THE PICK INTERPOLATION THEOREM FOR FINITELY CONNECTED DOMAINS

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Let D be the open unit disk, let $z_1, \ldots, z_n$ be distinct points in D, and let $w_1, \ldots, w_n$ be complex numbers. A theorem of Pick asserts that there is an analytic function ϕ on D satisfying $|\phi(z)| \leq 1$ for z in D and $\phi(z_i) = w_i$ for $i = 1, \ldots, n$ if and only if the matrix

$$\left[\frac{1 - w_i \bar{w}_j}{1 - z_i \bar{z}_j}\right]$$

is nonnegative (positive semidefinite); moreover, the interpolating function ϕ is unique if and only if the determinant of this matrix is zero [16]. The purpose of this paper is to generalize this theorem with D replaced by a finitely connected domain in the plane.

To state the general result, let R be a bounded domain in the plane whose boundary consists of $p + 1$ disjoint analytic Jordan curves, let ∂R denote the boundary of R, let ρ be a nonnegative Borel measurable function on ∂R which is bounded and bounded away from zero, let μ be the measure $d\mu(z) = \rho(z)\,d|z|$, and let $\Lambda = \{(\alpha_1, \ldots, \alpha_p) : |\alpha_k| = 1 \text{ for } k = 1, \ldots, p\}$ be the p-torus. For α in Λ, there is a Hardy space $H^2_\alpha(R)$ of multiple-valued analytic functions on R which are modulus automorphic of index α. These spaces arise in questions on factorization [25], invariant subspaces [18], [23], [24], subnormal operators [2], and extremal polynomials [26]. The space $H^2_\alpha(R)$ can be viewed as a closed subspace of $L^2(\mu)$ and, using the norm in $L^2(\mu)$, the space $H^2_\alpha(R)$ is a functional Hilbert Space over R. Thus, there is a kernel function $k^\alpha(s,t)$ on $R \times R$ such that for f in $H^2_\alpha(R)$ $f(t) = \langle f, k^\alpha_t \rangle$ where $k^\alpha_t(s) = k^\alpha(s,t)$.

THEOREM. *Let* $z_1, \ldots, z_n$ *be distinct points in* R *and let* $w_1, \ldots, w_n$ *be complex numbers. There is an analytic function* ϕ *on* R *satisfying* $|\phi(z)| \leq 1$ *for* z *in* R *and* $\phi(z_i) = w_i$ *for* $i = 1, \ldots, n$ *if and only if the matrix*

$$[(1 - w_i \bar{w}_j)\, k^\alpha(z_i, z_j)]$$

is nonnegative for each α *in* Λ. *The interpolating function* ϕ *is unique if and only if the determinant of this matrix is zero for some* α.

Note that if R is the unit disk and if $\rho \equiv 1$, then Λ consists of one point and the one kernel function involved is the Szegö kernel $k(s,t) = (2\pi)^{-1}(1 - s\bar{t})^{-1}$.

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It follows that the interpolation theorem of Pick is a special case of the theorem above. Also, it should be noted that the kernel functions k^α do depend upon the weight function ρ. However, the interpolation criterion of the theorem, that the matrix $[(1 - w_i \bar{w}_j) k^\alpha (z_i z_j)]$ is nonnegative for each α, is independent of ρ.

Some historical observations may help to place this theorem in perspective. The problem of determining conditions on points $z_1, \ldots, z_n$ in the disk and on points $w_1, \ldots, w_n$ such that these values can be interpolated by an analytic function on the disk bounded by one is known as the Nevanlinna-Pick interpolation problem. The solution mentioned above due to Pick appeared in 1916 [16] while a rather different solution due to Nevanlinna appeared in 1919 [15]. Since that time, several proofs and extensions of the Pick theorem have been found using different blends of function theory and functional analysis [6], [7], [12], [13], [14], [19], [22].

For multiply connected domains, the Nevanlinna-Pick problem was considered by certain authors in the 1940's. The papers of Garabedian [8] and Heins [10] relate geometric properties of the set of n-tuples $(w_1, \ldots, w_n)$ which can be interpolated to the question of uniqueness of the interpolating function. Furthermore, the interpolating function is described geometrically in the case when it is unique. Also, an early paper of Heins shows that the Nevanlinna-Pick problem on an annulus is equivalent to a corresponding Nevanlinna-Pick problem on the disk obtained by means of a universal covering map [9].

