THE LOGARITHMIC DERIVATIVE OF MULTIVALENT FUNCTIONS

W. K. Hayman

1. INTRODUCTION

Suppose that

$$f(z) = a_0 + a_1 z + \ldots \tag{1.1}$$

is mean $p -$ valent in $\Delta: |z| < 1$, where $0 < p < \infty$. This means [3, p. 23], that the area of the image of Δ by $f(z)$ covers any disk $|w| < R$ at most p times on the average, with due count of multiplicity. In this paper we investigate the restriction which this assumption places on the mean square of the logarithmic derivative $f'(z)/f(z)$. If p is a positive integer then $f(z)$ is said to be p-valent if $f(z)$ assumes no value more than p times. Our counter examples will be p-valent in this case and will show that our estimates are fairly sharp in the narrower class also.

Since $f'(z)/f(z) = \infty$, wherever $f(z)$ has a zero, we need to exclude neighbourhoods of zeros from our integrals. We deal with this difficulty as follows. Suppose that $\zeta_1, \zeta_2, \ldots \zeta_q$ are the zeros of $f(z)$, so that [3, p. 25], $q \leq p$. Set

$$\Pi(z) = \prod_{\nu=1}^{q} \frac{z - \zeta_\nu}{1 - \bar{\zeta}_\nu z}. \tag{1.2}$$

We write, for $0 \leq \delta < 1$, $0 < r < 1$

$$A(r, f, \delta) = \iint \left|\frac{f'(z)}{f(z)}\right|^2 dxdy, \tag{1.3}$$

where the integral is taken over all those points of $|z| < r$, where $|\Pi(z)| > \delta^q$. If $\delta = 0$, so that we integrate over the whole disk $|z| < r$, we write $A(r,f)$.

We shall denote by K any absolute constant not necessarily the same each time, and by K_1, $K_2 \ldots$ particular constants. Constants depending on p, q, etc., will be denoted by $K(p)$, $K(p,q)$ etc. It is of interest that our basic inequality requires no normalisation.

THEOREM 1. *If $f(z)$ is mean p-valent in Δ, then with the above notation*

$$A(r, f, \delta) < 2\pi p \left\{4p \log \frac{1}{1-r} + q \log \frac{1}{\delta} + (p+1) K\right\}, 0 < r < 1. \tag{1.4}$$

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More generally, for $0 \leq r_1 < r_2 < 1$,

$$A(r_2, f, \delta) - A(r_1, f, \delta) < 2\pi p \left\{ p \log \frac{1}{1 - r_1} + 4p \log \frac{1 - r_1}{1 - r_2} + q \log \frac{1}{\delta} + (p + 1) K \right\}. \tag{1.5}$$

We now set

$$I_2(r, f'/f) = \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right|^2 d\theta \tag{1.6}$$

and shall deduce from Theorem 1

THEOREM 2. *If f is mean p-valent in Δ, then*

$$\overline{\lim_{r \to 1}} \frac{1 - r}{\log \frac{1}{1 - r}} I_2\left(r, \frac{f'}{f}\right) \leq p^2, \tag{1.7}$$

$$\underline{\lim_{r \to 1}} (1 - r) I_2\left(r, \frac{f'}{f}\right) \leq 4p^2. \tag{1.8}$$

If p is an integer and $f(z)$ has p zeros in Δ, we can replace the bound p^2 by $\frac{1}{2}p^2$ in (1.7) *and $4p^2$ by $2p^2$ in* (1.8).

Simple examples will show that the inequalities (1.4), (1.8) are essentially best possible. In (1.5) at any rate the coefficient of the term $\log \{1/1 - r_1\}$ cannot be reduced. We shall prove

THEOREM 3. *Given a positive number p, there exists $f_p(z)$, mean p-valent and nonvanishing in Δ and p-valent if p is an integer such that, for any $\epsilon > 0$, there are sequences r_k, r_k' tending to one, with $(1 - r_k)/(1 - r_k') < K(\epsilon)$, while*

$$A(r_k', f_p) - A(r_k, f_p) > (1 - \epsilon)\, 2\pi p^2 \log \frac{1}{1 - r_k}. \tag{1.9}$$

Thus

$$\overline{\lim_{r \to 1}} \frac{1 - r}{\log \frac{1}{1 - r}} I_2\left(r, \frac{f'}{f}\right) > K p^2. \tag{1.10}$$

1.1. The main difficulties in the proof of Theorem 1 arise from our aim to obtain results independent of the position of the zeros or any other normalisation, and also to obtain the sharp coefficient $2\pi p^2$ of $\log 1/(1 - r_1)$ in (1.5) and hence the bound p^2 in (1.7) (which is probably still not sharp). The following argument suggested by the referee, Prof. P. L. Duren, shows what can be achieved by elementary methods.

Suppose that

$$f(z) = a_0 + a_1 z + \dots$$

is univalent and $f(z) \neq 0$ in Δ. Let $m = \inf_{|z|=r} |f(z)|, M = \sup_{|z|=r} |f(z)|$, let $D(r)$ be the image of $|z| < r$ by $f(z)$ and write $f(z) = \rho e^{i\phi}$. Then

$$A(r,f) = \iint_{|z|<r} |f'(z)/f(z)|^2 \, dxdy = \iint_{D(r)} \frac{\rho d\rho d\phi}{\rho^2}$$
$$\leq 2\pi \int_m^M \frac{d\rho}{\rho} = 2\pi \log \frac{M}{m}.$$

Classical inequalities yield [3, p. 95]

$$m \geq |a_0| \left(\frac{1-r}{1+r}\right)^2, M \leq |a_0| \left(\frac{1+r}{1-r}\right)^2,$$

so that we obtain $A(r,f) \leq 8\pi \log\left(\frac{1+r}{1-r}\right) \leq 2\pi \left\{4 \log \frac{1}{1-r} + 4 \log 2\right\}$ in this case, which corresponds to (1.4).

Next, since $I_2(r,f)$ increases with r, we deduce that, for $0 < r_1 < r_2 < 1$

$$I_2\left(r_1, \frac{f'}{f}\right) \leq \frac{2}{r_2^2 - r_1^2} \int_{r_1}^{r_2} I_2\left(r, \frac{f'}{f}\right) rdr \leq \frac{2}{r_2^2 - r_1^2} \frac{A(r_2, f)}{2\pi}$$

$$\leq \frac{8}{r_2^2 - r_1^2} \left\{\log \frac{1}{1-r_2} + \log 2\right\}.$$

If we define r_2 by $\frac{1-r_1^2}{1-r_2^2} = \log \frac{32}{1-r_1^2}$, we obtain, after some calculations

$$I_2\left(r_1, \frac{f'}{f}\right) < \frac{4}{1-r_1} \left\{\log \frac{1}{1-r_1} + \log^+ \log \frac{1}{1-r_1} + 20\right\}$$

which yields

$$\overline{\lim_{r\to 1}} \frac{(1-r)}{\log \frac{1}{1-r}} I_2(r, f'/f) \leq 4.$$

However (1.7) shows that the left hand side is at most 1 in this case. Similarly if $f(z)$ is univalent in Δ and $f(0) = 0$, then an elementary method yields

$$A(r,f) < 4\pi \log \frac{1}{1-r} + O(1)$$

and hence

$$\overline{\lim_{r\to 1}} \frac{1-r}{\log \dfrac{1}{1-r}} I_2(r, f'/f) \leq 2$$

while Theorem 2, which is based on an improved version of (1.5), shows that the bound 2 may be replaced by $1/2$ in this case.

2. SIMPLE EXAMPLES

We give some examples to test the inequalities (1.4) and (1.8). Consider first

$$f(z) = \left(\frac{1+z}{1-z}\right)^{2p}.$$

This function maps $|z| < 1$ onto the (possibly self-overlapping) sector $|\arg z| < p\pi$. Thus $f(z)$ is a mean p-valent for any positive p and p-valent if p is an integer. Also

$$\frac{f'(z)}{f(z)} = \frac{4p}{1-z^2}.$$

We deduce that for $z = re^{i\theta}$

$$\left|\frac{f'(z)}{f(z)}\right|^2 = \frac{16p^2}{1-2r^2\cos 2\theta + r^4}.$$

Thus

$$I_2\left(r, \frac{f'}{f}\right) = \frac{1}{2\pi}\int_0^{2\pi} \left|\frac{f'(re^{i\theta})}{f(re^{i\theta})}\right|^2 d\theta$$

$$= \frac{16p^2}{1-r^4}\frac{1}{2\pi}\int_0^{2\pi} \frac{(1-r^4)\,d\theta}{1-2r^2\cos 2\theta + r^4} = \frac{16p^2}{1-r^4} = \frac{4p^2}{1-r} + O(1).$$

Also,

$$(2.1)\quad A(r,f) = 2\pi \int_0^r I_2\left(t,\frac{f'}{f}\right) tdt = 32\pi p^2 \int_0^r \frac{tdt}{1-t^4} = 8\pi p^2 \log\frac{1+r^2}{1-r^2}.$$

Thus (1.8) is sharp and so is the inequality (1.4) apart from a bounded term.

Similarly, if p is a positive integer, $f(z) = z^p(1-z)^{-2p}$ is p-valent in Δ and satisfies $I_2\left(r,\frac{f'}{f}\right) = \frac{2p^2}{1-r} + O(1)$.

