FREE HEEGAARD DIAGRAMS AND EXTENDED NIELSEN TRANSFORMATIONS, I

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This is the first of two papers (see also [7]) devoted to the study of a free analogue, called here a free splitting homomorphism, of the algebraic formalism for Heegaard splittings of 3-manifolds due to Stallings [30] and Jaco [10]. Roughly, free splitting homomorphisms come from replacing surface groups by free groups in the Stallings-Jaco formalism. A free splitting homomorphism has the form, $\psi = \psi_1 \times \psi_2 \colon G^m \to X^n \times Y^n$, where G^m, X^n, and Y^n are free groups of ranks m, n, and n respectively, and each of the factor homomorphisms ψ_1 and ψ_2 is surjective. We will show in Theorem 4.1 that, after allowances are made for stabilization, the free splitting homomorphism theory is equivalent to the theory of extended Nielsen transformations [2]. (Extended Nielsen transformations will be described in Section 1.)

The connection between the two theories will be made by normalizing a free splitting homomorphism ψ as above so that for free bases $\{g_i \colon i \leq m\}$ and $\{x_i \colon i \leq n\}$ for G^m and X^n, ψ has the form $\psi(g_i) = (x_i, v_i)$ $(i \leq n)$ and $\psi(g_{i+n}) = (1, r_i)$. One will then have an associated group presentation $\mathscr{P}(\psi) = \langle Y^n \colon (r_i) \rangle$. Theorem 4.1 states that two free splitting homomorphisms ψ and ϕ are stably equivalent if and only if, after normalization, the associated group presentations $\mathscr{P}(\psi)$ and $\mathscr{P}(\phi)$ are equivalent in the sense of extended Nielsen transformations including stabilization.

Two applications of Theorem 4.1 will then be given. The first of these, Theorem 5.2, concerns simplifying a group presentation $\langle Y^n \colon (r_i) \rangle$ to a presentation $\langle Y^q \colon (s_i) \rangle$ $(q < n)$ through the use of extended Nielsen transformations including stabilization. Theorem 5.2 says that this can be done if there are q elements $w_1, \ldots, w_q$ in Y^n such that $\{w_1\} \cup \{r_i\}$ generates Y^n. The second application, Theorem 5.3, shows that the stable form of the Andrews-Curtis conjecture on presentations for the trivial group holds if and only if, in a stable sense, the conclusion of the Grusko-Neumann theorem (see [9], [22], [11], [29], [12], [17]) holds for surjective homomorphisms of the form, $G^{2n} \to X^n \times Y^n$ where G^{2n}, X^n, and Y^n are free groups as before.

The second of the two papers will apply methods due to Rapaport [26] to compare equivalence for free splitting homomorphisms with equivalence under extended Nielsen transformations. Another criterion for simplifying a presentation will be given which mostly involves normal closure properties. Finally, some examples will be proposed which we suspect distinguish between two different kinds of extended Nielsen equivalence classes for balanced presentations of the trivial group.

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Here is an outline of the paper. In Section 1 we discuss some notation and we review extended Nielsen transformations and the Andrews-Curtis conjecture. In Section 2 we describe the free splitting homomorphism theory. Section 3 is devoted to obtaining two normal forms for free splitting homomorphisms. In Section 4 we prove the main classification theorem relating the two theories. The two applications described above are given in Section 5. We close the paper in Section 6 with some remarks and questions.

At about the same time that this paper and its sequel [7] were written, Wolfgang Metzler produced, independently, a manuscript on extended Nielsen transformations [19]. There is a small overlap of the results in the papers. The interested reader should compare Theorem 4 in [19] with Theorem 5.3 here and Theorem 4.3 in [7].

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SECTION 1. EXTENDED NIELSEN TRANSFORMATIONS

Throughout this paper G^m, X^n, and Y^n $(m = 1,2,\ldots,n = 1,2,\ldots)$ will denote free groups of ranks m, n, and n. It will be assumed that these free groups have canonical free bases $\{g_i: 1 \le i \le m\}$, $\{x_i: 1 \le i \le n\}$, and $\{y_i: 1 \le i \le n\}$. In particular then, for G^m, X^n, or Y^n, it will be assumed that these bases are nested for different values of m or n.

If $\{r_i\}$ is a set of elements in a free group W, then $Gp(\{r_i\})$ or $Gp_W(\{r_i\})$ will denote the subgroup of W generated by the elements r_i. Similarly $Cl(\{r_i\})$ or $Cl_W(\{r_i\})$ will denote the smallest normal subgroup of W containing the elements r_i. We will use the expression, id (identity), to denote the identity automorphism on a group, and in order to avoid confusion with the identity element of a group we will use 0 to denote any group homomorphism $\lambda: A \to B$ such that $\lambda(A) = 1 \in B$.

Let (r_i) be a p-tuple of elements in a free group W. By a *Nielsen transformation* on the p-tuple, denoted by $(r_i) \to (s_i)$, we mean any finite composition of elementary transformations of the following two types:

$$\begin{array}{lll} \textit{Type } 1\text{:} & s_i = r_i & (i \neq k) \\ & s_k = r_k^{-1}. & \\ \textit{Type } 2\text{:} & s_i = r_i & (i \neq k) \\ & s_k = r_k r_j & (j \neq k). \end{array}$$

Let $\{w_i: 1 \le i \le n\}$ be a free basis for the free group W, and let $w \in W$. Then w is expressed uniquely as a freely reduced product $w = \prod_{j=1}^{q} w_{i_j}^{\varepsilon_j}$ $(\varepsilon_j = \pm 1)$. The *length* $L(w)$ of w relative to the alphabet $\{w_i\}$ is defined to be the number of syllables, q, in the expression above. A Nielsen transformation $(r_i) \to (s_i)$ on p-tuples of elements of W is said to be *length reducing* (*length preserving*) with respect to the alphabet $\{w_i\}$ if $\sum L(s_i) < \sum L(r_i)$ $\left(\sum L(s_i) = \sum L(r_i)\right)$, and a p-tuple (r_i) is said to be *Nielsen reduced* with respect to $\{w_i\}$ if there exists no Nielsen transformation diminishing the sum of the lengths of the elements of (r_i). Section 3.2 in Magnus, Karass, and Solitar's book [15] and Section 2 of Chapter 1 in Lyndon and Schupp's book [13] are good references for the following results of Nielsen's on Nielsen transformations which we will need on several occasions in the two papers: First, any p-tuple (r_i) of elements in W can be transformed to a Nielsen reduced p-tuple (s_i) by a sequence of elementary Nielsen transformations that are length preserving or length reducing. Second, the non-trivial elements in the Nielsen reduced p-tuple (s_i) freely generate a subgroup S of W. Third, if $S = W$, then (s_i) is up to inversions a permutation of the p-tuple $(w_1, \ldots, w_n, 1, \ldots, 1)$ where the number of 1's is $(p - n)$. Now permutations of elements are Nielsen transformations (see Lemma 1.1_2), but they are not elementary Nielsen transformations. It thus follows from the above that any p-tuple (r_i) for which $\{r_i\}$ generates W, can be transformed to $(w_1, \ldots, w_n, 1, \ldots, 1)$ by a sequence of length preserving and length reducing Nielsen transformations.

Consider now the following six types of elementary transformations $(r_i) \to (s_i)$ on a p-tuple of elements in Y^n:

Type 1: Nielsen Type 1.

Type 2: Nielsen Type 2.

Type 3:
$$s_i = r_i \qquad (i \neq k)$$
$$s_k = y_j^{-\varepsilon} r_k y_j^{\varepsilon} \qquad (y_j \in \{y_i\}, \varepsilon = \pm 1).$$

Type 4: $s_i = \lambda(r_i)$ $(i \leq p)$ where λ is an automorphism of Y^n.

Type 5: Replace Y^n by Y^{n+1} and augment (r_i) to the $(p + 1)$-tuple
$$(s_i) = ((r_i), y_{n+1}).$$

Type 6: If $r_p = y_n$ and if each r_i $(i < p)$ belongs to Y^{n-1}, replace Y^n by Y^{n-1} and diminish (r_i) to the $(p - 1)$-tuple $(r_1, \ldots, r_{p-1})$.

By an *extended Nielsen transformation* (compare [1]) we mean any finite composition of transformations of Types 1–6. An equivalent set of transformations that avoids automorphisms of Y^n is described in the Appendix to this paper. We prefer to follow language of Rapaport's [26] and Metzler's [18] for describing various classes of extended Nielsen transformations: By a Q, Q^*, resp. Q^{**}*-transformation,* $(r_i) \to (s_i)$ we mean any finite composition of transformations of Types 1–3, Types 1–4, resp. Types 1–6. We emphasize that Q^{**}-transformations can change the sizes of tuples whereas Q and Q^*-transformations cannot change sizes. The definition of a Type 4 transformation appears to differ slightly from the one Metzler uses. He uses automorphisms of Y^n that are induced by elementary Nielsen transformations on the free basis $\{y_i\}$, but by [23] all automorphisms of Y^n are induced by compositions of the elementary transformations just mentioned; so the two definitions of Q^*-transformations are equivalent. Two tuples (r_i) and (s_i) are defined to be Q, Q^*, resp. Q^{**}*-equivalent* if one is a Q, Q^*, resp. Q^{**}-transform of the other. It is easy to check that these are equivalence relations (see the following lemma). We will speak interchangeably of Q, Q^*, or Q^{**}-equivalence of tuples (r_i) and (s_i) and Q, Q^*, or Q^{**}-equivalence of the corresponding group presentations $\langle Y^{n_1}:(r_i)\rangle$ and $\langle Y^{n_2}:(s_i)\rangle$ for the factor groups $Y^{n_1}/Cl(\{r_i\})$ and $Y^{n_2}/Cl(\{s_i\})$.

LEMMA 1.1. *Let (r_i) be a p-tuple of elements in Y^n.*

(1) *If $(r_i)\to(s_i)$ is a Nielsen, Q, Q^*, or Q^{**}-transformation, then $(s_i)\to(r_i)$ is a transformation of the same type.*

(2) *The transformation $(r_i)\to(s_i)$, $s_i=r_i$ $(i\neq j,k)$, $s_j=r_k$, $s_k=r_j$ is a Nielsen transformation.*

(3) *If w is an element of Y^n, then the transformation $(r_i)\to((r_i),y_{n+1}w)$ is a Q^{**}-transformation.*

Proof. We leave the verification of (1) to the reader. The transformation in (2) is effected as follows:

(a) $r_k\to r_k^{-1}\to r_k^{-1}r_j\to r_j^{-1}r_k=s_k(1)$.

(b) $r_j\to r_jr_j^{-1}r_k=r_k=s_j$.