The theorem in this paper gives intrinsic necessary and sufficient conditions for existence and uniqueness of an interpolating function on a finitely connected domain. Here, the term intrinsic means that the condition depends on the points $z_1, \ldots, z_n$ and $w_1, \ldots, w_n$ and on the kernel functions k^α which are natural to the domain R. Explicitness and computability are lacking in these conditions because the kernel functions k^α are not known explicitly. The only positive result along these lines is when R is the annulus $r < |z| < 1$ and $\rho \equiv 1$. In this case $k^\alpha(s,t)$ can be computed in terms of the orthogonal basis $\{z^{a+n}: n \text{ an integer}\}$ where $e^{2\pi a i} = \alpha$; one obtains $k^\alpha(s,t) = (2\pi)^{-1} \sum s^{a+n} \bar{t}^{a+n} (1 + r^{2a+2n+1})^{-1}$.

Finally, it should be noted that a normal families argument extends the interpolation theorem of Pick to a theorem on the existence of an analytic extension of a function on an arbitrary subset of the disk [7, Chapter XI], [13]. As was pointed out to the author by Marvin Rosenblum, the same proof applies in this case and one obtains the following theorem: a function f on a subset K of R extends to an analytic function on R bounded by one if and only if the function $(1 - f(s)\bar{f}(t)) k^\alpha(s,t)$ is positive definite on $K \times K$ for each α in Λ.

The proofs in this paper are modeled after those for the unit disk due to Sarason [19]. The existence assertion of the theorem is proved in Section 2 after an introductory section on the $H^2_\alpha(R)$ spaces. The uniqueness assertion is proved in Section 3 and Section 4 presents an example which exhibits the need for considering the spaces $H^2_\alpha(R)$ rather than just the Hardy space $H^2(R)$.

The author would like to thank Donald Marshall for simplifying the proof of the main theorem. In a private communication to the author, Marshall formulated Lemma 5 in Section 2 and indicated how it contains the essential ingredients of a somewhat lengthy development in an earlier draft of this paper.

1. THE HILBERT SPACE $H^2_\alpha(R)$.

Let $C_1,\ldots,C_p$ be pairwise disjoint analytic cuts in the region R such that the complement in R of the union $C = \bigcup \{C_k : k = 1,\ldots,p\}$ is simply connected. For $k = 1,\ldots,p$, let U_k and V_k be open sets in R such that $U_k \cap V_k = \emptyset$ and

$$\partial U_k \cap C = C_k = \partial V_k \cap C.$$

One way to obtain the cuts C_k and the open sets U_k and V_k is to carry out the construction explicitly for an annulus with concentric slits and then map conformally to the domain R [4, Chapter 6, Theorem 10].

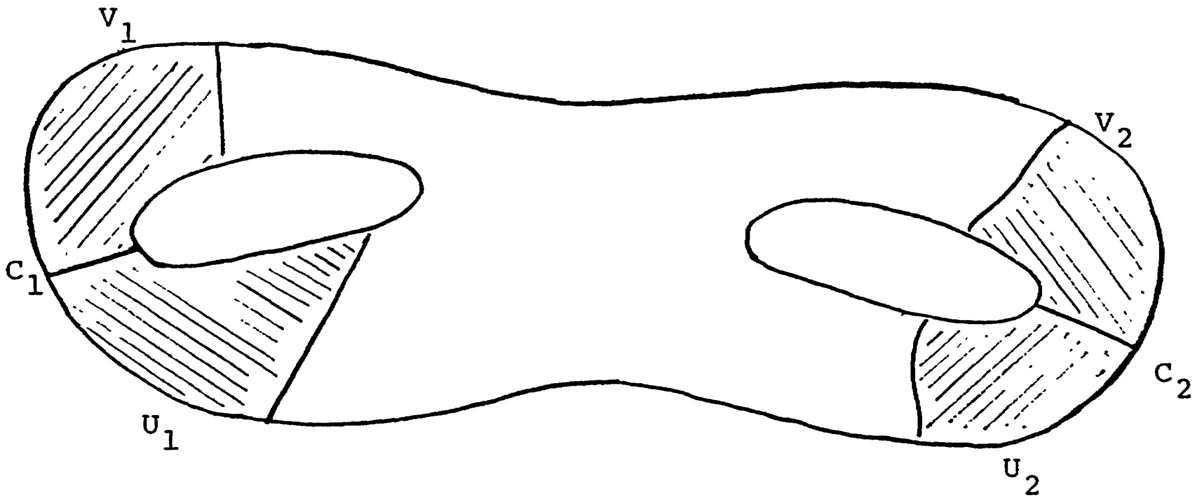

Figure 1.