Thus the sharpened version of (1.8) for functions with p zeros is also best possible.

Next we show that for small p the term Kp is necessary in (1.4). To see this, suppose that $0 < a < 1/2$, and set

$$w = f(z) = 1 + az.$$

Then $f(z)$ maps Δ onto the disk $|w-1| < a$, which has area πa^2. Hence if $R \leqslant 1/2$, the image of $f(z)$ does not meet $|w| < R$, while if $R > 1/2$, the area of intersection of this image with $|w| < R$ is, at most $\pi a^2 = \pi R^2 a^2/R^2 < 4a^2 \pi R^1$. Thus $f(z)$ is mean p-valent with $p = 4a^2$. Again

$$\left|\frac{f'(z)}{f(z)}\right| = \left|\frac{a}{1+az}\right| \geqslant \frac{a}{2}.$$

Thus for $0 < r < 1$

$$I_2\left(r,\frac{f'}{f}\right) \geqslant \frac{a^2}{4} = \frac{p}{16}.$$

$$A(r,f,\delta) \geqslant \frac{\pi r^2 p}{16},\quad 0 < r < 1.$$

Thus if r is fixed and p tends to zero, (1.4) shows the correct order of magnitude p for $A(r,f,\delta)$.

Finally, consider $f(z) = z^p$, where p is a positive integer. Then $f(z)$ is p-valent and $\frac{f'(z)}{f(z)} = \frac{p}{z}$.

Thus for $\delta < r$

$$A(r,f,\delta) = \int_0^{2\pi} d\theta \int_\delta^r \frac{p^2}{t^2} tdt = 2\pi p^2 \log\frac{r}{\delta}.$$

Hence (1.4) displays the right order of magnitude as $\delta \to 0$ for fixed p and r, at least when q is a positive integer and $p = q$.

The example of Theorem 3 lies considerably deeper and we defer it to the end of the paper.

3. PROOF OF THEOREM 1

3.1. *Preliminary results.* In order to prove Theorem 1, we need some inequalities concerning the growth of functions $f(z)$ in the unit disk in terms of the average number of roots of the equation $f(z) = w$ for varying w. We suppose that $f(z)$, given by (1.1), is regular in Δ and has q zeros there, where q is a positive integer. We define $\Pi(z)$ by (1.2) and set

$$f_0(z) = f(z)/\Pi(z). \tag{3.1}$$

Let $n(w)$ be the number of roots of the equation $f(z) = w$ in Δ and define

$$p(R) = \frac{1}{2\pi}\int_0^{2\pi} n(Re^{i\phi})\, d\phi. \tag{3.2}$$

Then we have

THEOREM 4. *With the above notation, suppose that* $|z_1| < 1$, $|z_2| < 1$ *and set*

$$r = \left|\frac{z_2 - z_1}{1 - \bar{z}_1 z_2}\right|.$$

Then

$$\int_{R_1}^{R_2} \frac{dR}{Rp(R)} < 2\log\frac{1+r}{1-r} + K_1, \tag{3.3}$$

where $R_1 = |f_0(z_1)|$, $R_2 = K_2^{-q}|f_0(z_2)|$, *and we assume* $R_1 \leqslant R_2$.

We shall deduce Theorem 4 from a similar result of Jenkins and Oikawa [4], who used a somewhat different normalisation.They proved

LEMMA 1. *The inequality* (3.3) *holds with* $K_1 = \pi^2$ *and*

$$R_2 = \sup_{|z|=r} |f(z)|, \quad R_1 = \sup_{0\leqslant t<1}\ \inf_{|z|=t} |f(z)|.$$

The inequality is stated by the authors with a slightly larger value of R_1 but the proof goes through when R_1 has the form given above. To deduce Theorem 4 from Lemma 1, we need another result.

LEMMA 2. *If* $z \in \Delta$, *then*

$$|f_0(z)| \leqslant (12e)^q \sup_{|\zeta|<(1/3)(2+|z|)} |f(\zeta)|.$$

We note that by Cartan's Lemma [2] we have

$$\prod_{\nu=1}^{q} |z - \zeta_\nu| \geq h^q$$

outside a set of disks the sum of whose radii is at most $2eh$. We choose

$$h = (8e)^{-1},$$

and deduce that there exists r_1, such that $0 \leq r_1 \leq 1/2$, and

$$\prod_{\nu=1}^{q} |z - \zeta_\nu| \geq (8e)^{-q}, \; |z| = r_1.$$

Thus

$$|\Pi(z)| = \prod_{\nu=1}^{q} \left| \frac{z - \zeta_\nu}{1 - \bar{z}\,\zeta_\nu} \right| \geq (1 + r_1)^{-q} (8e)^{-q} \geq (12e)^{-q}, \; |z| = r_1.$$

In view of the invariance of $\Pi(z)$ under bilinear self maps of Δ, we deduce that for any z_1 in Δ, there exists r_1, such that $0 \leq r_1 \leq 1/2$ and

$$|\Pi(z)| \geq (12e)^{-q}, \text{ if } \left| \frac{z - z_1}{1 - \bar{z}_1 z} \right| = r_1.$$

Also if

$$\left| \frac{z - z_1}{1 - \bar{z}_1 z} \right| = r_1,$$

we have

$$|z| \leq \frac{r_1 + |z_1|}{1 + r_1 |z_1|} \leq \frac{|z_1| + 1/2}{1 + \frac{1}{2}|z_1|} \leq \frac{1}{3}(2 + |z_1|).$$

Thus

$$\begin{aligned}(12e)^{-q} \; |f_0(z_1)| &\leq (12e)^{-q} \max_{|(z-z_1)/(1-\bar{z}_1 z)|=r_1} |f_0(z)| \\ &\leq \max_{|(z-z_1)/(1-\bar{z}_1 z)|=r_1} |f(z)| \leq \max_{|z|=(1/3)(z+|z_1|)} |f(z)|.\end{aligned}$$

This proves Lemma 2.

We can now prove Theorem 4. We assume first $z_1 = 0$. Then, since $f_0(z) \neq 0$ in $|z| < |$, and $|\Pi(z)| < 1$, we deduce

$$\inf_{|z|=t} |f(z)| \leqslant \inf_{|z|=t} |f_0(z)| \leqslant |f_0(0)|, \quad 0 < t < 1.$$

Thus we may choose $R_1 = |f_0(0)|$ in Lemma 1.

We next suppose given r, such that $0 < r < 1$, choose $r_1 = \dfrac{1}{3}(2 + r)$, and apply Lemma 1, with r_1 instead of r,

$$R_2 = \sup_{|z|=r_1} |f(z)|.$$

Then in view of Lemma 2, we have for $|z| = r$

$$|f_0(z)| \leqslant (12e)^q \max_{|z|=r_1} |f(z)| = (12e)^q R_2.$$

Thus we may apply (3.3) with r_1, instead of r, $R_1 = |f_0(0)|$ and $R_2 = (12e)^{-q}|f_0(z)|$. This gives

$$\int_{R_1}^{R_2} \frac{dR}{Rp(R)} < 2\log\frac{1+r_1}{1-r_1} + \pi^2 = 2\log\frac{3(1+r_1)}{1-r} + \pi^2$$
$$< 2\log\frac{1+r}{1-r} + \pi^2 + \log 36.$$

This proves Theorem 4, with $K_1 = \pi^2 + \log 36$, and $K_2 = 12e$, when $z_1 = 0$.

In the general case we use a bilinear transformation $z = \ell(Z)$ of Δ onto itself such that $\ell(0) = z_1$. We set

$$F(z) = f\{\ell(z)\}$$

and apply the above argument to $F(z)$. Suppose that $\ell(Z_2) = z_2$ and define $F_0(z)$ in terms of $F(z)$ as $f_0(z)$ was defined in terms of $f(z)$. Thus

$$|F_0(Z_2)| = F(Z_2)/\Pi\left|\frac{Z_2 - \zeta_\nu'}{1 - \bar{\zeta}_\nu' Z_2}\right|,$$

where $\ell(\zeta_\nu') = \zeta_\nu$. Also

$$\Pi\left|\frac{Z_2 - \zeta_\nu'}{1 - \bar{\zeta}_\nu' Z_2}\right| = \Pi\left|\frac{\ell(Z_2) - \ell(\zeta_\nu')}{1 - \overline{\ell(\zeta_\nu')}\,\ell(Z_2)}\right| = \Pi\left|\frac{z_2 - \zeta_\nu}{1 - \bar{z}_2\zeta_\nu}\right| = \Pi\ (z_2).$$

Hence $|F_0(Z_2)| = f_0(z_2)$. Also $p(R)$ is clearly the same for $f(z)$ and $F(z)$. Thus we may apply Theorem 4 with 0, Z_2, instead of z_1, z_2,

$$|f_0(z_2)| = |F_0(Z_2)|, \; |f_0(z_1)| = |F_0(0)|, \; r = |Z_2| = \left| \frac{z_2 - z_1}{1 - \bar{z}_1 z_2} \right|.$$

This completes the proof of Theorem 4.