(c) $s_k(1)\to(s_k(1)s_j^{-1})^{-1}=r_j=s_k$.

The transformation in (3) is effected by first activating y_{n+1} via the Type 5 transformation $(r_i)\to((r_i),y_{n+1})$. Then an automorphism $\lambda: Y^{n+1}\to Y^{n+1}$ is defined by $\lambda(y_i)=y_i(i\leq n)$ and $\lambda(y_{n+1})=y_{n+1}w$. This induces a Type 4 transformation from $((r_i),y_{n+1})$ to $((r_i),y_{n+1}w)$.

Clearly a necessary condition for Q^{**}-equivalence of p-tuples (r_i) and (s_i) of elements in Y^n is that the groups $Y^n/Cl(\{r_i\})$ and $Y^n/Cl(\{s_i\})$ be isomorphic, for Q^{**}-transformations are compositions of Tietze transformations. But even for p-tuples satisfying this condition, the situation is complex: Metzler, [18] gives some examples in this situation where the tuples are Q^{**}-equivalent but not Q^*-equivalent and other examples where the tuples are not even Q^{**}-equivalent. Other examples of this phenomenon are described in [14] and [8]. The case where $Y^n/Cl(\{r_i\})=1$ is untouched at present and is the subject of the Andrews-Curtis conjecture to be discussed presently.

An interesting simplification of Q^{**}-equivalence classes results from the addition of trivial relators to tuples. This is described by the following theorem that appears in [3, Le. 5]. In translating to the Q^{**}-language, the reader should bear in mind Lemma 1.1.

THEOREM 1.2. *Let (r_i) and (s_i) be p-tuples of elements in Y^n, and consider the $(2p+n)$-tuples $((r_i),1,\dots,1)$ and $((s_i),1,\dots,1)$.*

Then the two $(2p+n)$-tuples are Q^{**}-equivalent if and only if the groups $Y^n/Cl(\{r_i\})$ and $Y^n/Cl(\{s_i\})$ are isomorphic.

Below are listed four conjectures. When p is specialized to n in the first two, one obtains different forms of the Andrews-Curtis conjecture [1] and [2]. The last two conjectures are related to the Grusko-Neumann theorem.

$A_{p,n}$: Let (r_i) be any p-tuple of elements in Y^n such that $Cl(\{r_i\}) = Y^n$. Then (r_i) is Q-equivalent to $(y_1, \ldots, y_n, 1, \ldots, 1)$.

$B_{p,n}$: In the situation of $A_{p,n}$, the tuple (r_i) is Q^{**}-equivalent to $((y_i), 1, \ldots, 1)$.

$C_{m,n}$: Let $\psi = \psi_1 \times \psi_2 : G^m \to X^n \times Y^n$ be a surjective homomorphism. Then G^m decomposes as a free product $G_X * G_Y$ such that

$$\psi(G_X) = X^n \times 1 \quad \text{and} \quad \psi(G_Y) = 1 \times Y^n.$$

$D_{m,n}$: In the situation of $C_{m,n}$, there is an integer ℓ, and there is a surjective homomorphism $\phi = \phi_1 \times \phi_2 : G^{2\ell} \to X^\ell \times Y^\ell$ satisfying the conclusion of $C_{2\ell,\ell}$ so that the homomorphism $(\psi_1 * \phi_1) \times (\psi_2 * \phi_2) : G^m * G^{2\ell} \to (X^n * X^\ell) \times (Y^n * Y^\ell)$ satisfies the conclusion of $C_{m+2\ell, n+\ell}$ when the three free products are identified with $G^{m+2\ell}$, $X^{n+\ell}$, and $Y^{n+\ell}$.

We will show in Theorem 5.3 that for $m = p + n$, the two conjectures $B_{p,n}$ and $D_{m,n}$ are equivalent.

SECTION 2. FREE SPLITTING HOMOMORPHISMS AND INVARIANTS OF GEOMETRIC HEEGAARD SPLITTINGS

In this section we describe free splitting homomorphisms. To motivate their use, and to show that the objects we end up with provide topological invariants for 3-manifolds, we begin by reviewing the Stallings-Jaco formalism ([30] and [10]) and some facts about free presentations for surface groups and isomorphisms between surface groups.

By [30] and [10] it is known that Heegaard splittings can be studied formally via the group homomorphisms they give rise to. Thus, for our purposes here, we may take a *Heegaard splitting of genus n* for a 3-manifold M to be a homomorphism

$$\psi = \psi_1 \times \psi_2 : \pi_1(Q) \to X^n \times Y^n \tag{1}$$

where Q is an orientable surface of genus n, X^n and Y^n are the free groups mentioned in Section 1, and each of the factor homomorphisms ψ_1 and ψ_2 is surjective. Two splitting homomorphisms of genus n, ψ and ϕ, are defined to be *equivalent* if there are isomorphisms $\eta : \pi_1(Q) \to \pi_1(Q)$, $\eta_1 : X^n \to X^n$, and $\eta_2 : Y^n \to Y^n$ such that the diagram below is commutative:

$$\begin{array}{ccc} \phi : \pi_1(Q) & \longrightarrow & X^n \times Y^n \\ \downarrow \eta & & \downarrow (\eta_1, \eta_2) \\ \psi : \pi_1(Q) & \longrightarrow & X^n \times Y^n \end{array} \tag{2}$$

We are intentionally ignoring conditions on orientation because they disappear anyway in the passage to free splitting homomorphisms. Jaco allows an interchange of the factors X^n and Y^n in his definition of equivalence. The correspondence with geometric Heegaard splittings stands regardless of whether or not this interchange is allowed.

Consider the group G^{2n}. Let q be an element of G^{2n} satisfying the two conditions,

(i) for some free basis $\{e_i\}$ for G^{2n}, $q = \prod_{j=1}^{4n} e_{i_j}^{\varepsilon_j}$ $(\varepsilon_j = \pm 1)$ where each generator e_i appears twice, once as e_i^1 and once as e_i^{-1}, and

(ii) the length $4n$ of the spelling of q with respect to $\{e_i\}$ is minimal over all free bases for G^{2n}.

Then the fundamental group $\pi_1(Q)$ of an orientable surface Q of genus n has a presentation $\pi_1(Q) = \langle G^{2n} : q \rangle$. This follows from standard cut and paste arguments as in [17, Ch. 2, Secs. 5-7]. The presentation arises in the following way: Start with a graph that is the wedge of $2n$ circles identified with the generators e_i and then attach one boundary component of an annulus to the graph reading the cyclic word $q = \prod e_{i_j}^{\varepsilon_j}$ to get a punctured surface $E(q)$ whose boundary is a simple closed curve. Then attach a disk $D(q)$ to $E(q)$ along $\mathrm{Bd}E(q)$ to get a closed orientable surface of genus n. By the classification theorem for surfaces (see for example the reference [17] above), we may assume that $E(q)$ and $D(q)$ are contained in Q. Thus there is a natural homomorphism $\mu: G^{2n} \to \pi_1(Q)$ induced by the composition $G^{2n} \to \pi_1(E(q)) \to \pi_1(Q)$. By the Seifert-van Kampen theorem (see [17, Ch. 4]) ker μ is the normal closure of q.

Let $\langle G^{2n} : q \rangle$ and $\langle G^{2n} : q' \rangle$ be presentations for $\pi_1(Q)$ as in the preceding paragraph corresponding to partitions $(E(q), D(q))$ and $(E(q'), D(q'))$ and homomorphisms $\mu: G^{2n} \to \pi_1(Q)$ and $\nu: G^{2n} \to \pi_1(Q)$. Let $\eta: \pi_1(Q) \to \pi_1(Q)$ be an automorphism. By [23] and [16, Th. 2], η is induced by a homeomorphism h of Q, and by isotopically modifying h holding the basepoint fixed we may assume that $h(D(q')) = D(q)$. Thus $h(E(q')) = E(q)$ and so there is an automorphism $\eta_G: G^{2n} \to G^{2n}$ corresponding to the automorphism $(h \mid E(q'))_*$ on $\pi_1(E(q'))$ so that $\nu\eta_G = \mu\eta$. Thus there is associated with the equivalence (2) a commutative diagram,

(3)
$$\begin{tikzcd}
G^{2n} \arrow[dr, "\nu"] \arrow[drr, bend left, "\phi_G = \phi\nu"] \arrow[ddd, "\eta_G"'] & & \\
& \pi_1(Q) \arrow[r, "\phi"] \arrow[d, "\eta"] & X^n \times Y^n \arrow[d, "(\eta_1, \eta_2)"] \\
& \pi_1(Q) \arrow[r, "\psi"] & X^n \times Y^n \\
G^{2n} \arrow[ur, "\mu"] \arrow[urr, bend right, "\psi_G = \psi\mu"'] & &
\end{tikzcd}$$

We will refer to the homomorphisms ψ_G and ϕ_G as *free presentations* for ψ and ϕ, and we will model equivalences in the free splitting homomorphism theory on the outer block in (3). This will enable us to treat invariants of free presentations for Heegaard splitting homomorphisms as invariants of geometric Heegaard diagrams.

We define now a *free splitting homomorphism* to be a homomorphism of the form $\psi = \psi_1 \times \psi_2 : G^m \to X^n \times Y^n$ where each of the factor maps ψ_1 and ψ_2 is surjective. We do not require that m equal $2n$ or that m be even. This generality will enable us to set up criteria for Q^* and Q^{**}-equivalence of unbalanced group presentations. Let ψ and ϕ be two free splitting homomorphisms from G^m to $X^n \times Y^n$. We define ψ and ϕ to be *equivalent* if there are isomorphisms $\eta: G^m \to G^m$, $\eta_1: X^n \to X^n$, and $\eta_2: Y^n \to Y^n$ such that the diagram below is commutative:

$$\begin{array}{ccccc} \phi & : & G^m & \longrightarrow & X^n \times Y^n \\ & & \downarrow \eta & & \downarrow (\eta_1, \eta_2) \\ \psi & : & G^m & \longrightarrow & X^n \times Y^n \end{array} \tag{4}$$

We define a standard free splitting homomorphism $\chi_n : G^{2n} \to X^n \times Y^n$ by $\chi_n(g_i) = (x_i, y_i)$ $(i \le n)$ and $\chi_n(g_{i+n}) = (1, y_i)$. The particular description given for χ_n has been chosen to correspond to the normal forms that we will develop in the next section. Let $\psi : G^{m_1} \to X^{n_1} \times Y^{n_1}$ and $\phi : G^{m_2} \to X^{n_2} \times Y^{n_2}$ be two splitting homomorphisms. We define the *sum* $\psi \# \phi$ of ψ and ϕ to be a homomorphism from $G^{m_1+m_2}$ to $X^{n_1+n_2} \times Y^{n_1+n_2}$ defined as follows: First define

$$\lambda : X^{n_2} \times Y^{n_2} \to X^{n_1+n_2} \times Y^{n_1+n_2}$$

by $\lambda((x_i, 1)) = (x_{i+n_1}, 1)$ and $\lambda((1, y_i)) = (1, y_{i+n_1})$ Then set,

$$\psi \# \phi(g_i) = \psi(g_i) \quad (i \le n_1) \qquad \psi \# \phi(g_{i+n_1}) = \lambda\phi(g_i) \quad (i \le n_2)$$

$$\begin{aligned} \psi \# \phi(g_{i+n_1+n_2}) &= \psi(g_{i+n_1}) \quad (i \le m_1 - n_1) \\ \psi \# \phi(g_{i+m_1+n_2}) &= \lambda\phi(g_{i+n_2}) \quad (i \le m_2 - n_2). \end{aligned}$$

Note that the sum $\psi \# \phi$ is just a rearrangement of the homomorphism

$$(\psi_1 * \phi_1) \times (\psi_2 * \phi_2) : G^{m_1} * G^{m_2} \to (X^{n_1} * X^{n_2}) \times (Y^{n_1} * Y^{n_2}).$$

Finally, we define two splitting homomorphisms ψ and ϕ to be *stably equivalent* if there are integers p and q such that $\psi \# \chi_p$ and $\phi \# \chi_q$ are equivalent.