For $\alpha = (\alpha_1,\ldots,\alpha_p)$ in the p-torus Λ, let $H_\alpha(R)$ be the set of complex functions f on R such that f is analytic on $R\setminus C$, for z in V_k and w in C_k, the function f satisfies $\lim_{z\to w} f(z) = f(w)$, and, for z in U_k and w in C_k, the function f satisfies $\lim_{z\to w} f(z) = \alpha_k f(w)$. Thus, the space $H_\alpha(R)$ is a space of complex functions on R which are analytic except for certain systematic jump discontinuities across the cuts $C_1,\ldots,C_p$. If α is the identity in Λ, that is, if $\alpha = (1,1,\ldots,1)$, then $H_\alpha(R)$ is the space $H(R)$ of all analytic functions on R.

If f is in $H_\alpha(R)$, then f can be extended by analytic continuation to a multiple-valued analytic function F on R which has a single-valued modulus. Such a multiple-valued function is said to be modulus automorphic and it is easily verified that any modulus automorphic function on R can be obtained in this way. Thus, the function f in $H_\alpha(R)$ is single-valued with discontinuities across the cuts and it uniquely determines a multiple-valued function F on R without discontinuities. One refers to the function f in $H_\alpha(R)$ as being modulus automorphic, although this is a slight abuse of the language. The connection between f and α is indicated by saying that the index of f is α, denoted $\mathrm{Index}(f) = \alpha$.

If f and g are modulus automorphic of index α and if a and b are complex numbers, then it is immediate from the definitions that $af + bg$ is modulus automorphic of index α, hence, the space $H_\alpha(R)$ is linear. Furthermore, if f and g are modulus automorphic, then so is fg and Index (fg) = (Index f) $\times$ (Index g). It follows from this equation that $H_\alpha(R)$ is a module over H(R).

The space $H^2_\alpha(R)$ consists of all functions f in $H_\alpha(R)$ such that $|f|^2 \leq u$ with u harmonic on R. For $\alpha = (1,1,\dots,1)$, the space $H^2_\alpha(R)$ is the usual Hardy space $H^2(R)$ which has been studied in depth [1], [3], [11], [17], [18], [20], [26]. The following lemma establishes a close relationship between $H^2_\alpha(R)$ and $H^2(R)$; it is proved in many places, for example [1, Proposition 1.15].

LEMMA 1. *There is a function* E_α *in* $H_\alpha(R)$ *such that* $E_\alpha H^2(R) = H^2_\alpha(R)$. *The function* E_α *is bounded, bounded away from zero, and can be continued analytically across any point of the boundary of* R *which is not an endpoint of one of the cuts* C_k.

It follows from Lemma 1 and known facts about $H^2(R)$ [17] that a function f in $H^2_\alpha(R)$ determines via non-tangential limits a boundary function f^* in $L^2(\mu)$. An inner product is defined on $H^2_\alpha(R)$ by setting $\langle f, g\rangle = \langle f^*, g^*\rangle$. The following lemma is a consequence of Lemma 1 and the corresponding assertions for $H^2(R)$ [17], [26].

LEMMA 2. *The space* $H^2_\alpha(R)$ *is a Hilbert space and the function* $f \to f(t)$ *is a bounded linear functional on* $H^2_\alpha(R)$ *for every* t *in* R.

In light of Lemma 2 and the Riesz representation theorem for bounded linear functionals on a Hilbert space, there is for each t in R a function k^α_t in $H^2_\alpha(R)$ such that $f(t) = \langle f, k^\alpha_t\rangle$. The kernel function for $H^2_\alpha(R)$ is the function on $R \times R$ defined by the equation $k^\alpha(s,t) = k^\alpha_t(s)$. The space $\{f^*: f \text{ in } H^2_\alpha(R)\}$ shall be denoted H^2_α and the space $\{f^*: f \text{ in } H^2(R)\}$ shall be denoted H^2. In the following sections, no distinction will be made between a function f in $H^2_\alpha(R)$ and its boundary function f^* in H^2_α. Thus, for instance, the kernel function k^α_t shall be considered both as a function in $H^2_\alpha(R)$ and as an element in H^2_α.

