FUNCTIONAL INTEGRALS RELATED TO A NONCONTRACTION SEMIGROUP

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1. INTRODUCTION

Since the controversial Feynman path integral (see [3,6]) appeared, functional integral representations for solutions of certain initial value problem $u_t = Au + vu$ have been extensively studied. There are essentially two types of integral representations. In one type of representation, A is the infinitesimal generator of a contraction semigroup S_t such that $S_t 1 = 1$, $\|S_t\| = 1$. The measure involved in the integration is a probability measure which is associated with a diffusion process (see [5,1]). For generalization to the nonhomogeneous case, see [10]. In other types of representations, A is associated with a semigroup of operators S_t such that $S_t 1 = 1$ and $\|S_t\| \equiv c > 1$. The integration is carried out with respect to a finitely additive set function (see [7,2]). We will study here a type where A generates a semigroup of operators S_t such that $S_t 1 = 1$ and $\|S_t\| \leq e^{\alpha t}$ for some $\alpha \in R^+$. The measure to be used in the integration is a measure, perhaps complex or signed, with total measure 1. Note that the condition $S_t 1 = 1$ is indispensable in the construction of measures or set functions on function spaces. The difference between the above three cases is the norm of S_t.

Throughout this article, unless otherwise specified, X denotes a compact metric space with metric ρ, $C = C(X)$ denotes the space of all real continuous functions on X with supremum norm and A denotes a closed linear operator on C with domain $\mathscr{D}(A)$ dense in C and containing all constant functions.

It will be shown that if A satisfies the following conditions:

(1.1) $$Af(x_0) \leq \alpha f(x_0) \text{ if } f(x_0) = \|f\|, \text{ where } \alpha \in R^+,$$

(1.2) $$\lambda - A \text{ maps } \mathscr{D}(A) \text{ onto } C \text{ for each } \lambda > \alpha,$$

(1.3) $$A1 = 0,$$

then the solution of the initial value problem

(1.4) $$\begin{cases} u_t(t,x) = Au(t,x) + v(x)u(t,x), \\ u(0,x) = f(x), \end{cases}$$

$0 \leq t \leq T < \infty$, $v \in C$, $f \in \mathscr{D}(A)$, has a functional integral representation. The solution can also be expressed as

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$$u(t,x) = \int_X Q(t,x,dy) f(y) \tag{1.5}$$

where $Q(t,x,dy)$ is a transition function, that is, it is measurable in (t,x) for fixed dy, a measure in dy for fixed (t,x), and satisfies the Chapman-Kolmogorov equation.

The above result can be extended to the case where C is the space of complex continuous functions. It can also be extended to the case where X is a separable locally compact Hausdorff space by applying compactification and metrization. However, then, the space C should be modified such that every element in C has a continuous extension when X is compactified. For example, if $X = R$, then C should be the space of all continuous functions on R which have limits as $|x|$ approaches infinity.

Conditions (1.1) and (1.2) are standard in the theory of semigroups of operators. They permit A to generate a semigroup of linear operators. If A is not closed, one has to assume that $R(\lambda - A)$ is dense in C. Condition (1.3) makes it possible to construct a measure on the function space $\Omega = X^{[0,T]}$. In section 2, it is shown that A is associated with a process $x(t,\omega)$ which is Markov in the sense defined in [9]. One of the main tasks of this article is to show that the integration of a measurable function of $x(t,\omega)$ with respect to dt makes sense. This is done in section 3. Indeed, it is proved there that $x(t,\omega)$ is stochastically equivalent to a process which is right continuous and has no second kind of discontinuities. In the last section, the functional integral representation of $u(t,x)$ is derived. The transition function $Q(t,x,dy)$ is defined as

$$Q(t,x,F) = E_{0,x}\left\{ I_F(x(t)) \exp\left[\int_0^t v(x(r))\, dr \right] \right\} \tag{1.6}$$

where I_F denotes the indicator function of Borel set $F \subset X$.

Operators which satisfy conditions (1.1) – (1.3) can be found easily. For example, let $X = \{0,1,2,\ldots\}$, $C = \{\text{all convergent sequences}\}$, $\gamma \in R$ and n be a positive integer. Then the operator A defined by

$$Af(x) = \gamma \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(x+k)$$

satisfies condition (1.1) – (1.3) with $\alpha = 2^n |\gamma|$. For other examples, see [8].