The following lemma shows that free splitting homomorphisms provide 3-manifold invariants (compare Statement (6) in [19]):

LEMMA 2.1. *Let ψ_G and ϕ_G be free presentations for Heegaard splitting homomorphisms ψ and ϕ (as in (1)) corresponding to 3-manifolds M and N.*

Then ψ and ϕ are equivalent or stably equivalent only if ψ_G and ϕ_G are equivalent or stably equivalent. In particular, any stable invariants of ψ_G and ϕ_G are topological invariants of the manifolds M and N.

Proof. From (3) equivalences of Heegaard splitting homomorphisms induce equivalences for free presentations for these homomorphisms. Similarly, from [30] and [10] sums and stabilizations of Heegaard splitting homomorphisms induce corresponding sums and stabilizations for the free presentations. Thus the first assertion in the conclusion of the lemma holds. The second assertion now follows from the Reidemeister-Singer theorem on stable equivalence of Heegaard splittings of a 3-manifold [27], [28] (see also [6]).

Examples. Below are described two free splitting homomorphisms from G^5 to $X^2 \times Y^2$. According to Metzler [18, p. 9] the two 3-tuples $(y_1^5, y_2^5, [y_1, y_2])$ and $(y_1^5, y_2^5, [y_1, y_2^2])$ are Q^{**}-equivalent but not Q^*-equivalent. By Theorem 4.1 the two free splitting homomorphisms are stably equivalent. It would be somewhat surprising if the two turned out to be equivalent.

$$\begin{array}{llll}
g_i \to (x_i, y_i) \quad (i \le 2) & & g_i \to (x_i, y_i) \quad (i \le 2) \\
g_3 \to (1, y_1^5) & & g_3 \to (1, y_1^5) \\
g_4 \to (1, y_2^5) & & g_4 \to (1, y_2^5) \\
g_5 \to (1, [y_1, y_2]) & & g_5 \to (1, [y_1, y_2^2]).
\end{array}$$

SECTION 3. NORMAL FORMS FOR FREE SPLITTING HOMOMORPHISMS

In this section we develop two normal forms for free splitting homomorphisms. As indicated in the introduction these will provide us with a means to relate the free splitting homomorphism theory with the theory of extended Nielsen transformations.

LEMMA 3.1. (Normal form). *Let ψ be a free splitting homomorphism from G^m to $X^n \times Y^n$.*

Then ψ is equivalent to a splitting homomorphism ϕ such that $\phi(g_i) = (x_i, v_i)$ $(i \le n)$ and $\phi(g_{i+n}) = (1, r_i)$. Here $\{v_i\}$ may or may not be a free basis for Y^n.

Proof. The factor homomorphism ψ_1 is surjective so $\psi_1(g_i)$ generates X^n. By the Nielsen reduction theorem described in Section 1, there is a sequence of length preserving and length reducing Nielsen transformations converting the m-tuple $(\psi_1(g_i))$ to $(x_1, \ldots, x_n, 1, \ldots, 1)$. Carry out the corresponding sequence of transformations on $(g_1, \ldots, g_m)$ to get a new m-tuple $(g_1', \ldots, g_m')$ such that $\{g_1', \ldots, g_m'\}$ is a free basis for G^m. We then have $\psi(g_i') = (x_i, v_i)$ $(i \le n)$ and $\psi(g_{i+n}') = (1, r_i)$. Define an automorphism $\eta: G^m \to G^m$ by $\eta(g_i) = g_i'$ $(i \le m)$. Then the promised ϕ is given by $\psi\eta$.

Remark. By a length argument we may suppose that the 1's in the m-tuple $(\psi_1(g_i))$ are at most permuted and are not otherwise involved in the sequence of Nielsen transformations. Thus if $\psi_1(g_i) = 1$, then $\psi_2(g_i)$ ends up being one of the elements r_j. We will need to use this observation in Section 5.

We will refer to a free splitting homomorphism with the form of ϕ in Lemma 3.1 as being in *normal form.* We denote the group presentation $\langle Y^n : (r_i)\rangle$ by $\mathscr{P}(\phi)$. Now every finite group presentation $\mathscr{P} = \langle Y^n : (r_i)\rangle$ is associated as $\mathscr{P}(\psi)$ for some free splitting homomorphism ψ in normal form. To see this, set $m = n + p$, and define $\psi : G^n \to X^n \times Y^n$ by $\psi(g_i) = (x_i, y_i)(i \leq n)$ and $\psi(g_{i+n}) = (1, r_i)$. This construction is due to Mihailova [20] (see also the books by Miller [21, pp. 35–42] and Lyndon and Schupp [13, pp. 193–195]). The interest of these authors' in this particular form was in showing that the generalized word problem and similar problems are recursively unsolvable in direct products of free groups. We will refer to any free splitting homomorphism with the form ψ defined above as a *Mihailova map.* The following lemma, together with Lemma 3.1, shows that any free splitting homomorphism is stably equivalent to a Mihailova map:

LEMMA 3.2 (Mihailova normal form). *Let $\psi: G^m \to X^n \times Y^n$ be a free splitting homomorphism in normal form and with associated group presentation $\mathscr{P}(\psi) = \langle Y^n : (r_i)\rangle$.*

Then $\psi \# \chi_n$ is equivalent to a Mihailova map $\phi: G^{m+2n} \to X^{2n} \times Y^{2n}$ whose associated group presentation $\mathscr{P}(\phi)$ is $\langle Y^{2n} : ((r_i), y_{n+1}, \ldots, y_{2n})\rangle$.

Proof. We will be changing ψ through several steps. To simplify notation we will regard ψ as being redefined at each step and so use the symbol ψ to denote the new homomorphism each time.

Step 1. Stabilize ψ to $\psi \# \chi_n$ to get a new ψ defined by,

$$\psi(g_i) = (x_i, v_i) \quad (i \leq n) \qquad \psi(g_{i+n}) = (x_{i+n}, y_{i+n}) \quad (i \leq n)$$

$$\psi(g_{i+2n}) = (1, r_i) \quad (i \leq m - n) \qquad (\psi(g_{i+m+n}) = (1, y_{i+n}).$$

From now on each step will modify ψ by redefining it as $(\eta_1^{-1}, \mathrm{id})\,\psi\eta$ where η, η_1, and $\eta_2 = \mathrm{id}$ are the isomorphisms in (4), the model for equivalence.

In any given step at most one of η and η_1 will be different from the identity. We will specify the one that is not the identity and not mention the other.

Step 2. Define $\eta: G^{m+2n} \to G^{m+2n}$ by $\eta(g_{i+n}) = g_{i+n}g_{i+m+n}^{-1}$ $(i \leq n)$ and $\eta(g_i) = g_i$ otherwise. Now redefine ψ as indicated at the end of Step 1 to give ψ the form,

$$\psi(g_i) = (x_i, v_i) \quad (i \leq n) \qquad \psi(g_{i+n}) = (x_{i+n}, 1) \quad (i \leq n)$$

$$\psi(g_{i+2n}) = (1, r_i) \quad (i \leq m - n) \qquad \psi(g_{i+m+n}) = (1, y_{i+n}).$$

Step 3. Let G_A, G_B, G_C, and G_D denote the respective subgroups of G^{m+2n}, $Gp(\{g_i : i \leq n\})$, $Gp(\{g_i : n + 1 \leq i \leq 2n\})$, $Gp(\{g_i : 2n + 1 \leq i \leq m + n)$, and $Gp(\{g_i : m + n + 1 \leq i\})$. Note that G^{m+2n} decomposes as $G_A * G_B * G_C * G_D$.

Now ψ_2 is surjective so for each $y_i (i \leq n)$ there is an element $d_i \in G_A * G_C$ such that $\psi(d_i) = (w_i, v_i^{-1}y_i)$. Define another automorphism η of G^{m+2n} by $\eta(g_{i+n}) = g_{i+n}d_i$ $(i \leq n)$ and $\eta(g_i) = g_i$ otherwise. Redefine ψ as before to give ψ the new form,

$$\psi(g_i) = (x_i, v_i) \qquad \psi(g_{i+n}) = (x_{i+n} w_i, v_i^{-1} y_i)$$

$$\psi(g_{i+2n}) = (1, r_i) \qquad \psi(g_{i+m+n}) = (1, y_{i+n}).$$

Step 4. Note that $\{x_i, x_{i+n} w_i\}$ is a free basis for X^{2n}. Thus there is an automorphism η_1^{-1}: $X^{2n} \to X^{2n}$ given by

$$\eta_1^{-1}(x_i) = x_i \ (i \leq n) \quad \text{and} \quad \eta_1^{-1}(x_{i+n} w_i) = x_{i+n}.$$