3.2. *An area estimate.* We need one other result. This is

LEMMA 3. *Suppose that $f(z)$ is a mean p-valent in a domain D, and let $p(R)$ be defined by (3.2) where $n(w)$ denotes the number of roots of the equation $f(z) = w$ in D. Then if ρ_1, ρ_2 denote the infimum and supremum of $|f(z)|$ in D, we have*

$$A = \int_D \left| \frac{f'(z)}{f(z)} \right|^2 dxdy = 2\pi \int_{\rho_1}^{\rho_2} p(\zeta) \frac{d\zeta}{\zeta} \leq 2\pi p \left\{ \log \frac{\rho_2}{\rho_1} + \frac{1}{2} \right\}. \tag{3.4}$$

We divide D into subdomains, D_ν, such that $\log f(z)$ is univalent in each D_ν, by suitable cuts in D, (see e.g. [3, p. 20]). Then, in each D_ν, $s = g(z) = \log f(z)$ is well defined and maps D_ν onto a set in the s plane, whose area is A_ν say. Further

$$A = \int_D \left| \frac{f'(z)}{f(z)} \right|^2 dxdy = \sum \int_{D_\nu} \left| \frac{f'(z)}{f(z)} \right|^2 dxdy$$

$$= \sum \int_{D_\nu} |g'(z)|^2 \, dxdy = \sum A_\nu$$

say. Thus we need to estimate the areas A_ν.

Let $n(w)$ be the number of roots in D of the equation $f(z) = w$, and write

$$p(R) = \frac{1}{2\pi} \int_0^{2\pi} n(Re^{i\phi}) \, d\phi.$$

Then [3, p. 19] $2\pi p(R)$ is the total variation of $\arg f(z)$ on all the level curves $|f(z)| = R$ in D_ν, and so the total length of all the intersections with the line $\sigma = \log R$ of all the images of the domains D_ν by $s = g(z)$. Thus

$$A = 2\pi \int_{\rho_1}^{\rho_2} p(R) \, d\sigma = 2\pi \int_{\rho_1}^{\rho_2} p(R) \frac{dR}{R}.$$

This proves the first identity in (3.4).

We now use the fact that $f(z)$ is mean p-valent. This can be written as [3, pp. 22, 23]

$$\int_0^R p(\rho) \, d(\rho^2) \leq pR^2, \quad 0 < R < \infty.$$

We set $p(\rho) = p + h(\rho)$,

$$H(R) = \int_0^R h(\rho)\, d(\rho^2),$$

and deduce that $-pR^2 \leqslant H(R) \leqslant 0$, $0 < R < \infty$. Hence

$$\int_{R_1}^{R_2} \frac{h(R)\, dR}{R} = \int_{R_1}^{R_2} \frac{dH(R)}{2R^2} = \frac{H(R_2)}{2R_2^2} - \frac{H(R_1)}{2R_1^2} + \int_{R_1}^{R_2} \frac{H(R)\, dR}{R^3}$$
$$\leqslant \frac{-H(R_1)}{2R_1^2} \leqslant \frac{p}{2}.$$

Thus

$$A = 2\pi \int_{\rho_1}^{\rho_2} \frac{p(R)\, dR}{R} = 2\pi p \log\frac{\rho_2}{\rho_1} + 2\pi \int_{\rho_1}^{\rho_2} \frac{h(R)\, dR}{R} \leqslant 2\pi p \left\{\log\frac{\rho_2}{\rho_1} + \frac{1}{2}\right\},$$

and this proves Lemma 3.

3.3. *Completion of proof of Theorem* 1. We now suppose that

$$\tfrac{1}{2} \leqslant r_1 < r_2 < 1,$$

set $\rho_1 = r_1^2$, define $f_0(z)$ as in (3.1) and set $\mu = f_0(0)$,

$$m_2 = \inf_{|z| \leqslant r_2} |f_0(z)|, \quad M_2 = \sup_{|z| \leqslant r_2} |f_0(z)|,$$

$$m_1 = \inf_{|z| \leqslant \rho_1} |f_0(z)|, \quad M_1 = K_3^{-q}\mu \sup_{|z| \leqslant \rho_1} |f_0(z)|, \quad \text{where } K_3 = \frac{7}{2} K_2.$$

Let $n_1(w)$, $n_2(w)$ denote the number of roots of $f(z) = w$ in $|z| < r_1$ and $r_1 \leqslant |z| < 1$ respectively and define

$$p_j(R) = \frac{1}{2\pi} \int_0^{2\pi} n_j(Re^{i\theta})\, d\theta, \quad p(R) = p_1(R) + p_2(R).$$

We proceed to develop various inequalities relating the above quantities. We have first

$$\int_{\mu}^{M_1} \frac{dR}{Rp_1(R)} < 2\log\frac{1+r_1}{1-r_1} + K_1, \quad \int_{m_1}^{K_3^{-q}\mu} \frac{dR}{Rp_1(R)} < 2\log\frac{1+r_1}{1-r_1} + K_1. \tag{3.5}$$

To see this we apply (3.3) with $F(z) = f(r_1 z)$ instead of $f(z)$. Let z_j, $j = 1,2$ be such that $|z_j| = \rho_1 = r_1^2$, and

$$|f_0(z_1)| = m_1, \quad |f_0(z_2)| = M_1 K_3^q.$$

Let z, z' be points in $|z| < 1$, set $\zeta = z/r_1$, $\zeta' = z'/r_1$ and consider

$$\phi(z,z') = \left(\frac{\zeta' - \zeta}{1 - \bar{\zeta}\zeta'}\right) \Big/ \left(\frac{z - z'}{1 - \bar{z}z'}\right).$$

Then for $|z| = r_1$, $|z'| \leqslant r_1^2$, we have $|\zeta| = 1$, $|\zeta'| \leqslant r_1$ and so

$$1 < |\phi(z,z')| = \left|\frac{1 - \bar{z}z'}{z - z'}\right| \leqslant \frac{1 - r_1^3}{r_1 - r_1^2} = \frac{1 + r_1 + r_1^2}{r_1}$$
$$\leqslant \frac{1}{2} + 1 + 2 = 3\frac{1}{2}.$$

In view of the maximum modulus principle these inequalities for $|\phi(z,z')|$ remain valid for $|z| < r_1$, $|z'| \leqslant r_1^2$. Again, if $1 > |z| > r_1$, $|z'| \leqslant r_1^2$, then

$$1 < \left|\frac{1 - \bar{z}z'}{z - z'}\right| < 3\tfrac{1}{2}.$$

Thus if z_ν, $\nu = 1$ to q are the zeros of f in $|z| < 1$, and $\zeta_\nu = z_\nu/r_1$, the corresponding zeros of $F(\zeta) = f(r_1\zeta)$, we deduce that for $|z'| \leqslant r_1^2$, we have

$$0 \leqslant \sum_{\nu=1}^{q} \log^+ \left|\frac{1 - \bar{z}_\nu z'}{z_\nu - z'}\right| - \sum_{\nu=1}^{q} \log^+ \frac{|1 - \bar{\zeta}_\nu\zeta'|}{|\zeta_\nu - \zeta'|} \leqslant q \log\left(\frac{7}{2}\right).$$

Thus for $|z'| \leqslant r_1^2$, $\zeta' = z'/r_1$, we have

$$|F_0(\zeta')| \leqslant |f_0(z')| \leqslant \left(\frac{7}{2}\right)^q |F_0(\zeta')|.$$

Hence if $\zeta_j = z_j/r_1$, we have

$$K_2^{-q}\,|F_0(\zeta_2)| \geqslant \left(\frac{7}{2}K_2\right)^{-q} |f_0(z_2)| = K_3^{-q}\,|f_0(z_2)| = M_1,$$
$$|F_0(0)| \leqslant \mu = |f_0(0)| \leqslant \left(\frac{7}{2}\right)^q |F_0(0)|,$$

and

$$|F_0(\zeta_1)| \leqslant |f_0(z_1)| = m_1.$$

Also $|\zeta_j| = r_1$, $j = 1,2$. We apply Theorem 4 to $F(z)$ in turn with $(0,\zeta_2)$ and $(\zeta_1, 0)$ instead of z_1, z_2. This yields

$$\int_{\mu}^{M_1} \frac{dR}{Rp_1(R)} < 2 \log \frac{1 + r_1}{1 - r_1} + K_1.$$

This is the first inequality in (3.5). A second application of Theorem 4 yields

$$\int_{m_1}^{K_3^{-q}\mu} \frac{dR}{Rp_1(R)} \leqslant 2 \log \frac{1 \div r_1}{1 - r_1} + K_1,$$

which is the second inequality in (3.5).

Next we note that

$$\log M_2 < \log M_1 + 2p \log \frac{1 - r_1}{1 - r_2} + K_4(p + 1), \tag{3.6}$$

$$\log m_1 < \log m_2 + 2p \log \frac{1 - r_1}{1 - r_2} + K_4(p + 1). \tag{3.7}$$

To prove (3.6), choose $z_2 = r_2 e^{i\theta}$, so that $|f_0(z_2)| = M_2$. Then if $z_1 = \rho_1 e^{i\theta}$, we have $|f_0(z_1)| \leqslant K_3^q M_1$. We set $R_1 = |f_0(z_1)|$, $R_2 = K_2^{-q} |f_0(z_2)|$, and deduce from (3.3), and since f is mean p-valent, [3, p.23]

$$\frac{1}{p}\left(\log \frac{R_2}{R_1} - \frac{1}{2}\right) < \int_{R_1}^{R_2} \frac{dR}{Rp(R)} < 2 \log \frac{1 + r}{1 - r} + K_1,$$

where $r = \dfrac{r_2 - \rho_1}{1 - r_2\rho_1}$,

$$\frac{1 + r}{1 - r} = \frac{(1 + r_2)(1 - \rho_1)}{(1 - r_2)(1 + \rho_1)} = \frac{(1 + r_2)(1 + r_1)(1 - r_1)}{(1 - r_2)(1 + r_1^2)} < \frac{4(1 - r_1)}{1 - r_2}.$$

Thus

$$\begin{aligned} \log \frac{M_2}{M_1} &\leqslant \log \frac{R_2}{R_1} + q \log (K_2 K_3) \\ &\leqslant 2p \log \frac{4(1 - r_1)}{1 - r_2} + \frac{1}{2} + pK_1 + q \log (K_2 K_3), \end{aligned}$$

and this proves (3.6) with $K_4 = K_1 + 1/2 + 2 \log 4 + \log (K_2 K_3)$. The proof of (3.7) is similar.

