 2 depends indirectly (by way of [7]), but apparently unavoidably, on the deep results of Carleson and Hunt [3] concerning the convergence of Fourier series.

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