Redefine ψ as indicated before to give ψ the new form,

$$\psi(g_i) = (x_i, v_i) \qquad \psi(g_{i+n}) = (x_{i+n}, v_i^{-1} y_i)$$

$$\psi(g_{i+2n}) = (1, r_i) \qquad \psi(g_{i+m+n}) = (1, y_{i+n}).$$

Step 5. Define another automorphism η of G^{m+2n} by $\eta(g_i) = g_i g_{i+n}$ $(i \leq n)$ and $\eta(g_i) = g_i$ otherwise. Redefine ψ as before to get the new form,

$$\psi(g_i) = (x_i x_{i+n}, y_i) \qquad \psi(g_{i+n}) = (x_{i+n}, v_i^{-1} y_i)$$

$$\psi(g_{i+2n}) = (1, r_i) \qquad \psi(g_{i+m+n}) = (1, y_{i+n}).$$

Step 6. Define an automorphism $\eta_1^{-1}: X^{2n} \to X^{2n}$ by $\eta_1^{-1}(x_i x_{i+n}) = x_i$ $(i \leq n)$ and $\eta_1^{-1}(x_{i+n}) = x_{i+n}$. Redefine ψ as before to get the new form,

$$\psi(g_i) = (x_i, y_i) \qquad \psi(g_{i+n}) = (x_{i+n}, v_i^{-1} y_i)$$

$$\psi(g_{i+2n}) = (1, r_i) \qquad \psi(g_{i+m+n}) = (1, y_{i+n}).$$

Step 7. For each $i \leq n$, let e_i be an element of G_A such that $\psi_2(e_i) = v_i^{-1} y_i$. Define an automorphism η of G^{m+2n} by $\eta(g_{i+n}) = g_{i+n} e_i^{-1} g_{i+m+n}$ $(i \leq n)$ and $\eta(g_i) = g_i$ otherwise. Set $f_i = \psi_1(e_i^{-1})$ $(i \leq n)$. Redefine ψ as before to get the new form,

$$\psi(g_i) = (x_i, y_i) \qquad \psi(g_{i+n}) = (x_{i+n} f_i, y_{i+n})$$

$$\psi(g_{i+2n}) = (1, r_i) \qquad \psi(g_{i+m+n}) = (1, y_{i+n}).$$

Step 8. The set $\{x_i, x_{i+n} f_i\}$ is a free basis for X^{2n}. Define an automorphism η_1^{-1} of X^{2n} by $\eta_1^{-1}(x_i) = x_i$ $(i \leq n)$ and $\eta_1^{-1}(x_{i+n} f_i) = x_{i+n}$. Then redefine ψ as before to get, finally, the desired form,

$$\psi(g_i) = (x_i, y_i) \qquad \psi(g_{i+n}) = (x_{i+n}, y_{i+n})$$

$$\psi(g_{i+2n}) = (1, r_i) \qquad \psi(g_{i+m+n}) = (1, y_{i+n}).$$

The following lemma, due essentially to Mihailova [20], reveals the connection between surjectivity of splitting homomorphisms and presentations for the trivial group. The reader should compare [30], Th. 1].

LEMMA 3.3. *Let $\psi: G^m \to X^n \times Y^n$ be a free splitting homomorphism in normal form with associated group presentation $\mathscr{P}(\psi)$.*

Then ψ is surjective if and only if $\mathscr{P}(\psi)$ presents the trivial group.

Proof. Stabilization does not change the surjectivity or lack of it; so by Lemma 3.2 we may assume that ψ is a Mihailova map. But now the lemma follows from [20], [21, Ch. 3, Le. 18], or [13, Ch. 4, Le. 4.2].

SECTION 4. THE MAIN CLASSIFICATION THEOREM

We come now to the main classification theorem, Theorem 4.1, relating the two theories. This theorem is stated below but its proof is delayed so that we may introduce some geometric machinery to prove it. The geometric approach here uses mapping cylinders for geometric maps corresponding to free splitting homomorphisms and is essentially the free counterpart of the mapping cylinder construction used by Jaco [10] to analyze constrained Heegaard splitting homomorphisms. Formal deformations (to be discussed presently) will be used to analyze these mapping cylinders. In the sequel to this paper we will give a purely algebraic proof of Theorem 4.1 based on Rapaport's techniques [26]. The geometric approach used here seems to be of some interest in its own right, for as we will point out in Theorem 4.6, this approach enables us to characterize equivalence of 2-dimensional polyhedra under formal 3-deformations in terms of relative homotopy equivalence of other 2-dimensional polyhedra.

THEOREM 4.1. *Let ψ and ϕ be free splitting homomorphisms in normal form and with associated group presentations $\mathscr{P}(\psi)$ and $\mathscr{P}(\phi)$.*

Then ψ and ϕ are stably equivalent if and only if $\mathscr{P}(\psi)$ and $\mathscr{P}(\phi)$ are Q^{**}-equivalent.

We will be dealing with polyhedra in this section. We will assume that all maps mentioned are piecewise linear (pwl). We begin our preparations for the proof of Theorem 4.1 by recalling that there is a well known method for associating with any compact, connected, 2-dimensional polyhedron A, a finite presentation $\mathscr{P}(A) = \langle Y^n : (r_i) \rangle$ for the fundamental group $\pi_1(A)$. This is based on a decomposition of A as a cell complex. See [34] for example. Next, let A and B be polyhedra. We say that A *formally j-deforms to B*, the deformation denoted by $A \underset{j}{\nearrow\!\searrow} B$, provided that A can be transformed to B by a sequence of abstract elementary polyhedral expansions ($\overset{e}{\nearrow}$) and elementary polyhedral collapses ($\overset{e}{\searrow}$) such that the dimension of any intervening polyhedron in the process does not exceed j. See [31], [32], [33], and [34] for a fuller discussion. There is an obvious extension to tuples of polyhedra and subpolyhedra defined by the requirement that there be consistent sequences of deformations on the items in the tuple. We emphasize that the collapses and expansions here are polyhedral. The reason for this restriction is so that we may have available the following folk theorem due to Wright [34, Cor. 3.1] (see also [19, Sec. 1]):

THEOREM 4.2 (Wright). *Let A and B be compact, connected 2-dimensional polyhedra with associated group presentations $\langle Y^{n_1}: (r_i)\rangle$ and $\langle Y^{n_2}: (s_i)\rangle$.*

Then $A \underset{3}{\nearrow\!\!\searrow} B$ if and only if (r_i) and (s_i) are Q^{**}-equivalent.

Remark. It may not be obvious to a person reading [34] that the results there imply Theorem 4.2. In particular, it is not apparent what corresponds in [34] to transformations of Type 4. For this reason we are including an appendix which shows how to make the transition from [34, Cor. 3.1] to Theorem 4.2.

Any formal deformation $A \underset{j}{\nearrow\!\!\searrow} B$ induces a homotopy equivalence between A and B. It is enough to note how this works for a single elementary expansion $A \overset{e}{\nearrow} B = A \cup E$ where E is a ball with $E \cap A$ a ball in BdE of dimension one less. One has an inclusion map $A \to B$ and a retraction $r: B \to A$ defined so that $r(E) = E \cap A$. These two maps define the homotopy equivalence.

Next we provide a geometric interpretation for free splitting homomorphisms. Let $\psi = \psi_1 \times \psi_2$ be a free splitting homomorphism from G^m to $X^n \times Y^n$. Let (A, a), (B, b), and (C, c) be finite, connected, pointed polyhedral graphs with fundamental groups isomorphic to G^m, X^n, and Y^n respectively via isomorphisms

$$\lambda: G^m \to \pi_1(A, a), \quad \rho: X^n \to \pi_1(B, b), \quad \text{and} \quad \omega: Y^n \to \pi_1(C, c).$$

Let $f: (A, a) \to (B, b)$ and $g: (A, a) \to (C, c)$ be maps such that $\psi_1 = \rho^{-1} f_* \lambda$ and $\psi_2 = \omega^{-1} g_* \lambda$ where $*$ denotes the induced homomorphism on the fundamental group. Consider the mapping cylinders,

$$_fM = A \times [-1, 0] \underset{(x,-1)=f(x)}{+} B$$

$$M_g = A \times [0, 1] \underset{(x,1)=g(x)}{+} C.$$

Here $\underset{x=y}{+}$ denotes disjoint union with identification. We have a double mapping cylinder defined by

$$_fMM_g = {_fM} \underset{(x,0)=(x,0)}{+} M_g.$$

We say that the triple

$$T = ((_fMM_g, a \times [-1, 1]), (_fM, a \times [-1, 0]), (M_g, a \times [0, 1]))$$

is a *geometric realization* for the free splitting homomorphism ψ. The realization T is given a polyhedral structure as follows: Take triangulations $(J, J(a))$, $(K, K(b))$, and $(L, L(c))$ of (A, a), (B, b), and (C, c) so that f and g are simplicial with respect to these triangulations. Then regard the mapping cylinders as the simplicial mapping cylinders as defined by Whitehead [33] or Cohen [5, Sec. 4]. By the Hauptvermutung for 2-complexes (see [25] or [4]), the polyhedral structures on T defined by different triangulations are equivalent so it does not matter really which triangulations we use.

The following lemma shows that geometric realizations for free splitting homomorphisms are unique up to formal deformation. The use of Theorem 7.1_A of [5] in the proof here was suggested by Marshall Cohen.

LEMMA 4.3. *Let T and T' be geometric realizations for equivalent free splitting homomorphisms ψ and ϕ.*

Then $T \underset{3}{\nearrow\!\!\searrow} T'$.

Proof. Let T and T' be built from graphs $(A,a),\ldots,(C',c')$, maps f, g, f', g', and isomorphisms $\lambda,\ldots,\omega'$. The proof of the lemma is based on a sequence of simplifications on the assumptions about the items above. Each step in the proof proceeds on the basis of the preceding simplifications.

Step 1. *We may assume that* $\psi = \phi$. Given an equivalence between ψ and ϕ, construct a geometric realization T'' for ϕ using the same graphs $(A,a),\ldots,(C,c)$ and the same maps f, g as for T but using different isomorphisms λ'', ρ'', ω'' that reflect the equivalence between ψ and ϕ. But then $T = T''$ so we may now regard T as a geometric realization for ϕ.

Step 2. *We may assume that* $(B,b) = (B',b')$, $(C,c) = (C',c')$, $\rho = \rho'$, *and* $\omega = \omega'$. Consider, for the moment, just the $[-1,0]$ side in the triple T. The map $\rho'\rho^{-1}$ induces a homotopy equivalence from B to B'. Identify b with b' by taking a different copy of B if necessary. By taking formal 2-deformations that reflect Nielsen transformations, it is possible to construct a formal 2-deformation $B \underset{2}{\nearrow\!\!\searrow} B'$ that holds b fixed and effects the homomorphism $\rho'\rho^{-1}$ via a sequence of inclusions and retractions induced by the elementary steps in the deformation. This deformation may be assumed to be divided into pairs of steps: elementary 2-expansion of a 1-polyhedron $\left(\overset{e}{\underset{2}{\nearrow}}\right)$ followed by an elementary 2-collapse $\left(\overset{e}{\underset{2}{\searrow}}\right)$ of the resulting 2-polyhedron to a new 1-polyhedron. Thus it is enough to consider the case $B \overset{e}{\underset{2}{\nearrow}} B(1) \overset{e}{\underset{2}{\searrow}} B'$.