2. EXISTENCE OF AN INTERPOLATING FUNCTION

The existence proof makes use of the following three lemmas. Lemma 3 is an elementary result which is valid in general functional Hilbert spaces [21, proof of Lemma 4]. Lemma 4 is well known, see for example [1, Theorem 1.7]. Lemma 5 is the key factorization result needed in the proof. This lemma stems from the inner-outer factorization for a function in $H^2(R)$ due to Voichick and Zalcman [25]. A closely related lemma was established by the author in [1, Lemma 4.4]. Let P_α be the orthogonal projection from $L^2(\mu)$ onto H^2_α and let $\mathscr{M}^\perp$ be the orthogonal complement of a subspace $\mathscr{M}$ in $L^2(\mu)$. Let H^∞ be the subspace of $L^\infty(\mu)$ consisting of boundary functions of bounded analytic functions on R; a function in H^∞ shall also be viewed as an analytic function on R.

LEMMA 3. *For* ϕ *in* H^∞, $P_\alpha(\bar{\phi} k^\alpha_t) = \bar{\phi}(t) k^\alpha_t$.

LEMMA 4. *The linear manifold* $H^{2\perp} \cap L^\infty(\mu)$ *is dense in* $H^{2\perp}$.

LEMMA 5. *Let* w *be an invertible function in* $L^\infty(\mu)$. *If* f *is in* $(wH^2)^\perp \cap L^\infty(\mu)$, *then there is an* α *in* Λ *such that* $f = \bar{g}h$ *with* g *in* H^2_α, h *in* $(wH^2_\alpha)^\perp$, *and*

$$|f| = |g|^2 = |h|^2.$$

Proof. Suppose that f is in $(wH^2)^\perp \cap L^\infty(\mu)$. Since $(wH^2)^\perp = \bar{w}^{-1}H^{2\perp}$, the function f satisfies $\int \log|f|\,d|z| < \infty$ [1, Theorem 1.7 and Theorem 1.18]. It follows that there is an α and an outer function g in H^2_α with $|g|^2 = |f|$ [1, Theorem 1.12]. Set $h = f/\bar{g}$. Since $|h|^2 = |f|^2/|g|^2 = |f|$, it remains to show that h is orthogonal to wH^2_α. Since g is outer and in $H^2_\alpha \cap L^\infty(\mu)$, the space gH^2 is a dense linear manifold in H^2_α [25, Theorem 3]. Thus, it is sufficient to show that h is orthogonal to wgH^2. For this, take k in H^2. Since f is in $(wH^2)^\perp$,

$$0 = \langle f, wk\rangle = \langle f/\bar{g}, wgk\rangle = \langle h, wgk\rangle$$

and this completes the proof of the lemma.

THEOREM 1. *Let* $z_1, \ldots, z_n$ *be distinct points in* R, *let* $w_1, \ldots, w_n$ *be complex numbers, and let* A *be a nonnegative real number. There is an analytic function* ϕ *on* R *satisfying* $|\phi(z)| \le A$ *for* z *in* R *and* $\phi(z_i) = w_i$ *for* $i = 1, \ldots, n$ *if and only if the matrix*

$$[(A^2 - w_i\bar{w}_j)\,k^\alpha(z_i, z_j)]$$

is nonnegative for each α *in* Λ.

Proof. Assume that ϕ is in H^∞ and that $F = \{z_1, \ldots, z_n\}$. For s in F, let C_s be a complex number and set

$$k = \sum_s \bar{C}_s k^\alpha_s. \tag{1}$$

The following two calculations are elementary; equation (3) uses Lemma 3.

$$\|k\|^2 = \sum_{s,t} C_s \bar{C}_t k^\alpha(s, t). \tag{2}$$

$$\|P_\alpha(\bar{\phi}k)\|^2 = \sum_{s,t} C_s \bar{C}_t \phi(s)\bar{\phi}(t) k^\alpha(s, t). \tag{3}$$

Assume further that $\phi(z_i) = w_i$ for $i = 1, \ldots, n$ and let $\mathscr{M}_\alpha$ be the set of functions k as defined by Equation (1). Equations (2) and (3) show that the assertion

$$\|P_\alpha(\bar{\phi}k)\|^2 \le A^2\|k\|^2 \tag{4}$$

for all k in $\mathscr{M}_\alpha$ is equivalent to the assertion

$$[(A^2 - w_i \bar{w}_j)\, k^\alpha(z_i, z_j)] \geq 0. \tag{5}$$

If $\|\phi\| \leq A$, then (4) is evidently true, which implies (5) and this proves one direction of Theorem 1.