We now write

$$I = \frac{1}{2\pi p}\left\{A(r_2, f, \delta) - A(r_1, f, \delta)\right\} = \frac{1}{2\pi p}\int_E \left|\frac{f'(z)}{f(z)}\right|^2 dx\,dy,$$

where the integral is taken over the set E of points in $r_1 < |z| < r_2$, where $|\Pi(z) > \delta^q$, and $\Pi(z)$ is given by (1.2). Thus f is mean p-valent in E and

$$m_2\,\delta^q \leqslant |f(z)| \leqslant M_2$$

there. Thus Lemma 3 yields

$$I \leqslant \frac{1}{p}\int_{m_2\delta^q}^{M_2} p_2(\rho)\frac{d\rho}{\rho} = \frac{1}{p}\left(\int_{m_1}^{M_1} + \int_{m_2\delta^q}^{m_1} + \int_{M_1}^{M_2}\right) p_2(\rho)\frac{d\rho}{\rho}$$

$$\leqslant \frac{1}{p}\int_{m_1}^{M_1} p_2(\rho)\frac{d\rho}{\rho} + \log\frac{M_2}{M_1} + \log\frac{m_1}{m_2} + q\log\frac{1}{\delta} + 1. \tag{3.8}$$

Next we note that by Schwarz's inequality

$$\left(\log\frac{M_1}{\mu}\right)^2 \leqslant \int_{\mu}^{M_1}\frac{dR}{p_1(R)}\int_{\mu}^{M_1} p_1(R)\frac{dR}{R}. \tag{3.9}$$

Also $p_2(\rho) \leqslant p(\rho) - p_1(\rho)$, and so, using (3.4) again, we have

$$\int_{\mu}^{M_1} p_2(\rho)\frac{d\rho}{\rho} \leqslant \int_{\mu}^{M_1} p(\rho)\frac{d\rho}{\rho} - \int_{\mu}^{M_1} p_1(\rho)\frac{d\rho}{\rho}$$

$$\leqslant p\left(\log\frac{M_1}{\mu} + \frac{1}{2}\right) - \int_{\mu}^{M_1} p_1(\rho)\frac{d\rho}{\rho}.$$

We combine this with (3.9) and write $t = \log(M_1/\mu)$. We deduce

$$\frac{1}{p}\int_{\mu}^{M_1} p_2(\rho)\frac{d\rho}{\rho} \leqslant t + \frac{1}{2} - t^2\Big/\left\{p\int_{\mu}^{M_1}\frac{d\rho}{\rho p_1(\rho)}\right\} \tag{3.10}$$

$$\leqslant \frac{1}{2} + \frac{p}{4}\int_{\mu}^{M_1}\frac{d\rho}{\rho p_1(\rho)} < \frac{p}{2}\log\frac{1+r_1}{1-r_1} + K_1\frac{p}{4} + \frac{1}{2},$$

in view of (3.5) and the inequality $t - at^2 \leqslant (4a)^{-1}$, for $a > 0$.

Similarly

$$\frac{1}{p}\int_{m_1}^{K_3^{-q}\mu} p_2(\rho)\frac{d\rho}{\rho} \leqslant \frac{1}{2} + \frac{p}{4}\int_{m_1}^{K_3^{-q}\mu}\frac{d\rho}{\rho p_1(\rho)} \tag{3.11}$$

$$< \frac{1}{2}p\log\frac{1+r_1}{1-r_1} + K_1\frac{p}{4} + \frac{1}{2}.$$

Also (3.4) yields

$$\frac{1}{p}\int_{K_3^{-q}\mu}^{\mu} p_2(\rho)\frac{d\rho}{\rho} < \log(K_3{}^q) + \frac{1}{2} < K(p+1). \tag{3.12}$$

On combining (3.6) to (3.8) and (3.10) to (3.12), we deduce that

$$I \leq p \log \frac{1+r_1}{1-r_1} + 4p \log \frac{1-r_1}{1-r_2} + q \log \frac{1}{\delta} + K(p+1),$$

which yields (1.5), if $r_1 \geq \frac{1}{2}$.

We next prove (1.4). Suppose that z_1, z_2 are chosen so that $|z_1| \leq r_2$, $|z_2| \leq r_2$ and

$$|f_0(z_1)| = m_2 = \inf_{|z| \leq r_2} f_0(z), \qquad f_0(z_2)| = M_2 = \sup_{|z| \leq r_2} |f(z)|.$$

Then an application of Theorem 4 and [3, Lemma 2.1, p. 23] yields

$$\frac{1}{p}\left\{\log\left(\frac{M_2 K_2^{-q}}{m_2}\right) - \frac{1}{2}\right\} < \int_{m_2}^{M_2 K_2^{-q}} \frac{dR}{Rp(R)} < 2 \log \frac{1+r}{1-r} + K_1,$$

where $r = \left|\dfrac{z_2 - z_1}{1 - \bar{z}_1 z_2}\right| \leq \dfrac{2r_2}{1+r_2^2}$. Thus $\dfrac{1+r}{1-r} \leq \left(\dfrac{1+r_2}{1-r_2}\right)^2$, and we deduce that

$$\log \frac{M_2}{m_2} < 4p \log \frac{1+r_2}{1-r_2} + K(p+1).$$

Also if E is the subset of $|z| < r_2$, where $|\Pi(z)| > \delta^q$, we have $m_2 \delta^q < |f(z)| < M_2$ in E. Thus Lemma 3 yields

$$A(r_2, f, \delta) = \int_E \left|\frac{f'(z)}{f(z)}\right|^2 dxdy < 2\pi p \left\{\log \frac{M_2}{m_2 \delta^q} + \frac{1}{2}\right\},$$

and (1.4) follows.

Finally if $0 \leq r_1 \leq \frac{1}{2}$, we have

$$A(r_2, f, \delta) - A(r_1, f, \delta) \leq A(r_2, f, \delta),$$

and now (1.5) follows from (1.4).

3.4. *Mean p-valent functions with p zeros.* If $p = q$ in Theorem 1, we can sharpen (1.5), but the improved estimate necessarily depends on the position of the zeros. We prove

THEOREM 5. *Suppose that with the hypotheses of Theorem* 1, $f(z)$ *has* p *zeros. Then we have with the notation of Theorem* 1

(3.14) $$A(r_2, f, \delta) - A(r_1, f, \delta) < \pi p^2 \left\{ \log \frac{1}{1-r_1} + 4 \log \frac{1-r_1}{1-r_2} \right\} + O(1)$$

as $r_1 \to 1$, *while* $r_1 < r_2 < 1$.

With the hypotheses of Theorem 5, $f(z)$ assumes all sufficiently small values exactly once in the neighbourhood of each zero. In other words there exists $\eta > 0$, and $R < 1$, so that for $|w| < \eta$, the equation $f(z) = w$ has exactly p roots *in* $|z| < R$. Thus, since f is mean p-valent, this equation can have no other roots in $|z| < 1$. In particular $|f(z)| > \eta$, $R < |z| < 1$. Hence (3.8) can be sharpened to

(3.15) $$I \leq \frac{1}{p} \int_{\eta}^{M_1} p_2(\rho) \frac{d\rho}{\rho} + \log \frac{M_2}{M_1} + O(1),$$

if $R \leqslant r_1 < r_2 < 1$. We deduce from Lemma 3, that

$$\int_{\eta}^{\mu} p_2(\rho) \frac{d\rho}{\rho} < O(1).$$

On combining this with (3.6), (3.10) and (3.15), we obtain

$$I \leqslant \frac{1}{2} p \log \frac{1}{1-r_1} + 2p \log \frac{1-r_1}{1-r_2} + O(1),$$

which is Theorem 5.

We give an example to show that the term $O(1)$ in Theorem 5 cannot be bounded independently of the position of the zeros. To see this, suppose that $0 < r < 1$ and consider

$$f(z) = \frac{(z+r)(1+rz)}{(1-z)^2(1-r)^2} = \left(\frac{z+r}{1+rz}\right) \Big/ \left(1 - \frac{z+r}{1+rz}\right)^2.$$

Then $f(z)$ is univalent in $|z| < 1$, with a zero at $z = -r$. Also

$$\frac{f'(z)}{f(z)} = \frac{1}{z+r} + \frac{r}{1+rz} + \frac{2}{1-z} \to \frac{2}{1+z} + \frac{2}{1-z}, \text{ as } r \to 1.$$

Thus if ρ is fixed and $r \to 1$

$$A(\rho, f) \to A\left\{\rho, \left(\frac{1+z}{1-z}\right)^2\right\} = 8\pi \log \frac{1+\rho^2}{1-\rho^2}$$

in view of (2.1). This would contradict (3.14) with a bounded $O(1)$ for varying

r, since we could take $1 - r_2$ very small compared with $1 - r_1$. We shall show at the end of the paper that the constant π in (3.14) cannot be replaced by any smaller number.