Define a new triple $T(1)$ by using the old maps but replacing B by $B(1)$. Clearly $T \underset{2}{\nearrow} T(1)$. Let $r\colon B(1) \to B'$ be a pwl retraction induced by the collapse $B(1) \searrow B'$. Note that $\rho' = r_*\rho$. Define a new triple $T(2)$ from $T(1)$ by replacing $B(1)$ by B', f by rf, and ρ by ρ'. We will show that $T(1) \underset{3}{\nearrow\!\!\searrow} T(2)$.

Observe first that f and rf are homotopic maps from (A,a) to $(B(1),b)$. To avoid confusion with the mapping cylinder notation we define a polyhedron A^* equivalent to the product $A \times [0,1]$ with subpolyhedra $A^*(0)$, $A^*(1)$, and

$$A^*(a \times [0,1])$$

corresponding to $A \times 0$, $A \times 1$, and $a \times [0,1]$. By taking a pwl approximation to the original homotopy between f and rf, we obtain a pwl map $h: A^* \to B(1)$ such that $h \mid A^*(0)$ represents f (via $A = A^*(0)$) and $h \mid A^*(1)$ represents rf (via $A = A^*(1)$). We assume that h is chosen so that $h(A^*(a \times [0,1])) = b$. Define a product map k from A^* to C so that k corresponds to the projection of $A \times [0,1]$ to A followed by g. The map k is clearly pwl. Let J, K, and L be complexes triangulating A^*, $B(1)$, and C so that h and k are simplicial maps with respect to these triangulations and so that $A^*(a \times [0,1])$ is triangulated as a subcomplex $J(a \times [0,1])$ of J. Let $J(0)$ and $J(1)$ denote the subcomplexes of J triangulating $A^*(0)$ and $A^*(1)$.

Define a new triple $T(3)$ from $T(1)$ by using A^* in place of A, $A^*(a \times [0,1])$ in place of a, and h and k in place of f and g. Regard $T(1)$ and $T(2)$ as being defined by the restrictions of A^* to $A^*(0)$ and $A^*(1)$, and give all three triples $T(1)$, $T(2)$, and $T(3)$ polyhedral structures by regarding the mapping cylinders ${}_hM$ and M_k as being the simplicial mapping cylinders defined by Cohen [5, Sec. 4] for the complexes J, K, and L.

It is well known that for complexes of dimension less than or equal to 2, polyhedral collapses of complexes to subcomplexes (in which each elementary collapse involves a ball that is a subcomplex) can be effected simplicially. Thus we have a simplicial formal deformation:

$$J(0) \underset{1}{\overset{s}{\nearrow}} J(0) \cup J(a \times [0,1]) \underset{2}{\overset{s}{\nearrow}} J \underset{2}{\overset{s}{\searrow}} J(1) \cup J(a \times [0,1]) \underset{1}{\overset{s}{\searrow}} J(1). \tag{5}$$

Consider a simplex σ of J. By [5, Th. 7.1_A] the polyhedra $(\sigma \times [-1,0])/h$ and $(\sigma \times [0,1])/k$ are balls where $/h$ and $/k$ indicate identification of points in $\sigma \times -1$ and $\sigma \times 1$ according to h and k. By mimicking the expansions and collapses in (5) by expansions and collapses of the balls $(\sigma \times [-1,0])/h$ and $(\sigma \times [0,1])/k$ along the walls of the mapping cylinders ${}_hM$ and M_k and then collapsing the base $B(1) \underset{2}{\searrow} B'$, we obtain a formal deformation (recall that $T(1)$ and $T(2)$ are identified with restrictions of $T(3)$):

$$ {}_fM \underset{2}{\nearrow} {}_fM \cup A^*(a \times [0,1]) \underset{2}{\nearrow} {}_fM \cup (A^*(a \times [0,1]) \times [-1,0])/h$$

$$\underset{3}{\nearrow} {}_hM \underset{3}{\searrow} {}_{rf}M \cup (A^*(a \times [0,1]) \times [-1,0])/h \underset{2}{\searrow} {}_{rf}M \cup A^*(a \times [0,1]) \underset{2}{\searrow} {}_{rf}M.$$

There is a similar deformation $M_g \underset{3}{\nearrow\!\searrow} M_g$ where the two copies of M_g are identified with restrictions of M_k associated with $A^*(0)$ and $A^*(1)$. Combining these two deformations with the deformation $T \underset{2}{\nearrow} T(1)$ obtained previously, we obtain the formal 3-deformation

$$T \underset{2}{\nearrow} T(1) \underset{3}{\nearrow\!\searrow} T(3) \underset{3}{\nearrow\!\searrow} T(2).$$

The change $C \to C'$, $\omega \to \omega'$ is effected in the identical manner. Finally we end up with a triple T'' associated with the graphs (A,a), (B',b'), (C',c'), maps f'' and g'', and isomorphisms λ'', ρ', and ω'. It is easy to check that the maps f'' and g'' still represent ψ_1 and ψ_2 in the revised model. Now use T'' *for* T.

Step 3. *We may assume that* $A = A'$ *and* $\lambda = \lambda'$. Reversing the roles of A and A' we construct, as in Step 2, a formal deformation $A' \underset{2}{\nearrow\!\searrow} A$ fixing a and effecting the map $\lambda(\lambda')^{-1}$. We will construct a new triple T'' using A' and λ' in place of A and λ, and we will construct a formal deformation $T \underset{3}{\nearrow\!\searrow} T''$. As in Step 2 we may assume that the deformation $A' \underset{2}{\nearrow\!\searrow} A$ is made up of the two steps $A' \overset{e}{\underset{2}{\nearrow}} A(1) \overset{e}{\underset{2}{\searrow}} A$. Let r: $A(1) \to A$ be a retraction associated with the collapse $A(1) \searrow A$ so that $\lambda(\lambda')^{-1} = r|A'$. We will use the retraction r to define new mapping cylinders and new triples $T(1)$ and T''.

Let h and k be maps from $A(1)$ to B and C defined by $h = fr$ and $k = gr$. Because f, g, and r are pwl, h and k are also pwl. Let f'' and g'' denote the restrictions $h|A'$ and $k|A'$. Define a new triple $T(1)$ from T by using $A(1)$ in place of A and h and k in place of f and g. Then define a triple T'' from $T(1)$ by taking the restriction associated with A', f'', and g''. It is easy to check that f'' and g'' represent ψ_1 and ψ_2 in the new model. Choose triangulations J, K, and L of $A(1)$, B, and C so that with respect to these triangulations the maps f, g, h, k, f'', and g'' are simplicial. We suppose also that $a = a'$ is a vertex of J. Let $J(A)$ and $J(A')$ denote the subcomplexes of J triangulating A and A'. Let the polyhedral structures on T, $T(1)$, and T'' be defined as in Step 2.

As in Step 2 we have a simplicial deformation $J(A) \overset{s}{\underset{2}{\nearrow}} J \overset{s}{\underset{2}{\searrow}} J(A')$, and as in Step 2 this leads to a deformation $T \underset{3}{\nearrow} T(1) \underset{3}{\searrow} T''$. Now use T'' in place of T.

Step 4. *We may assume that* $f = f'$ *and* $g = g'$. There are homotopies from f to f' and from g to g' both fixing the image of a. The analysis of this step proceeds now exactly as in Step 2.

We have, at this point, reduced the problem to the trivial case $T = T'$, and the proof of the lemma is complete.

The following theorem, together with Theorem 4.1, will show that Q^{**}-equivalence of p-tuples can be posed as a problem in relative homotopy equivalence of geometric realizations for splitting homomorphisms (see Theorem 4.6). This result appears to have some similarity with the results in Section 2 of [19].

THEOREM 4.4. *Let ψ and ϕ be free splitting homomorphisms with geometric realizations $T(\psi)$ and $T(\phi)$.*

Then ψ and ϕ are equivalent if and only if $T(\psi)$ and $T(\phi)$ are homotopy equivalent triples.

Proof. Formal deformation is a strong form of homotopy equivalence; so if ψ and ϕ are equivalent, then $T(\psi)$ and $T(\phi)$ are homotopy equivalent by Lemma 4.3. On the other hand, suppose that $T(\psi)$ and $T(\phi)$ are homotopy equivalent. Let $A(\psi), \ldots, C(\phi)$ be the associated graphs. Then, because the homotopy equivalence is a relative one, $A(\psi) \times 0$ must be taken to $A(\phi) \times 0$ and vice versa, and the homotopies describing the equivalence must restrict to homotopies of the 0-sections $A(\psi) \times 0$ and $A(\phi) \times 0$. By adjustments along the walls of the mapping cylinders, we may suppose that the equivalence exchanges $B(\psi)$ with $B(\phi)$ and similarly exchanges $C(\psi)$ with $C(\phi)$. But then the equivalence between ψ and ϕ can be read off by substituting in (4) the fundamental group maps induced by restrictions of the homotopy equivalence $T(\phi) \to T(\psi)$ to $A(\phi)$, $B(\phi)$, and $C(\phi)$.

LEMMA 4.5. *Let $\psi: G^m \to X^n \times Y^n$ be a free splitting homomorphism in normal form, and let $\mathscr{P}(\psi) = \langle Y^n: (r_i)\rangle$ be its associated group presentation.*

Let $T = (({}_fMM_g, a \times [-1,1]), \ldots)$ be any geometric realization of ψ. Finally let D be any compact, connected 2-dimensional polyhedron whose fundamental group is read as $\mathscr{P}(D) = \mathscr{P}(\psi)$ in the manner previously described.

Then ${}_fMM_g \underset{3}{\nearrow\!\searrow} D$.

Proof. By Lemmas 4.2 and 4.3, it is sufficient to find one T realizing ψ and one D realizing $\mathscr{P}(\psi)$ for which the conclusion of the lemma holds. We construct T according to the recipe for geometric realizations but subject to the refinements described below:

First let A be a wedge of m circles with wedge point a, and think of A as a wedge $A_L \vee A_H$ corresponding to the free product decomposition $G^m = G_L * G_H$ where $G_L = Gp(\{g_i: i \leq n\})$ and $G_H = Gp(\{g_{i+n}\})$. Choose (B,b), (C,c), f, and g so that:

(i) f maps A_H to the base point $b \in B$,

(ii) f maps A_L homeomorphically onto B,

(iii) g attaches the circles in A_H to C according to the words r_i in the group presentation $\mathscr{P}(\psi) = \langle Y^n: (r_i)\rangle$,

(iv) g attaches the circles in A_L to C according to the words $v_i = \psi_2(g_i)(i \leq n)$.