To prove the converse, assume (5) for each α in Λ and let ϕ be a polynomial such that $\phi(z_i) = w_i$ for $i = 1, \ldots, n$. The discussion above shows that the validity of (5) implies that of (4). Let w be the polynomial $w(z) = (z - z_1) \ldots (z - z_n)$, let g be in H^2_α, and let h be in $(wH^2_\alpha)^\perp$. It is easily verified that

$$H^2_\alpha = \mathscr{M}_\alpha \oplus wH^2_\alpha \tag{6}$$

and it follows from (6) that $k = P_\alpha(h)$ is in $\mathscr{M}_\alpha$. Hence, by (4),

$$\begin{aligned} \left| \int \bar{\phi}\, \bar{g}\, h\, d\mu \right| &= |\langle h, \phi g\rangle| = |\langle P_\alpha(h), \phi g\rangle| \\ &= |\langle k, \phi g\rangle| = |\langle P_\alpha(\bar{\phi} k), g\rangle| \\ &\leq \|P_\alpha(\bar{\phi} k)\| \|g\| \leq A \|k\| \|g\| \leq A \|h\| \|g\|. \end{aligned} \tag{7}$$

Assertion (7) and Lemma (3) imply that

$$\left| \int \bar{\phi} f\, d\mu \right| \leq A \|f\|_1 \tag{8}$$

for all f in $(wH^2)^\perp \cap L^\infty(\mu)$. It follows from (8) and the Hahn-Banach Theorem, that there is a function ψ in $L^\infty(\mu)$ such that $\|\psi\|_\infty \leq A$ and

$$\int \bar{\phi} f\, d\mu = \int \bar{\psi} f\, d\mu \tag{9}$$

for all f in $(wH^2)^\perp \cap L^\infty(\mu)$. In particular, the function $\phi - \psi$ is orthogonal in $L^2(\mu)$ to $(wH^2)^\perp$ and therefore there is a function η in H^2 with $\phi - \psi = w\eta$. It follows that $\psi(z_i) = \phi(z_i) = w_i$ for $i = 1, \ldots, n$ and this completes the proof of the theorem.

3. UNIQUENESS OF THE INTERPOLATING FUNCTION

Let $z_1, \ldots, z_n$ be distinct points in R, let $w_1, \ldots, w_n$ be complex numbers, and assume that there is an analytic function ϕ on R with $\phi(z_i) = w_i$ for $i = 1, \ldots, n$ and $|\phi(z)| \leq A$ for z in R. This section considers the uniqueness of the function ϕ. A theorem of Garabedian asserts that if ϕ is unique, then ϕ extends analytically across the boundary of R, $|\phi(z)| = A$ for z in the boundary of R, and the number of zeros of ϕ in R (counting multiplicities) is less than $n + p$ [8, Theorem 4; 9]. Theorem 2 below gives a criterion for uniqueness. This theorem generalizes a result for the unit disk due to Pick [7, Chapter XI], [14], [16]. The proof of Theorem 2 makes use of the following lemma due to Widom [26, Theorem 7.3].

LEMMA 6. *For* z *and* w *in* R, *the function* $\alpha \to k^{\alpha}(z,w)$ *is continuous on* Λ.

THEOREM 2. *The interpolating function is unique if and only if*

$$\det\,[(A^2 - w_i\bar{w}_j)\,k^{\alpha}(z_i, z_j)] = 0$$

for some α.

Proof. Let M_α be the matrix $[(A^2 - w_i\bar{w}_j)\,k^{\alpha}(z_i,z_j)]$. The assumption that an interpolating function exists implies by Theorem 1 that $M_\alpha \geq 0$ for each α. Suppose that $\det M_\alpha \neq 0$ for each α. Then each M_α is an invertible positive matrix and the function $\alpha \to M_\alpha$ is continuous by Lemma 6. Since the p-torus Λ is compact, it follows that there is an $\varepsilon > 0$ such that $M_\alpha \geq \varepsilon\, I$ for each α where I is the $n \times n$ identity matrix. Let A_α be the matrix $[k^{\alpha}(z_i,z_j)]$. Then each A_α is a positive matrix and the function $\alpha \to A_\alpha$ is continuous by Lemma 6. The compactness of Λ implies the existence of a $\delta > 0$ such that $\varepsilon I \geq \delta A_\alpha$ for each α. Thus, one obtains $M_\alpha \geq \delta A_\alpha$ for each α. This implies that $M_\alpha - \delta A_\alpha \geq 0$ which says that $[(A^2 - \delta - w_i\bar{w}_j)\,k^{\alpha}(z_i,z_j)] \geq 0$. Theorem 1 implies the existence of a function ϕ in H^∞ with $\|\phi\|_\infty \leq \sqrt{A^2 - \delta}$ and $\phi(z_i) = w_i$ for $i = 1,\ldots,n$. Thus, if