4. PROOF OF THEOREM 2

We note that $f(z)$ has no zeros in some annulus $r_0 < |z| < 1$. If $r_1 = \frac{1}{2}(1 + r_0)$, then we deduce that for $|z| = r > r_1$ and any zeros ζ_ν of $f(z)$

$$\left|\frac{z - \zeta_\nu}{1 - \bar{z}\zeta_\nu}\right| \geqslant \frac{r_1 - r_0}{1 - r_0 r_1} = \delta$$

say. Thus $|\Pi(z)| \geqslant \delta^p$, for $|z| \geqslant r_1$, and (1.4) yields for $r > r_1$

$$2\pi \int_{r_1}^{r} I_2\left(t, \frac{f'}{f}\right) t dt \leqslant 8\pi p^2 \left\{\log \frac{1}{1 - r} + O(1)\right\} \text{ as } r \to 1. \tag{4.1}$$

Suppose now that (1.8) is false. Then we can choose r_1 so near 1, that we have for $r_1 \leqslant t < 1$

$$I_2\left(t, \frac{f'}{f}\right) > \frac{C}{1 - t},$$

where $Cr_1 > 4p^2$. This yields

$$\int_{r_1}^{r} I_2\left(t, \frac{f'}{f}\right) t dt \geqslant Cr_1 \int_{r_1}^{r} \frac{dt}{1 - t} = Cr_1 \log \frac{1}{1 - r} + O(1),$$

as $r \to 1$, and this contradicts (4.1). Thus (1.8) must hold.

Next, since f'/f is regular for $|z| > r_0$, $I_2(t, f'/f)$ is a convex function of $\log t$ for $t > r_0$. Thus $I_2(t, f'/f)$ is either bounded as $t \to 1$, or increases with t for $r_2 \leqslant t < 1$, where $r_2 < 1$. In the former case (1.7) holds trivially. Thus we may suppose that $I_2(t)$ is finally increasing. In this case, suppose that $0 < \epsilon < 1$ and, given r_1 near 1, define r_2 by $(1 - r_2) = \epsilon(1 - r_1)$. Then (1.5) yields

$$\begin{aligned} 2\pi p^2 \left\{\log \frac{1}{1 - r_1} + O(1)\right\} &\geqslant 2\pi \int_{r_1}^{r_2} I_2\left(t, \frac{f'}{f}\right) t dt \\ &\geqslant 2\pi r_1 (r_2 - r_1) I_2\left(r_1, \frac{f'}{f}\right) \\ &= 2\pi r_1 (1 - \epsilon)(1 - r_1) I_2\left(r_1, \frac{f'}{f}\right). \end{aligned}$$

Thus, for r_1 sufficiently near one we have

$$I_2\left(r_1, \frac{f'}{f}\right) < \frac{p^2}{(1-\epsilon)(1-r_1)r_1}\left\{\log\frac{1}{1-r_1} + O(1)\right\}.$$

Since ϵ may be chosen as small as we please we deduce (1.7).

If $f(z)$ has p zeros in $|z| < 1$, we may replace (1.5) by (3.14) and this yields the required improvements in (1.7) and (1.8). This completes the proof of Theorem 2.

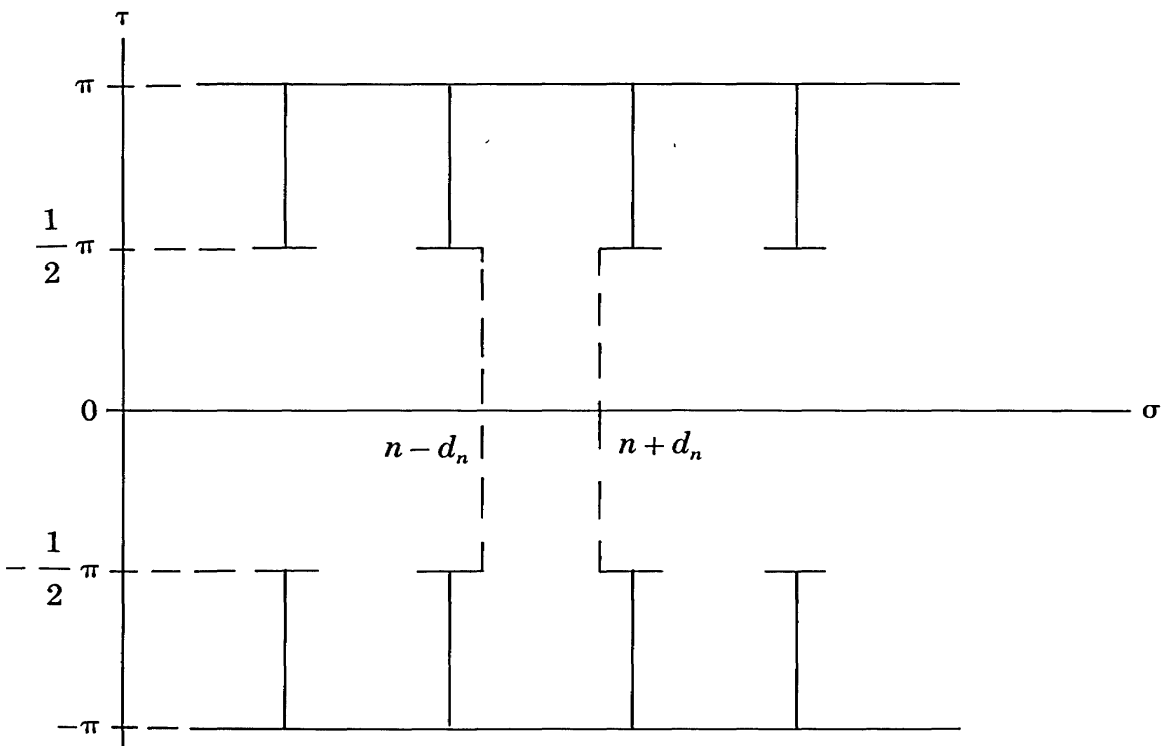
5. PROOF OF THEOREM 3; THE FUNDAMENTAL CONFORMAL MAPPING

Let D be the domain in the s plane, $s = \sigma + i\tau$, defined as follows, where d_n is a sequence of numbers, $0 < d_n < 1/2$, defined below. D consists of the union of all sets S_τ, $-\pi < \tau < \pi$, where

(5.1) $$S_\tau = \{s \mid -\infty < \sigma < \infty\},\ |\tau| < \frac{\pi}{2}$$

(5.2) $$S_\tau = \{s \mid |n - \sigma| < d_n, \quad \text{for some integer } n\}, \quad |\tau| = \pi/2$$

(5.3) $$S_\tau = \{s \mid \sigma + (1/2) \quad \text{is not an integer}\}, \quad (\pi/2) < |\tau| < \pi.$$



D

The quantities d_n are defined as follows. We set

(5.4) $$d_0 = 1/e,$$

(5.5) $$d_{-n} = d_n,\ n \geqslant 1.$$

Suppose next that

(5.6) $$2^{(k-1)^2} \leqslant n < 2^{k^2},$$

where k is a positive integer. Then we define

(5.7) $$2 \log \frac{1}{d_n} = 2^{k^2} - n + k.$$

Thus $d_n \leq e^{-1} < 1/2$ for all n. Clearly (5.1) to (5.7) define D as a simply-connected domain in the s plane consisting of a central strip given by (5.1) connected to a series of rectangular boxes, given by (5.3), by the narrow slits defined by (5.2), subject to (5.4) to (5.6). Evidently D is a symmetrical about the real and imaginary axes.

We can map Δ onto D by a function

(5.8) $$s = g(z),\ g(0) = 0,\ g'(0) > 0.$$

Then $g(z)$ is real and increasing on the interval $(-1,1)$ and maps this interval onto the whole real s axis. Given a positive integer p, we define

(5.9) $$f_p(z) = \exp\{pg(z)\}.$$

We shall show that $f_p(z)$ has the properties asserted in Theorem 3. Suppose first that $p = 1$. Then $f_1(z)$ is clearly univalent since e^s is univalent in the strip $|\tau| < \pi$, which contains Δ. In the general case $f_p(z) = \{f_1(z)\}^p$, and this shows that $f(z)$ is p-valent if p is a positive integer, and (circumferentially) mean p-valent if $p > 0$ [3, p. 94]. Thus we have to prove (1.9). Since

$$A(r, f_p) = p^2 A(r, f_1), \quad p > 0$$

it is sufficient to consider the case $p = 1$.