Consider the subpolyhedra E, F, and E_0 of ${}_fMM_g$ defined by,

$$E = (A_L \times [-1,1])/f,g \cup B \cup C$$
$$F = (A_H \times [-1,1])/f,g \cup C$$
$$E_0 = a \times [-1,1] \cup C.$$

As in the proof of Lemma 4.3, $/f,g$ indicates that identifications are made at the -1 and 1 levels.

Now ${}_fMM_g = E \cup F$, and $E \cap F = E_0$. But, by the definition of f, E is a product up to $(A_L \times 1)/g$; so E collapses to E_0 and thus ${}_fMM_g$ collapses to F. But F is made up of 2-cells attached to C according to the recipe $\mathscr{P}(\psi)$ where the 2-cells are then bound together along the arc $a \times [-1, 1]$. Thus the group reading $\mathscr{P}(F)$ may be taken to be $\mathscr{P}(\psi)$, and we may take D to be F to finish the proof of the lemma.

Proof of Theorem 4.1. First suppose that the presentations $\mathscr{P}(\psi)$ and $\mathscr{P}(\phi)$ are Q^{**}-equivalent. By stabilizing ψ and ϕ we may as well assume that $\mathscr{P}(\psi)$ and $\mathscr{P}(\phi)$ are Q^*-equivalent. By Lemma 3.3 we may assume that ψ and ϕ are Mihailova maps from G^m to $X^n \times Y^n$ and that

$$\mathscr{P}(\psi) = \langle Y^n : (r_i) \rangle \quad \text{and} \quad \mathscr{P}(\phi) = \langle Y^n : (s_i) \rangle$$

are still Q^*-equivalent. Now, without loss in generality, we may suppose that (s_i) results from (r_i) by a single transformation of Type 1–4. We consider the possible types separately and we leave consideration of Type 1 to the reader:

Type 2. Let $s_k = r_k r_j$ and $s_i = r_i$ otherwise. Define an isomorphism $\eta: G^m \to G^m$ by $\eta(g_{k+n}) = g_{k+n} g_{j+n}$ and $\eta(g_i) = g_i$ otherwise. Set $\eta_1 = \mathrm{id}$ and $\eta_2 = \mathrm{id}$ in (4). Then (4) defines an equivalence between ψ and ϕ.

Type 3. Let $s_k = y_j^{-\varepsilon} r_k y_j^{\varepsilon}$ and $s_i = r_i$ otherwise. Define $\eta: G^m \to G^m$ by $\eta(g_{k+n}) = g_j^{-\varepsilon} g_{k+n} g_j^{\varepsilon}$ and $\eta(g_i) = g_i$ otherwise. Set $\eta_1 = \mathrm{id}$ and $\eta_2 = \mathrm{id}$. Then we have an equivalence defined by (4) as in the Type 2 case.

Type 4. Let $s_i = \lambda(r_i)$ where $\lambda: Y^n \to Y^n$ is an automorphism. First let $\nu: X^n \to Y^n$ be the isomorphism defined by $\nu(x_i) = y_i$ $(i \le n)$. Define isomorphisms $\eta_1: X^n \to X^n$ and $\eta_2: Y^n \to Y^n$ by $\eta_2 = \lambda^{-1}$ and $\eta_1 = \nu^{-1} \lambda^{-1} \nu$. Now observe that ψ_1 and ϕ_1 both take G_L isomorphically onto X^n where $G_L * G_H$ is the free product decomposition of G^m described in the proof of Lemma 4.5. Thus for each x_i $(i \le n)$ there is an element $e_i \in G_L$ such that $\psi_1(e_i) = \eta_1(x_i)$. Clearly $\{e_i\}$ is a free basis for G_L. Define an isomorphism $\eta: G^m \to G^m$ by $\eta \mid G_H = \mathrm{id}$ and $\eta(g_i) = e_i$ $(i \le n)$. Then η, η_1, and η_2 define an equivalence in (4) between ψ and ϕ.

Suppose now that ψ and ϕ are stably equivalent. Then as before we can replace ψ and ϕ by stable versions of themselves so that ψ and ϕ are equivalent and so that the Q^{**}-equivalence classes of the new presentations $\mathscr{P}(\psi)$ and $\mathscr{P}(\phi)$ are the same as the old classes. Let $T = (({}_fMM_g, \ldots)$ and $T' = (({}_{f'}MM_{g'}, \ldots)$ be geometric realizations for ψ and ϕ, and let D and D' be compact, connected 2-dimensional polyhedra with associated group readings $\mathscr{P}(\psi)$ and $\mathscr{P}(\phi)$. By Lemmas 4.3 and 4.5 we have a sequence of formal deformations,

$$D \underset{3}{\curvearrowright} {}_fMM_g \underset{3}{\curvearrowright} {}_{f'}MM_{g'} \underset{3}{\curvearrowright} D'. \tag{6}$$

Thus by Wright's theorem, Theorem 4.2, $\mathscr{P}(\psi)$ and $\mathscr{P}(\phi)$ are Q^{**}-equivalent.

The theorem below shows how to re-express equivalence of 2-dimensional polyhedra under formal 3-deformations in terms of relative homotopy equivalences.

THEOREM 4.6. *Let D and E be compact, connected 2-dimensional polyhedra with associated group presentations*

$$\mathscr{P}(D) = \langle Y^{n_1}\colon (r_1, \ldots, r_p)\rangle \quad \textit{and} \quad \mathscr{P}(E) = \langle Y^{n_2}\colon (s_1, \ldots, s_q)\rangle$$

for $\pi_1(D)$ *and* $\pi_1(E)$. *Set* $m_1 = n_1 + p$ *and* $m_2 = n_2 + q$, *and let* ψ *and* ϕ *denote the Mihailova maps* $\psi\colon G^{m_1} \to X^{n_1} \times Y^{n_1}$ *and* $\phi\colon G^{m_2} \to X^{n_2} \times Y^{n_2}$ *defined by*

$$\psi(g_i) = (x_i, y_i) \quad (i \leq n_1), \qquad \psi(g_{i+n_1}) = (1, r_i),$$
$$\phi(g_i) = (x_i, y_i) \quad (i \leq n_2), \quad \textit{and} \quad \phi(g_{i+n_2}) = (1, s_i).$$

Then $D \underset{3}{\nearrow\!\!\searrow} E$ *if and only if for some integers* k *and* ℓ, *the geometric realizations* $T(\psi \,\#\, \chi_k)$ *and* $T(\phi \,\#\, \chi_\ell)$ *for* $\psi \,\#\, \chi_k$ *and* $\phi \,\#\, \chi_\ell$ *are homotopy equivalent.*

Proof. By Lemma 4.2, $D \underset{3}{\nearrow\!\!\searrow} E$ if and only if (r_i) and (s_i) are Q^{**}-equivalent. By Theorem 4.1, (r_i) and (s_i) are Q^{**}-equivalent if and only if $\psi \,\#\, \chi_k$ and $\phi \,\#\, \chi_\ell$ are equivalent for some k and ℓ. By Theorem 4.4, $\psi \,\#\, \chi_k$ and $\phi \,\#\, \chi_\ell$ are equivalent if and only if their geometric realizations, $T(\psi \,\#\, \chi_k)$ and $T(\phi \,\#\, \chi_\ell)$, are homotopy equivalent.

SECTION 5. APPLICATIONS

In this section we give two applications of Theorem 4.1. The first of these gives a sufficient condition for simplifying a group presentation by Q^{**}-transformations. The second relates the Andrews-Curtis conjecture (see Section 1) to the Grusko-Neumann theorem.

LEMMA 5.1. *Consider the homomorphism* $\nu\colon X^n \to Y^n$ *given by* $\nu(x_i) = y_i$ $(i \leq n)$. *Let* ψ *and* ϕ *be two free splitting homomorphisms from* G^m *to* $X^n \times Y^n$ *defined by* $\psi(g_i) = (x_i, v_i)$ $(i \leq n)$, $\psi(g_{i+n}) = (1, r_i)$, $\phi(g_i) = (u_i, y_i)$ *and*

$$\phi(g_{i+n}) = (s_i, 1)$$

where $v_i = \nu(u_i)$ *and* $r_i = \nu(s_i)$.

Then ψ *and* ϕ *are stably equivalent. Moreover, if* ψ *is a Mihailova map then* ψ *and* ϕ *are equivalent.*

Proof. Suppose first that ψ is not a Mihailova map. By the proof of Lemma 3.2, applied to ψ and applied to ϕ with the roles of X^n and Y^n reversed, we can stabilize both ψ and ϕ and then renormalize them both so that ψ is a Mihailova map and so that the hypotheses of the lemma still hold. Thus it is sufficient to prove the lemma for the case where ψ is a Mihailova map.

The homomorphism ϕ_1 takes G_L isomorphically onto X^n where $G^m = G_L * G_H$ is the free product decomposition of G^m described in the proof of Lemma 4.5. For each $s_i \in X^n$ there is an element $e_i \in G_L$ such that $\phi_1(e_i) = s_i^{-1}$. Define an automorphism $\eta\colon G^m \to G^m$ by $\eta(g_i) = g_i$ $(i \leq n)$ and $\eta(g_{i+n}) = (e_i g_{i+n})^{-1}$. Set $\eta_1 = \mathrm{id}$ and $\eta_2 = \mathrm{id}$ in (4). Then η, η_1, η_2 define an equivalence in (4) between ψ and ϕ.

THEOREM 5.2. *Let (r_i) be a p-tuple of elements in Y^n. Suppose that for some $q < n$, there exist elements $w_1, \ldots, w_q$ in Y^n so that $\{w_i\} \cup \{r_i\}$ generates Y^n.*

Then (r_i) is Q^{**}-equivalent to a $(p - (n - q))$-tuple of elements $(t_1, \ldots, t_{p-(n-q)})$ in Y^q.

Proof. Set $m = n + p$ and define a homomorphism $\psi: G^m \to X^n \times Y^n$ by $\psi(g_i) = (x_i, w_i)$ $(i \le q)$, $\psi(g_i) = (x_i, 1)$ $(q < i \le n)$, and $\psi(g_{i+n}) = (1, r_i)$. By the hypothesis of the theorem, ψ is a bonafide free splitting homomorphism. Notice that ψ is in normal form.

Now carry out the normalization construction in the proof of Lemma 3.1 with the roles of X^n and Y^n reversed. Let ϕ denote the resulting homomorphism where $\phi(g_i) = (u_i, y_i)$ $(i \le n)$ and $\phi(g_{i+n}) = (s_i, 1)$. By the remark following Lemma 3.1 we may assume that the images $(x_i, 1)$ $(i > q)$ are left unchanged in the process. This fact has two consequences: First, by rearranging the terms s_i, we may suppose that the last $(n - q)$ terms in $(s_1, \ldots, s_p)$ are $x_{q+1}, \ldots, x_n$. Second, since the Nielsen transformations in the construction do not involve the 1's, it follows that the first $p - (n - q)$ terms in (s_i) do not involve $x_{q+1}, \ldots, x_n$; that is, these terms are elements of X^q.