$$w(z) = (z - z_1)\ldots(z - z_n),$$

then for each β with $|\beta| < A - \sqrt{A^2 - \delta}$, the function $\psi = \phi + \beta w/\|w\|_\infty$ satisfies $\|\psi\|_\infty \leq A$ and $\psi(z_i) = w_i$ for $i = 1,\ldots,n$. Hence, the interpolating function is not unique.

Conversely, let ϕ be a function in H^∞ with $\|\phi\|_\infty \leq A$ and $\phi(z_i) = w_i$ for $i = 1,\ldots,n$ and assume that $\det M_\alpha = 0$. It follows that there are complex numbers C_s not all zero such that

$$\sum C_s\bar{C}_t(1 - \phi(s)\overline{\phi(t)})\,k^{\alpha}(s,t) = 0. \tag{10}$$

If $k = \sum C_s k^{\alpha}_s$, then (2), (3), and (10) say that

$$\|P_\alpha(\bar{\phi}k)\|^2 = A^2\|k\|^2. \tag{11}$$

Since $\|\phi\|_\infty \leq A$, one has $\|P_\alpha(\bar{\phi}k)\|^2 \leq \|\bar{\phi}k\|^2 \leq A^2\|k\|^2$ and this combined with (11) implies that $\bar{\phi}k = g$ with g in H^2_α. Since the function k cannot vanish on a set of positive measure [1, Corollary 1.19], one has the representation $\phi = g/k$ which shows that ϕ is unique. This completes the proof of Theorem 2.

As for the unit disk [19, Proposition 5.1], the proof of Theorem 2 shows that if the interpolating function ϕ is unique, then $|\phi| = A$ $d\mu$-almost-everywhere and $\phi = g/k$ with both g and k in the space $\mathcal{M}_\alpha$, the set of linear combinations of the kernel functions $k^{\alpha}_{z_1},\ldots,k^{\alpha}_{z_n}$. It is possible to use these facts to deduce that ϕ is continuous across the boundary of R and has less than $n + p$ zeros in R, thus recovering the aforementioned theorem of Garabedian. This analysis requires a careful look at functions in $\mathcal{M}_\alpha$ and will not be carried out here.

4. AN EXAMPLE

Let $k(s,t)$ be the kernel function $k^\alpha(s,t)$ with $\alpha = (1,\ldots,1)$. If $\rho \equiv 1$ $(d\mu(z) = d|z|)$, then k is the Szegö kernel function for R and it is known that there are points z_1 and z_2 in R with $k(z_1,z_2) = 0$ [5, Chapter VII]. Given such z_1 and z_2 and given complex numbers w_1 and w_2, the matrix $[(1 - w_i\bar{w}_j)\, k(z_i,z_j)]$ is the diagonal matrix

$$\begin{bmatrix} (1 - |w_1|^2)\, k(z_1,z_1) & 0 \\ 0 & (1 - |w_2|^2)\, k(z_2,z_2) \end{bmatrix}.$$

Since $k(z_1,z_1)$ and $k(z_2,z_2)$ are positive, this matrix is nonnegative if and only if $|w_1| \le 1$ and $|w_2| \le 1$. This condition is clearly not sufficient to guarantee the existence of an analytic function ϕ on R with $\phi(z_1) = w_1$, $\phi(z_2) = w_2$, and $|\phi(z)| \le 1$ for all z in R, for example, consider $w_1 = 0$ and $w_1 = 1$. This example shows that it is not enough to consider only the Szegö kernel as in the case for the unit disk.