The idea of our proof is as follows. We define ρ_k by $\log \dfrac{1+\rho_k}{1-\rho_k} = 2^{k^2} + k$. Given $\epsilon > 0$, we shall show that if $K_1(\epsilon)$ is sufficiently large, a proportion at least $1 - \dfrac{1}{2}\epsilon$ of the area in the boxes (5.3) for

(5.10) $$2^{(k-1)^2} - 1/2 < |\sigma| < 2^{k^2} - 1/2$$

will correspond by $g(z)$ to points of Δ lying in the annulus

$$(5.11) \qquad \frac{1-\rho_k}{K_1(\epsilon)} < 1 - |z| < K_1(\epsilon)(1-\rho_k).$$

The total area of these boxes is

$$2\pi\left\{2^{k^2} - 2^{(k-1)^2}\right\} \sim 2\pi \log\frac{1+\rho_k}{1-\rho_k},$$

and (1.9) will follow, with $1 - r_k = K_1(\epsilon)(1-\rho_k)$, $1 - r_k' = (1-\rho_k)/K_1(\epsilon)$.

The hyperbolic distance $d(s_1, s_2; D)$ of two points s_1, s_2 in a simply connected domain D is defined [5, p. 48] as follows. We map Δ onto D in such a way that $z = 0$, r, correspond to $s = s_1, s_2$. If no ambiguity will result, we sometimes write just $d(s_1, s_2)$. Then

$$(5.12) \qquad d(s_1, s_2) = \frac{1}{2}\log\frac{1+r}{1-r}.$$

In this terminology let E_0^+ be the rectangle defined by

$$|\sigma| \leqslant \frac{1}{2}\left(1 - \frac{\epsilon}{4}\right), \quad \frac{\pi}{2}\left(1 + \frac{\epsilon}{8}\right) \leqslant \tau \leqslant \pi\left(1 - \frac{\epsilon}{16}\right).$$

Thus E_0^+ is a compact subset of D, with area $\frac{\pi}{2}\left(1 - \frac{\epsilon}{4}\right)^2 > \left(1 - \frac{\epsilon}{2}\right)\frac{\pi}{2}$. Let E_0^- be the reflection of E_0^+ in the real axis and let E_n^+, E_n^- be obtained from E_0^+, E_0^- respectively by a shift of n in the direction of the positive real axis. We shall prove that if

$$(5.13) \qquad s \in E_n^+ \quad \text{or} \quad s \in E_n^-, \quad 2^{(k-1)^2} \leqslant |n| < 2^{k^2},$$

then

$$(5.14) \qquad 2^{k^2} + k - O(1) < 2\,d(0, s; D) < 2^{k^2} + k + O(1),$$

where the constant implied by $O(1)$ depends only on ϵ. We recall that the area of E_n^+ or E_n^- is at least $\left(1 - \frac{1}{2}\epsilon\right)\frac{\pi}{2}$. Then it will follow that all points in E_n^+ and E_n^-, where n satisfies (5.13) will correspond in Δ to points satisfying (5.11), and this will yield (1.9), with $(1-\rho_k)K_1(\epsilon)$, $(1-\rho_k)/K_1(\epsilon)$ instead of $1 - r_k$, $1 - r_k'$. To deduce (1.10), we apply (1.9) with $\epsilon = 1/2$. This yields

$$2\pi \int_{r_k}^{r_k'} I_2\left(r, \frac{f'}{f}\right) r\,dr > \pi p^2 \log\frac{1}{1-r_k}.$$

Thus for some r with $r_k \leqslant r < r_k'$ we have

$$I_2\left(r, \frac{f'}{f}\right) \geq \frac{p^2}{r_k^2 - r_k'^2} \log \frac{1}{1 - r_k} > Kp^2 \frac{1}{1 - r_k'} \log \frac{1}{1 - r_k'}$$
$$> K_5 p^2 \frac{1}{1 - r} \log \frac{1}{1 - r},$$

which yields (1.10). Unfortunately the constant K_5 obtained thus will be rather small, since our estimates for the term $O(1)$ in (5.14) are not good.

6. SOME INEQUALITIES FOR HYPERBOLIC DISTANCE

The remainder of this paper will be devoted to a proof of (5.14). For this, we need some preliminary estimates concerning hyperbolic distance.

LEMMA 4. *Suppose that D is a simply-connected domain with a line of symmetry L; let w_1 be a point of D not in L and let δ be the minimum hyperbolic distance of w_1 from L, with respect to D. Then if w_2, w_3 are points on L, and*

$$d(w_1, w_2) \leq \delta + C, \tag{6.1}$$

then we have

$$d(w_2, w_3) + \delta - \log 2 - C \leq d(w_1, w_3) \leq d(w_2, w_3) + \delta + C. \tag{6.2}$$

The second inequality is obvious since hyperbolic distance satisfies the triangle inequality. The first inequality of (6.2) shows that in the hyperbolic plane Pythagoras' Theorem asserts that the length of the hypotenuse is "almost" equal to the sum of the other two sides.

We may suppose without loss of generality, that D is the unit disk Δ. We suppose initially that $C = 0$. In this case we may take the real axis for the line of symmetry, and write $w_1 = ir_1$, $w_3 = r_3$ and $w_2 = 0$, where $0 < r_1 < 1$, $0 < r_3 < 1$. Then

$$d(w_1, w_3) = \frac{1}{2} \log \frac{1 + \rho}{1 - \rho}, \tag{6.3}$$
$$d(w_1, 0) = \frac{1}{2} \log \frac{1 + r_1}{1 - r_1}, \quad d(0, w_3) = \frac{1}{2} \log \frac{1 + r_3}{1 - r_3},$$

where

$$\rho = \left| \frac{\bar{w}_3 - w_1}{1 - w_1 \bar{w}_3} \right| = \left| \frac{w_3 - w_1}{1 - w_1 w_3} \right|.$$

Thus

$$1 - \rho^2 = \frac{|1 - w_1 w_3|^2 - |w_3 - w_1|^2}{|1 - w_3 w_1|^2}$$

$$= \frac{1 + r_1^2 r_3^2 - r_3^2 - r_1^2}{1 + r_1^2 r_3^2}$$
$$= \frac{(1 - r_3^2)(1 - r_1^2)}{1 + r_1^2 r_3^2}.$$

Hence

$$\frac{1+\rho}{1-\rho} = \frac{(1+\rho)^2}{1-\rho^2} = \frac{(1+\rho)^2\,(1 + r_1^2 r_3^2)}{(1 - r_1^2)\,(1 - r_3^2)} \tag{6.4}$$
$$= \frac{(1+\rho)^2\,(1 + r_1^2 r_3^2)}{(1 + r_1)^2\,(1 + r_3)^2} \cdot \frac{1 + r_1}{1 - r_1} \cdot \frac{1 + r_3}{1 - r_3}.$$

We now write

$$\phi\,(r_1, r_3) = d\,(w_1, 0) + d\,(0, w_3) - d\,(w_1, w_3).$$

Then if $0 < r_3 < r_3' < 1$, we have

$$\begin{aligned}\phi\,(r_1, r_3') - \phi\,(r_1, r_3) &= d\,(0, r_3') - d\,(0, r_3) - d\,(w_1, r_3') + d\,(w_1, r_3)\\ &= d\,(w_1, r_3) + d\,(r_3, r_3') - d\,(w_1, r_3') \geqslant 0\end{aligned}$$

by the triangle inequality, which holds with equality for 3 points in order along the real axis. Thus $\phi\,(r_1, r_3)$ increases with r_3 and similarly with r_1. Also in view of (6.3), (6.4) we have, as $r_1 \to 1$ and $r_3 \to 1$,

$$\begin{aligned}\phi\,(r_1, r_3) &= \frac{1}{2}\log\frac{1 + r_1}{1 - r_1} + \frac{1}{2}\log\frac{1 + r_3}{1 - r_3} - \frac{1}{2}\log\frac{1+\rho}{1-\rho}\\ &= \log\left\{\frac{(1 + r_1)(1 + r_3)}{(1+\rho)(1 + r_1^2 r_3^2)^{1/2}}\right\} \to \frac{1}{2}\log 2.\end{aligned}$$

Thus

$$0 \leqslant \phi\,(r_1, r_3) \leqslant \frac{1}{2}\log 2, \quad 0 < r_1 < 1, \quad 0 < r_3 < 1. \tag{6.5}$$

This proves (6.2) if $C = 0$, with $\frac{1}{2}\log 2$ instead of $\log 2$.

In the general case we still write $w_1 = ir_1$, $w_3 = r_3$, so that (6.3) holds. We also write $w_2 = \mp r_2$. It now follows from (6.1) and (6.5) that

$$\delta + C \geqslant d\,(w_1, w_2) \geqslant \frac{1}{2}\log\frac{1 + r_1}{1 - r_1} + \frac{1}{2}\log\frac{1 + r_2}{1 - r_2} - \frac{1}{2}\log 2,$$

so that $d(0, w_2) = \frac{1}{2} \log \frac{1 + r_2}{1 - r_2} \leq C + \frac{1}{2} \log 2$. Thus

$$d(w_2, w_3) \leq d(0, w_3) + d(0, w_2) \leq d(0, w_3) + C + \frac{1}{2} \log 2$$

$$\leq d(w_1, w_3) - \delta + \frac{1}{2} \log 2 + C + \frac{1}{2} \log 2,$$

since (6.2) holds when $C = 0$, $w_2 = 0$, with $(1/2) \log 2$ instead of $\log 2$. This completes the proof of Lemma 4.