By Lemma 5.1, ϕ is stably equivalent to the free splitting homorphism $\phi': G^m \to X^n \times Y^n$ defined by $\phi'_1 = \nu^{-1}\phi_2$ and $\phi'_2 = \nu\phi_1$. Set $t_i = \nu(s_i)$ $(i \le p)$. Then since ψ is equivalent to ψ, ϕ' is, by transitivity, also equivalent to ψ. Thus by Theorem 4.1, (r_i) and $(t_1, \ldots, t_p)$ are Q^{**}-equivalent. But, by the preceding paragraph, $(t_1, \ldots, t_p)$ reduces to the $(p - (n - q))$ tuple $(t_1, \ldots, t_{p-(n-q)})$ of elements in Y^q by $(n - q)$ successive transformations of Type 6. Thus (r_i) and $(t_1, \ldots, t_{p-(n-q)})$ are Q^{**}-equivalent as desired.

The Grusko-Neumann theorem [9], [22], [11], [29], [12], [17, Ch. 4] states that if $\lambda: G \to A * B$ is an epimorphism from a free group G to any free product of groups $A * B$ then G admits a decomposition as a free product $G_A * G_B$ such that $\lambda(G_A) = A$ and $\lambda(G_B) = B$. The theorem which follows shows that, at the stable level, the existence of such free product decompositions for free splitting homomorphisms is equivalent to the conjecture, Conjecture B in Section 1, that presentations for the trivial group are standard (in the sense of Q^{**}-transformations).

THEOREM 5.3. *Let (r_i) be a p-tuple of elements in Y^n with $p \ge n$. Set $m = n + p$ and let ψ be any free splitting homomorphism in normal form with*

$$\mathscr{P}(\psi) = \langle Y^n : r_i \rangle.$$

Then (r_i) is Q^{**}-equivalent to $(y_1, \ldots, y_n, 1, \ldots, 1)$ if and only if for some number ℓ, the group $G^{m+2\ell}$ in the sum $(\psi \# \chi_\ell)$: $G^{m+2\ell} \to X^{n+\ell} \times Y^{n+\ell}$ decomposes as $G_X * G_Y$ so that $(\psi \# \chi_\ell)(G_X) = X^{n+\ell} \times 1$ and $(\psi \# \chi_\ell)(G_Y) = 1 \times Y^{n+\ell}$.

Proof. Let $\chi_{m,n}$ $(m \ge 2n)$ denote the free splitting homomorphism from G^m to $X^n \times Y^n$ given by $\chi_{m,n}(g_i) = (x_i, y_i)$ $(i \le n)$, $\chi_{m,n}(g_{i+n}) = (1, y_i)$ $(i \le n)$, and $\chi_{m,n}(g_{i+n}) = (1,1)$ $(i > n)$. That is, $\chi_{m,n}$ is the Mihailova map associated with the group presentation $\langle Y^n : (y_1, \ldots, y_n, 1, \ldots, 1)\rangle$. Notice that $\chi_{m,n} \# \chi_\ell$ is equivalent to $\chi_{m+\ell,n+\ell}$. By Theorem 4.1, (r_i) is Q^{**}-equivalent to $(y_1, \ldots, y_n, 1, \ldots, 1)$ if and only if for some ℓ, $\psi \# \chi_\ell$ and $\chi_{m+\ell,n+\ell}$ are equivalent. Now G^m in

$$\chi_{m,n}: G^m \to X^n \times Y^n$$

splits as the free product $G_X * G_Y$ where $G_X = Gp(\{g_i g_{i+n}^{-1}: i \leq n\})$ and $G_Y = Gp(\{g_{i+n}\})$, and this free product decomposition has the property indicated in the conclusion of the lemma. Thus if (r_i) is Q^{**}-equivalent to $(y_1, \ldots, y_n, 1, \ldots, 1)$ then by the above argument, the equivalence between $\psi \# \chi_\ell$ and $\chi_{m+\ell,n+\ell}$ can be used to pull back the desired free product decomposition of $G^{m+2\ell}$.

Suppose, on the other hand, that $G^{m+2\ell}$ admits the desired free product decomposition. Then under a change of basis, $\psi \# \chi_\ell$ is represented as the free splitting homomorphism

$$g_i \to (x_i, 1)(i \leq n + \ell) \quad g_{i+n+\ell} \to (1, y_i)(i \leq n + \ell) \quad g_{i+m+\ell} \to (1, 1)$$

By an obvious change of basis we can replace the 1's in the $(x_i, 1)$'s by y_i's to get images (x_i, y_i) $(i \leq n + \ell)$. But now we have a Mihailova map so by Theorem 4.1, (r_i) and $(y_1, \ldots, y_n, 1, \ldots, 1)$ are Q^{**}-equivalent.

The relation between the Andrews-Curtis conjecture and the Grusko-Neumann theorem is now seen by specializing Theorem 5.3 to the case $p = n$.

SECTION 6. CONCLUDING REMARKS

We return for a moment to free presentations for Heegaard splitting homomorphisms (Section 2). Let $\psi: G^{2n} \to X^n \times Y^n$ be a free presentation for a constrained splitting homomorphism $\pi_1(Q) \to X^n \times Y^n$ as in (1) where $\pi_1(Q) = G^{2n}/Cl(q)$. If ψ is in normal form with associated group presentation $\mathscr{P}(\psi) = \langle Y^n: (r_i)\rangle$, and if q has the form, $\prod g_{i_j} g_{i_j+n} + g_{i_j}^{-1} g_{i_j+n}^{-1}$, then the presentation $\mathscr{P}(\psi)$ is the geometric presentation described by Singer [28, Sec. 2] for the fundamental group of a 3-manifold M corresponding to the homomorphism $\pi_1(Q) \to X^n \times Y^n$. To see this use Jaco's construction to reconstruct from $\pi_1(Q) \to X^n \times Y^n$ a geometric Heegaard splitting $M = U +_{t_2 t_1^{-1}} V$ of M into cubes with handles U and V where U and V are identified along their boundaries via homeomorphisms t_1: $Q \to \mathrm{Bd}\,U$ and t_2: $Q \to \mathrm{Bd}\,V$. Here X^n is identified with $\pi_1(U)$ and Y^n with $\pi_1(V)$ so that if i and j denote the inclusion maps from $\mathrm{Bd}\,U$ to U and $\mathrm{Bd}\,V$ to V, then

$$i_* t_{1*} \times j_* t_{2*}: \pi_1(Q) \to X^n \times Y^n$$

is the splitting homomorphism described above. The generators g_{i+n} correspond to simple closed curves in Q that are mapped to meridians in $\mathrm{Bd}\,U$ by t_1, and under the identification above, $\mathscr{P}(\psi)$ is the presentation described by Singer.

It is, in fact, not difficult to show that every geometric presentation for the fundamental group of an arbitrary 3-manifold M that arises from a Heegaard splitting of M in the manner described by Singer is obtained as $\mathscr{P}(\psi)$ for some normalized free splitting homomorphism ψ that freely presents a Heegaard splitting homomorphism for M. Thus by Lemma 2.1 and Theorem 4.1, the geometric group presentations for the fundamental group of M are all Q^{**}-equivalent. We have then rederived the previously known result that the Q^{**}-class of a geometric group presentation for the fundamental group of a 3-manifold is a topological invariant of the 3-manifold. Metzler also makes this observation in [19, (6)] for presentations arising from spines of 3-manifolds. The Q^{**}-classes in the two cases are the same, for spines can be constructed from Heegaard splittings and vice versa so that the Q^{**}-classes of the group presentations are not changed.

But Lemma 2.1 and Theorem 4.1 show that the topological invariance referred to in the preceding paragraph applies to a much broader class of group presentations than just the geometric ones in Heegaard splittings, for there are many possible surface group relators q in the model above so that for these choices of q, the elements g_{i+n} in G^{2n} *do not* correspond to geometric simple closed curves in Q.

The discussion above suggests a new way to try to show that a balanced group presentation $\mathscr{P} = \langle Y^n : (r_i)\rangle$ is Q^{**}-equivalent to a geometric presentation for the fundamental group of a closed 3-manifold: Do not alter $\mathscr{P}$, except possibly to stabilize $\mathscr{P}$ to $\langle Y^{n+2\ell}: ((r_i),(y_{n+1},\dots,y_{n+\ell}))\rangle$. Consider free splitting homomorphisms of the form $\psi\colon G^{2n} \to X^n \times Y^n$ where $\psi(g_i) = (x_i, v_i)$ $(i \le n)$ and $\psi(g_{i+n}) = (1, r_i)$. Here the v_i's are arbitrary subject to the constriction that $\{v_i\} \cup \{r_i\}$ generate Y^n. Now search for true quadratic words q in G^{2n} under various free bases (each generator appears twice—once with exponent $+1$ and once with exponent -1) to see if one of these belongs to ker ψ. If one turns up, and if its spelling length, $4n$, is minimal over all free bases for G^{2n}, then ψ is a geometric realization of a Heegaard splitting homomorphism corresponding to a closed 3-manifold M, and $\mathscr{P}$ is Q^{**}-equivalent to a geometric presentation for $\pi_1(M)$. On the other hand, if $\mathscr{P}$ is Q^{**}-equivalent to a geometric 3-manifold presentation, then when unlimited stabilization is allowed as described above, a true quadratic word q as above must turn up in ker ψ and so the construction above must lead eventually to a geometric 3-manifold presentation Q^{**}-equivalent to $\mathscr{P}$.

One final question: Suppose that $\psi\colon G^{2n} \to X^n \times Y^n$ is a free presentation for a Heegaard splitting homomorphism associated with a homotopy 3-sphere M. By [30, Th. 1] or Lemma 3.3, ψ is surjective. Is ψ stably equivalent to χ_n? By Theorem 4.1 this is equivalent to asking whether a geometric group presentation for $\pi_1(M)$ is Q^{**}-equivalent to $\langle Y^n : (y_i)\rangle$.

APPENDIX. AN ALTERNATE DESCRIPTION OF EXTENDED NIELSEN TRANSFORMATIONS

Wright [34] and some others use a description of extended Nielsen transformations that does not involve isomorphisms of the base groups. We will show here that this alternate description leads to the same thing as Q^{**}-equivalence.