One can inquire about what subsets Λ_0 of Λ are sufficient for Theorem 1, that is, for which Λ_0 is the existence of an interpolating function equivalent to the nonnegativity of the matrices $[(1 - w_i w_j)\, k^\alpha(z_i,z_j)]$ for α in Λ_0. Theorem 1 asserts that $\Lambda_0 = \Lambda$ has this property. Lemma 6 implies that these matrices are continuous in α and therefore any dense set Λ_0 of Λ has this property. The example above is a case where Λ_0 contains exactly one point and Λ_0 does not have the property. The author conjectures that density of Λ_0 in Λ is needed; if Λ_0 omits a non-empty open subset of Λ, then the theorem fails.

REFERENCES

1. M. B. Abrahamse, *Toeplitz operators in multiply connected regions.* Amer. J. Math. 96 (1974), 261–297.
2. M. B. Abrahamse and R. G. Douglas, *A class of subnormal operators related to multiply-connected domains.* Advances in Math. 19 (1976), 106–148.
3. P. R. Ahern and D. Sarason, *The* H^p *spaces of a class of function algebras.* Acta Math. 117 (1967), 123–163.
4. L. V. Ahlfors, *Complex analysis: An introduction to the theory of analytic functions of one complex variable.* Second edition. McGraw-Hill Book Co., New York-Toronto, Ont.-London, 1966.
5. S. Bergman, *The kernel function and conformal mapping.* Mathematical Surveys, No. 5, American Mathematical Society, New York, N.Y., 1950.
6. D. N. Clark, *On matrices associated with generalized interpolation problems.* Pacific J. Math. 27 (1968), 241–253.
7. W. F. Donoghue, *Monotone matrix functions and analytic continuation.* Springer-Verlag, Berlin-New York, 1974.
8. P. R. Garabedian, *Schwarz's lemma and the Szegö kernel function.* Trans. Amer. Math. Soc. 67 (1949), 1–35.

9. M. H. Heins, *Extremal problems for functions analytic and single-valued in a doubly-connected region.* Amer. J. Math. 62 (1940), 91–106.

10. ———, *A lemma on positive harmonic functions.* Ann. of Math. (2) 52 (1950), 568–573.

11. ———, *Hardy classes on Riemann surfaces.* Lecture Notes in Mathematics, No. 98. Springer-Verlag, Berlin-New York, 1969.

12. A. Koranyi and B. Sz.-Nagy, *Relation d'un problème de Nevanlinna et Pick avec la théorie des opérateurs de l'espace hilbertien.* Acta Math. Acad. Sci. Hungar. 7 (1956), 295–303.

13. M. Krein and P. Rechtmann, *Sur une question de Nevanlinna-Pick.* Travaux de l'Univ. d'Odessa 2 (1938), 63–68.

14. D. E. Marshall, *An elementary proof of the Pick-Nevanlinna interpolation theorem.* Michigan Math. J. 21 (1974), 219–223.

15. R. Nevanlinna, *Über. beschränkte Funktionen, die in gegebenen Punkten vorgeschriebene Werte annehmen.* Ann. Acad. Sci. Fenn. Ser. A I Vol. 13 (1919), No. 1.

16. G. Pick, *Über die Beschrankungen analytischer Funktionen, welche durch vorgegebene Funktionswerte bewirkt werden.* Math. Ann. 77 (1916), 7–23.

17. W. Rudin, *Analytic functions of class* H_p. Trans. Amer. Math. Soc. 78 (1955), 46–66.

18. D. Sarason, *The* H^p *spaces of an annulus.* Mem. Amer. Math. Soc. No. 56 (1965).

19. ———, *Generalized interpolation in* H^∞. Trans. Amer. Math. Soc. 127 (1967), 179–203.

20. ———, *Representing measures for* R(X) *and their Green's functions.* J. Funct. Anal. 7 (1971), 359–385.

21. A. L. Shields and L. J. Wallen, *The commutants of certain Hilbert space operators.* Indiana Univ. Math. J. 20 (1970/71), 777–788.

22. A. Snyder, *Multipliers and the Nevanlinna-Pick Theorem.* Indiana Univ. Math. J. 25 (1976), 1159–1170.

23. M. Voichick, *Ideals and invariant subspaces of analytic functions.* Trans. Amer. Math. Soc. 111 (1964), 493–512.

24. ———, *Invariant subspaces on Riemann surfaces.* Canad. J. Math. 18 (1966), 399–403.

25. M. Voichick and L. Zalcman, *Inner and outer functions on Riemann surfaces.* Proc. Amer. Math. Soc. 16 (1965), 1200–1204.

26. H. Widom, *Extremal polynomials associated with a system of curves in the complex plane.* Advances in Math. 3 (1969), 127–232.

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