In order to estimate distances in D effectively we shall need a length area principle. (See e.g. Ahlfors [1, p. 8.]. We include the short proof for completeness.) This is

LEMMA 5. *Suppose that $w = f(z)$ maps an open set D_1 in the z plane* (1,1) *conformally into a set D_2 in the w plane. Let θ_x be the intersection of D_1 with the line $x =$ constant, let $\theta(x)$ be the length of θ_x, let $\ell(x)$ be the length of the image of θ_x in the w plane and let A_2 be the area of D_2. Then*

$$\int \frac{\ell(x)^2 \, dx}{\theta(x)} \leq A_2,$$

where the integral is taken over the set E of all x for which θ_x is not empty.

We note that, by Schwarz's inequality,

$$\ell(x)^2 = \left\{ \int_{\theta_x} |f'(x + iy)| \, dy \right\}^2 \leq \int_{\theta_x} dy \int_{\theta_x} |f'(x + iy)|^2 \, dy.$$

Thus

$$\frac{\ell(x)^2}{\theta(x)} \leq \int_{\theta_x} |f'(x + iy)|^2 \, dy,$$

$$\int_E \frac{\ell(x)^2}{\theta(x)} \, dx \leq \int_E dx \int_{\theta_x} |f'(x + iy)|^2 \, dy \leq A_2,$$

as required. We deduce

LEMMA 6. *Suppose that D_0 contains the rectangle*

$$R_0 = \left\{ s \mid s = \sigma + i\tau, \, |\tau| < \frac{\pi}{2}, \sigma_1 - \frac{\pi}{2} < \sigma < \sigma_2 + \frac{\pi}{2} \right\},$$

where $\sigma_1 < \sigma_2$. Then

$$d(\sigma_1, \sigma_2; D_0) \leqslant \frac{1}{2}(\sigma_2 - \sigma_1 + \pi). \tag{6.6}$$

Suppose further that the complement of D_0 *contains the points* $s = \sigma \mp i\dfrac{\pi}{2}$, *for* $\sigma_1 < \sigma < \sigma_2$, *except possibly for a set* F *of* σ *having measure* ℓ. *Then for* $|\tau_1| < \dfrac{\pi}{2}$, $|\tau_2| < \dfrac{\pi}{2}$, *we have*

$$d(\sigma_1 + i\tau_1, \sigma_2 + i\tau_2; D_0) \geqslant \frac{1}{2}(\sigma_2 - \sigma_1 - \ell - \pi). \tag{6.7}$$

To prove (6.6) we may assume without loss of generality that D is the rectangle R_0, since hyperbolic distance decreases with increasing domain. We then map R_0 onto the strip

$$S = \left\{ |y| < \frac{\pi}{4}, -\infty < x < +\infty \right\},$$

so that $s = \sigma_1, \sigma_2$ correspond to $z = 0, X$.

We now apply Lemma 2 to the inverse map of the rectangle $0 < x < X$, $|y| < \dfrac{\pi}{4}$ onto a subset of R_0. In this map θ_x corresponds to a curve meeting the segment $\sigma_1 < \sigma < \sigma_2$ of the real s axis, and going to the boundary of R_0 in both directions. Since the segment is distant $\dfrac{1}{2}\pi$ from this boundary we have

$$\ell(x) \geqslant \pi, \quad \theta(x) = \frac{\pi}{2}, \quad 0 < x < X.$$

Thus Lemma 5 yields

$$2\pi X \leqslant \int_0^X \frac{\ell(x)^2\, dx}{\theta(x)} \leqslant \pi(\sigma_2 - \sigma_1 + \pi),$$

i.e.

$$X \leqslant \frac{1}{2}(\sigma_2 - \sigma_1 + \pi).$$

The function

$$z = \frac{1}{2}\log\frac{1+\zeta}{1-\zeta}$$

maps Δ onto S, so that $\zeta = 0$, ρ, correspond to $z = 0$, X, where $X = \dfrac{1}{2}\log\dfrac{1+\rho}{1-\rho}$. Thus

$$X = d(0,\rho;\Delta) = d(0,X;S) = d(s_1,s_2;R_0),$$

and this gives (6.6).

To prove (6.7) we map D_0 onto the strip S so that $s_1 = \sigma_1 + i\tau_1$, $s_2 = \sigma_2 + i\tau_2$ correspond to $z = 0, X$. Then we apply Lemma 2 to this map, taking for D_2 the rectangle R_1 given by

$$-\frac{\pi}{4} < x < X + \frac{\pi}{4}, \quad |y| < \frac{\pi}{4},$$

and for D_1 the inverse image of that part of D_2 in the s plane which lies in the rectangle

$$\sigma_1 < \sigma < \sigma_2, \quad |\tau| < \frac{\pi}{2}.$$

For σ not in F the segment $|\tau| < \dfrac{\pi}{2}$ forms a crosscut θ in D_0 which separates s_1 from s_2 in D_0. Thus the image of θ is an arc separating 0 from X in the strip $|y| < \dfrac{1}{4}\pi$, and so this arc meets the segment $0 < x < X$, $y = 0$. Thus the length of the part of this image which is in R_1 is at least $\dfrac{\pi}{2}$. Thus in this case $\theta(\sigma) \leqslant \pi$, $\ell(\sigma) \geqslant \dfrac{\pi}{2}$ for σ not in F, and so $\dfrac{\ell(\sigma)^2}{\theta(\sigma)} \geqslant \dfrac{\pi}{4}$. Integrating over all σ not in F and satisfying $\sigma_1 \leqslant \sigma \leqslant \sigma_2$ we deduce from Lemma 5

$$\frac{\pi}{4}(\sigma_2 - \sigma_1 - \ell) \leqslant \frac{\pi}{2}\left(X + \frac{\pi}{2}\right), \text{ i.e. } X \geqslant \frac{1}{2}(\sigma_2 - \sigma_1 - \ell - \pi).$$

Thus

$$d(s_1, s_2; D_0) = d(0,X;S) = X \geqslant \frac{1}{2}(\sigma_2 - \sigma_1 - \ell - \pi).$$

which is (6.7). We deduce

LEMMA 7. *If D is the domain of section 5 and $s_j = \sigma_j$ is real for $j = 1,2$, then*

$$\frac{1}{2}(\sigma_2 - \sigma_1 - K) < d(s_1, s_2; D) < \frac{1}{2}(\sigma_2 - \sigma_1).$$

The second inequality is obvious since D contains the strip $|\tau| < \frac{1}{2}\pi$. We also note that the complement of D contains the points $s = \sigma \mp i\frac{\pi}{2}$, except when $|\sigma - n| < d_n$. The total length of these exceptional σ is

$$\sum_{n=-\infty}^{+\infty} 2d_n = 2d_0 + 2\sum_{n=1}^{\infty} 2d_n.$$

We write $n = 2^{k^2} - p$, for $1 \leqslant p \leqslant 2^{k^2} - 2^{(k-1)^2}$, and sum over the corresponding range of values of n. If this sum is denoted by Σ_k then by (5.7)

$$\Sigma_k d_n \leqslant \sum_{p=1}^{\infty} e^{-(1/2)(p+k)} < e^{-(1/2)k}/(1 - e^{-(1/2)})$$

Thus

$$\sum_{n=1}^{\infty} d_n \leqslant \sum_{k=1}^{\infty} \Sigma_k d_n \leqslant \frac{1}{1 - e^{-(1/2)}} \sum_{k=1}^{\infty} e^{-(1/2)k} = \frac{e^{-(1/2)}}{(1 - e^{-(1/2)})^2} = K,$$

and

$$\sum_{n=-\infty}^{\infty} d_n \leqslant d_0 + 2K = K_6.$$

Thus (6.7) gives

$$d(s_1, s_2; D) > \frac{1}{2}(\sigma_2 - \sigma_1 - 2K_6 - \pi),$$

as required. This completes the proof of Lemma 7.

We also need to estimate hyperbolic distances from points inside the boxes (5.3) to points on the real axis. We have

LEMMA 8. *Let D be the domain of section 5, set $s_n = n + i(\pi + 1)/2$ and let δ_n be the hyperbolic distance from s_n to the real axis, with respect to D. Then*

$$\delta_n \geqslant \log\frac{1}{d_n} - K_7 \tag{6.8}$$

and

$$d(s_n, n; D) \leqslant \log \frac{1}{d_n} + K_7. \tag{6.9}$$

We start by proving (6.8). We assume that

$$2d_n < \frac{1}{4} e^{-(1/2)\pi} = t_n, \text{ say,} \tag{6.10}$$

since otherwise (6.8) is trivial. Suppose that σ_n is a point on the real s axis which is nearest to s_n, so that

$$d(s_n, \sigma_n; D) = \delta_n.$$

Then we can map D onto Δ by a map $z = z(s)$ so that $s = s_n, \sigma_n$ correspond to $z = 0$, ρ_n, where

$$\delta_n = \frac{1}{2} \log \frac{1 + \rho_n}{1 - \rho_n}.$$

The segment $[0, \rho_n]$ in Δ corresponds to a Jordan arc γ_n from s_n to σ_n in D, which must cross the line $\tau = \frac{\pi}{2}$, where $|n - \sigma| \leqslant d_n$. Since γ_n remains in $\tau > \frac{\pi}{2}$, until it first crosses this line, we see that γ_n must meet the semicircles

$$\left| s - n + d_n - i\frac{\pi}{2} \right| = 2d_n, t_n, \quad \tau > \frac{\pi}{2},$$

where $t_n \geqslant 2d_n$ by (6.10), at points P_n, P_n' say, and γ_n contains an arc γ_n' joining P_n to P_n' and lying otherwise in the region

$$2d_n < \left| s - n + d_n - i\frac{\pi}{2} \right| < t_n, \quad \tau > \frac{\pi}{2}.$$