In the Appendix here we will understand a *finite group presentation* to be an expression of the form $\mathscr{P} = \langle y_{i_1}, \dots, y_{i_n} : (r_1, \dots, r_p)\rangle$ where $y_{i_1}, \dots, y_{i_n}$ are distinct elements in the infinite set $\{y_1, \dots, y_i, \dots\}$, and $(r_1, \dots, r_p)$ is a p-tuple of elements in the group $Gp(\{y_{i_1},\dots,y_{i_n}\})$. We will not distinguish between presentations that differ by a permutation on the order of the generators $y_{i_1},\dots,y_{i_n}$.

Consider the following six operations on a group presentation

$$\mathscr{P} = \langle y_{i_1},\dots,y_{i_n}\colon (r_1,\dots,r_p)\rangle:$$

(0) Interchange two relators r_j and r_k.

(1) Replace a relator r_k by $t^{-1}r_k t$ where $t \in Gp(\{y_{i_1},\dots,y_{i_n}\})$.

(2) Replace a relator r_k by r_k^{-1}.

(3) Add a new generator $y_{i_{n+1}} \notin \{y_{i_1},\dots,y_{i_n}\}$ and add a new relator $r_{p+1} = y_{i_{n+1}}w$ where $w \in Gp(\{y_{i_1},\dots,y_{i_n}\})$.

(4) Inverse of (3).

(5) Replace a relator r_k by $r_k r_j$ where $j \neq k$.

We define two presentations to be EN-*equivalent* if one can be obtained from the other by a finite sequence of operations of types (0)–(5). It is easy to check by reference to [34, Secs. 4–7] that EN-equivalence is, up to a change in notation, the equivalence relation used by Wright.

The following lemma shows that EN-equivalence is the same as Q^{**}-equivalence. With this lemma, Lemma 4.2 here becomes a restatement of Corollary 3.1 in [33].

LEMMA A.1. *Let (r_i) and (s_i) be tuples of elements in Y^{n_1} and Y^{n_2} respectively. Then (r_i) and (s_i) are Q^{**}-equivalent if and only if $\mathscr{P}(1) = \langle y_1,\dots,y_{n_1}\colon (r_i)\rangle$ and $\mathscr{P}(2) = \langle y_1,\dots,y_{n_2}\colon (s_i)\rangle$ are* EN-*equivalent.*

Proof. First consider the problem of converting an EN-equivalence to a Q^{**}-equivalence. For an arbitrary presentation $\mathscr{P} = \langle y_{i_1},\dots,y_{i_n}\colon (t_i)\rangle$, and an integer m such that $i_j \leq m$ $(1 \leq j \leq n)$, let $\mathscr{P}_m$ denote the presentation

$$\langle y_1,\dots,y_m\colon ((t_i),(y_{k_1},\dots))\rangle$$

obtained by activating via operations of type (3) each generator y_k $(k \leq m)$ such that $y_k \notin \{y_{i_1},\dots,y_{i_n}\}$ and adding each such y_k as a relator. Two facts are clear from this construction:

(I) If m_1 and m_2 are such that both $\mathscr{P}_{m_1}$ and $\mathscr{P}_{m_2}$ are defined, and if (u_i) and (v_i) are the relator tuples in $\mathscr{P}_{m_1}$ and $\mathscr{P}_{m_2}$, then (u_i) and (v_i) are Q^{**}-equivalent.

(II) Any $\mathscr{P}_m$ is EN-equivalent to $\mathscr{P}$.

Consider an elementary operation of type (0)–(5) converting a presentation $\mathscr{P}(a)$ to a presentation $\mathscr{P}(b)$. We claim that if m is sufficiently large so that both $\mathscr{P}(a)_m$ and $\mathscr{P}(b)_m$ are defined, and if (u_i) and (v_i) are the relator tuples in $\mathscr{P}(a)_m$ and $\mathscr{P}(b)_m$ respectively, then (u_i) and (v_i) are Q^{**}-equivalent. Except for the case where the operation is of type (3) or (4) this is clear since the other operations correspond to Q-transformations on (u_i). We need only consider a type (3) operation, for a type (4) operation is the inverse of a type (3) operation converting $\mathscr{P}(b)$ to $\mathscr{P}(a)$. Up to permutation, (v_i) results from (u_i), in the type (3) case, by replacing some element y_{i_j} with $y_{i_j}w$ where w is a word in $\{y_i : i \le m \text{ and } i \ne i_j\}$. Define an automorphism λ: $Y^m \to Y^m$ by $\lambda(y_i) = y_i$ $(i \ne i_j)$ and $\lambda(y_{i_j}) = y_{i_j}w$. Then $(\lambda(u_i))$ is a permutation of (v_i) so (u_i) and (v_i) are Q^{**}-equivalent by Lemma 1.1.

Suppose now that the presentations $\mathscr{P}(1)$ and $\mathscr{P}(2)$ in the hypotheses of the lemma are EN-equivalent. Let $\mathscr{P}(1) = \mathscr{P}(1,0) \dots \mathscr{P}(1,c) \dots \mathscr{P}(1,e) = \mathscr{P}(2)$ be a sequence of elementary operations converting $\mathscr{P}(1)$ to $\mathscr{P}(2)$. Let m be an integer larger than the index of any generator that is active in the sequence above. By the considerations in the preceding paragraph together with an induction argument, it follows that the tuples in the presentations $\mathscr{P}(1,c)_m$ $(0 \le c \le e)$ are all Q^{**}-equivalent. But $\mathscr{P}(1) = \mathscr{P}(1)_{n_1}$ and $\mathscr{P}(2) = \mathscr{P}(2)_{n_2}$ so by (I) the tuples (r_i) and (s_i) are Q^{**}-equivalent.

Second, consider the problem of converting Q^{**}-equivalences to EN-equivalences. Let (r_i) and (s_i) be Q^{**}-equivalent in the hypotheses of the lemma. We may suppose that (s_i) results from (r_i) by a single transformation of Type 1–6. Except for a Type 4 transformation, the EN-equivalence of $\mathscr{P}(1)$ and $\mathscr{P}(2)$ is clear, for the other transformations correspond to EN-operations. Let $(s_i) = (\lambda(r_i))$ where $\lambda: Y^{n_1} \to Y^{n_1}$ is an automorphism. By [23] we may suppose that λ is induced by an elementary Nielsen transformation. We consider the Type 2 case of a Nielsen transformation and leave the inversion case to the reader. Let $\lambda(y_i) = y_i$ $(i \ne k)$ and $\lambda(y_k) = y_k y_j$.

Consider the two presentations

$$\mathscr{P}(1) = \langle y_1, \dots, y_{n_1} : (r_i) \rangle \quad \text{and} \quad \mathscr{P}(2) = \langle y_1, \dots, y_{n_2} : (s_i) \rangle$$

where now $n_1 = n_2$. By activating the generators $y_{n_1+1}, \dots, y_{2n_1}$ via type (3) operations that add new relations $y_{n_1+i}y_i^{-1}$, convert $\mathscr{P}(1)$ to

$$\mathscr{P}(1,1) = \langle y_1, \dots, y_{2n_1} : ((r_i), (y_{n_1+1}y_i^{-1})) \rangle.$$

Now regard each r_i as a word in the alphabet $\{y_1, \dots, y_{n_1}\}$. Substitute $y^{\varepsilon}_{n_1+i}$ for each syllable y_i^{ε} $(1 \le i \le n_1)$ to convert (r_i) to a tuple (u_i) of elements in $Gp(\{y_{n_1+1}, \dots, y_{2n_1}\})$. This substitution induces a sequence of operations of types (0), (1), (2), and (5) converting $\mathscr{P}(1,1)$ to $\mathscr{P}(1,2) = \langle y_1, \dots, y_{2n_1} : ((u_i), (y_{n_1+i}y_i^{-1})) \rangle$. To see that this conversion can be made by EN-operations consider for example an expression uy_iv and the following sequence of EN-operations (with the unmodified terms ignored) converting uy_iv to $uy_{n_1+i}v$:

$$\begin{array}{rcll}
uy_iv & \to & vuy_i & \text{(conjugation)} \\
y_{n_1+i}y_i^{-1} & \to & y_i^{-1}y_{n_1+i} & \text{(conjugation)} \\
vuy_i & \to & vuy_iy_i^{-1}y_{n_1+i} & \text{(type (5))} \\
vuy_{n_1+i} & \to & uy_{n_1+i}v & \text{(conjugation)} \\
y_i^{-1}y_{n_1+i} & \to & y_{n_1+i}y_i^{-1} & \text{(conjugation)}
\end{array}$$

The substitution for inverses is similar.

Let $\{t_i : 1 \le i \le n_1\}$ be the set defined by $t_i = y_i$ $(i \ne k)$ and $t_k = y_k y_j$. In the same manner as with $\mathscr{P}(1)$, convert $\mathscr{P}(2)$ to $\mathscr{P}(2,1) = \langle y_1, \dots, y_{2n_1} : ((s_i), (y_{n_1+i}t_i^{-1})) \rangle$.

Then regard each s_i as being spelled in the alphabet $\{t_i\}$. As in the previous case, substitute $y^{\varepsilon}_{n_1+i}$ for each syllable t^{ε}_i $(1 \le i \le n_1)$ to convert $\mathscr{P}(2,1)$ to

$$\mathscr{P}(2,2) = \langle y_1, \ldots, y_{2n_1} : ((v_i), (y_{n_1+i} t_i^{-1})) \rangle .$$

But now (v_i) must be equal to (u_i) since the isomorphism λ creates a spelling correspondence between (r_i) in the alphabet $\{y_i\}$ and (s_i) in the alphabet $\{t_i\}$.

By inversions, convert $\mathscr{P}(1,2)$ to $\mathscr{P}(1,3) = \langle y_1, \ldots, y_{2n_1} : ((u_i), (y_i y^{-1}_{n_1+1})) \rangle$ and $\mathscr{P}(2,2)$ to $\mathscr{P}(2,3) = \langle y_1, \ldots, y_{2n_1} : ((v_i), (t_i y^{-1}_{n_1+i})) \rangle$. By operations of type (4) destabilize $\mathscr{P}(1,3)$ and $\mathscr{P}(2,3)$ to $\mathscr{P}(1,4) = \langle y_{n_1+1}, \ldots, y_{2n_1} : (u_i) \rangle$ and

$$\mathscr{P}(2,4) = \langle y_{n_1+1}, \ldots, y_{2n_1} : (v_i) \rangle .$$

In the case of $\mathscr{P}(2,3)$ we must drop y_k first and then the other generators. But $(u_i) = (v_i)$ so $\mathscr{P}(1,4) = \mathscr{P}(2,4)$ and by transitivity, $\mathscr{P}(1)$ and $\mathscr{P}(2)$ are EN-equivalent as desired.

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