Similarly γ_n contains an arc γ_n'' with end points Q_n, Q_n' on the semicircles

$$\left| s - n + d_n - i\frac{\pi}{2} \right| = 2d_n, t_n, \quad \tau < \frac{\pi}{2}$$

and lying otherwise in the region

$$2d_n < \left| s - n + d_n - i\frac{\pi}{2} \right| < t_n, \quad \tau < \frac{\pi}{2}.$$

Since γ_n is a geodesic, hyperbolic distances are additive along γ_n, and so

$$\delta_n \geq d(P_n, P_n'; D) + d(Q_n, Q_n'; D). \tag{6.11}$$

We now make a transformation

$$\zeta = \log\left(s_n - n + d_n - i\frac{\pi}{2}\right) - i\frac{\pi}{2} = \xi + i\eta,$$

so that D corresponds to a domain D_0 in the ζ plane. Suppose that P_n, P_n', correspond to p_n, p_n' say. Then

$$p_n' = \log(2d_n) + i\eta_n, \quad p_n = \log t_n + i\eta_n'$$

where $|\eta_n| < \frac{\pi}{2}$, $|\eta_n'| < \frac{\pi}{2}$ and, in view of (6.10), D_0 contains the rectangle R_0

$$\log(2d_n) - \frac{\pi}{2} < \xi < \log t_n + \frac{\pi}{2} = \log\frac{1}{4}, \quad |\eta| < \frac{\pi}{2}$$

while the boundary segments $\eta = \mp\frac{\pi}{2}$, $\log(2d_n) \leq \xi \leq \log t_n$ of R_0, lie in the complement of D_0. We now deduce from (6.7) that

$$d(P_n, P_n'; D) = d(p_n', p_n; D_0) \geq \frac{1}{2}\left\{\log\frac{t_n}{2d_n} - \pi\right\}.$$

Similarly

$$d(Q_n, Q_n'; D) \geq \frac{1}{2}\left\{\log\frac{t_n}{2d_n} - \pi\right\}.$$

Now (6.11) yields (6.8) with $K_7 = 3\left(\frac{\pi}{2} + \log 2\right)$. We next prove (6.9). We set

$$s_n' = n + \frac{i}{2}(\pi - d_n), \quad s_n'' = n + \frac{i}{2}(\pi + d_n).$$

Then

$$d(n, s_n; D) \leq d(n, s_n'; D) + d(s_n', s_n''; D) + d(s_n'', s_n; D). \tag{6.12}$$

We estimate these quantities in turn. Let D_1 be the disk $|s - n| < \pi/2$. Then D_1 contains n and s_n' and is contained in D. Thus

$$d(n, s_n'; D) \leqslant d(n, s_n'; D_1) = \frac{1}{2}\log\left(\frac{2 - d_n/\pi}{d_n/\pi}\right) < \frac{1}{2}\log\frac{2\pi}{d_n}. \tag{6.13}$$

For the function $z = \dfrac{2}{\pi}(s - n)$ maps D_1 onto Δ, so that $s = n$, s_n', correspond to $z = 0$, $i(1 - d_n/\pi)$.

Similarly since D contains the disk D_2 given by

$$|s - s_n''| < \frac{1}{2},$$

we have

$$d(s_n, s_n''; D) \leqslant d(s_n, s_n''; D_2) = \frac{1}{2}\log\frac{2 - d_n}{d_n} < \frac{1}{2}\log\frac{2}{d_n}. \tag{6.14}$$

Finally D contains the disk D_3 given by

$$\left|s - i\frac{\pi}{2}\right| < d_n,$$

and so

$$d(s_n', s_n''; D) \leqslant d(s_n', s_n''; D_3) = \log\frac{1 + 1/2}{1 - 1/2} = \log 3. \tag{6.15}$$

On combining (6.12) to (6.15) we obtain

$$d(n, s_n; D) \leqslant \log\frac{1}{d_n} + \frac{1}{2}\log(36\pi),$$

which yields (6.9) with $K_7 = (1/2)\log(36\pi)$.

7. COMPLETION OF PROOF OF THEOREM 3

We can now put together the results of Lemmas 4, 7, 8 to prove (5.14), thus completing the proof of Theorem 3. We start by proving that

$$|2d(s_n, 0; D) - 2^{k^2} - k| < K \tag{7.1}$$

for n in the range (5.6). We have from Lemma 7

$$\frac{1}{2}n - K < d(0,n) < \frac{1}{2}n.$$

On combining this with (6.9) and (5.7) we obtain

$$\begin{aligned} d(0,s_n) &\leqslant d(0,n) + d(n,s_n) \leqslant \frac{1}{2}n + \log\frac{1}{d_n} + K \\ &= \frac{1}{2}n + \frac{1}{2}(2^{k^2} - n + k) + K \\ &= \frac{1}{2}(2^{k^2} + k) + K. \end{aligned}$$

Next we apply Lemma 4, with s_n, n, 0 instead of w_1, w_2, w_3, $\delta = \delta_n$, and the real s axis instead of L. In view of (6.8), (6.9) we may take $C = 2K_7$ in (6.1) and hence in (6.2) and deduce that

$$\begin{aligned} d(0, s_n) &\geqslant d(n, 0) + \delta_n - 2K_7 - \log 2 \\ &\geqslant \frac{1}{2}n + \log\frac{1}{d_n} - K \\ &= \frac{1}{2}(2^{k^2} + k) - K. \end{aligned}$$

This yields (7.1).

Next let Q_n^+ be the rectangle

$$n - \frac{1}{2} < \sigma < n + \frac{1}{2}, \quad \frac{\pi}{2} < \tau < \pi.$$

If s_n is defined as in Lemma 8 and s_n' is any other point of E_n^+, then $s_0 = s_n - n$ and $s_0' = s_n' - n$ lie in E_0^+. Also

$$d(s_n, s_n'; D) \leqslant d(s_n, s_n'; Q_n^+) = d(s_0, s_0', Q_0) \leqslant K(\epsilon), \tag{7.2}$$

where the constant $K(\epsilon)$ depends only on the compact subset E_0^+ of Q_0^+, which itself depends on ϵ. For we may map Q_0 onto Δ so that s_0 corresponds to $z = 0$ and then E_0 maps onto a compact subset of Δ, whose points have a bounded hyperbolic distance from the origin. Further

$$d(0, s_n; D) - d(s_n, s_n'; D) \leqslant d(0, s_n'; D) \leqslant d(0, s_n; D) + d(s_n, s_n'; D).$$

On combining this with (7.1) and (7.2) we obtain (5.14) for any point $s = s_n'$ in E_n^+, where $n > 0$. By symmetry the same inequalities hold in E_n^- and also in E_{-n}^+, E_{-n}^-. This completes the proof of (5.14) and of Theorem 3.

8. NORMALISED p-VALENT FUNCTIONS

We can also show that the improved estimate of Theorem 5 for p-valent functions with p zeros in Δ is essentially best possible. Consider first the case $p = 1$. Let $f_1(z)$ be the function which we have just constructed to prove Theorem 3, defined by (5.9), so that $f_1(0) = 1$, and define

$$F_1(z) = \frac{f_1(z) - 1}{f_1'(0)} = z + \ldots .$$

Consider the total area A_k of all the E_n^+ and E_n^- for n lying in the range (5.13) and $n > 0$. If k is large we have

$$A_k > (1 - \epsilon)\, \pi 2^{k^2}.$$

Let Δ_k be the subset of Δ which corresponds by $f_1(z)$ to these E_n^+ and E_n^-. Then by (5.11) there exist numbers r_k and r_k' such that

$$0 < r_k < r_k' < 1, \quad 1 - r_k < K(\epsilon)\,(1 - r_k')$$

further

$$\log \frac{1}{1 - r_k} = 2^{k^2} + k + O(1),$$

and Δ_k lies entirely in the annulus $r_k < |z| < r_k'$. Thus

$$\int_{\Delta_k} \left| \frac{f_1'(z)}{f_1(z)} \right|^2 dxdy = \int_{\Delta_k} |g'(z)|^2\, dxdy = A_k.$$

Also in Δ_k we have

$$\log |f_1(z)| = \operatorname{Re} g(z) > 2^{(k-1)^2} - \frac{1}{2}.$$

Thus in Δ_k we have

$$F_1(z) = (1 + o(1))\, \frac{f_1(z)}{f_1'(0)}, \quad F_1'(z) = \frac{f_1'(z)}{f_1'(0)},$$

so that

$$\frac{F_1'(z)}{F_1(z)} = (1 + o(1))\, \frac{f_1'(z)}{f_1(z)}.$$

Hence for large k

$$\int_{\Delta_k} \left| \frac{F_1'(z)}{F_1(z)} \right|^2 dxdy > (1-2\epsilon)\pi 2^{k^2} > (1-3\epsilon)\pi \log\left(\frac{1}{1-r_k}\right).$$

Thus the constant π cannot be replaced by any smaller number in Theorem 5, when $p = 1$. If p is a general positive integer we obtain the same conclusion by considering $F_p(z) = F_1(z)^p$.

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Department of Mathematics
Imperial College,
London S.W. 7.