2. MARKOV PROCESS

Let B denote the space of all real bounded measurable functions on X with supremum norm. For $f_n \in B$, $n \geq 0$, denote $f_0 = w - \lim f_n$ if $q(f_0) = \lim q(f_n)$ for each finite measure q on X. Let A satisfy conditions (1.1) – (1.3).

LEMMA 2.1. *For each $\lambda > \alpha$, $R_\lambda = (\lambda - A)^{-1}$ is a well defined linear operator from C to $\mathscr{D}(A)$ such that $\|R_\lambda\| \le (\lambda - \alpha)^{-1}$. Moreover, if $\lambda, \mu > \alpha$, the resolvent equation $R_\lambda - R_\mu = (\mu - \lambda) R_\mu R_\lambda$ holds.*

Proof. Suppose that $(\lambda - A) f = 0$. Let $x_0 \in X$ be such that $|f(x_0)| = \|f\|$. If $f(x_0) = \|f\|$, then $\lambda f(x_0) = Af(x_0) \le \alpha f(x_0)$ by assumption (1.1). Since $\lambda > \alpha$, $\|f\| = f(x_0) = 0$. If $f(x_0) = -\|f\|$, replace f by $-f$ to obtain the same result. Therefore, $\lambda - A$ is one-to-one. From assumption (1.2), $R_\lambda = (\lambda - A)^{-1}$ is well defined on C. To show that $\|R_\lambda\| \le (\lambda - \alpha)^{-1}$, let $g = R_\lambda f$. Assume that $g(x_0) = \|g\|$. Then $\lambda g - Ag = f$ and $\lambda g(x_0) - f(x_0) = Ag(x_0) \le \alpha g(x_0)$. Hence

$$(\lambda - \alpha) g(x_0) \le f(x_0) \le \|f\|.$$

This implies that $(\lambda - \alpha)\|R_\lambda f\| \le \|f\|$. Again, if $g(x_0) = -\|g\|$, replace f by $-f$ and g by $-g$ to get the same inequality. Thus, $(\lambda - \alpha)\|R_\lambda\| \le 1$. This proves the first part of Lemma 2.1. The proof of the second part is routine.

From the assumptions imposed on A and Lemma 2.1, the Hille-Yosida theorem applies. The operator A generates a strongly continuous semigroup of linear operators S_t on C such that, for $t \ge 0$,

$$\|S_t\| \le e^{\alpha t}. \tag{2.1}$$

Furthermore, for each x, $R_\lambda f(x)$ is the Laplace transform of $S_t f(x)$. From assumption (1.3) and the uniqueness of the Laplace transform, it follows that, for each $t \ge 0$,

$$S_t 1 = 1. \tag{2.2}$$

For each (t, x), $S_t f(x)$ is a bounded linear functional on C. By the Riesz representation theorem, there exists a finite measure $P(t, x, dy)$ such that

$$S_t f(x) = \int_X P(t, x, dy) f(y) \tag{2.3}$$

for $f \in C$. Since $S_t f(x)$ is continuous in (t, x), it is measurable in (t, x). Therefore, it is clear that $P(t, x, dy)$ is measurable in (t, x). By (2.1),

$$|P|(t, x, X) \le e^{\alpha t} \tag{2.4}$$

for $t \ge 0$, $x \in X$ (where "$|\cdot|$" denotes the total variation of a measure). It is trivial from (2.3) that S_t can be extended to the space B such that inequality (2.1) still holds and that, for $f_n \in B$, $n \ge 0$,

$$S_t f_0 = w - \lim S_t f_n \quad \text{if} \quad f_0 = w - \lim f_n. \tag{2.5}$$

For fixed $0 < T < \infty$, (2.1), (2.2) and (2.5) imply that the following theorem is true (see [8, Theorem 2.1]).

THEOREM 2.2. *There exists a Markov process $x(t, \omega)$, $0 \le t \le T$, $\omega \in \Omega = X^{[0,T]}$, which has $P(t, x, dy)$ as its transition function and which satisfies*

$$E\{f(x(t+s))\,|\,\sigma(x(r), 0 \le r \le s)\} = S_t f(x(s)) \tag{2.6}$$

for $0 \le s \le t + s \le T$, $f \in B$.

Note that the conditional expectation used in (2.6) is defined in [9], which is a generalization of the ordinary conditional expectation.

3. INTEGRABILITY

In this section, it will be shown that the process $x(t, \omega)$ obtained in Theorem 2.2 is stochastically equivalent to a process whose sample functions are right continuous and have no discontinuities of the second kind. For $r > 0$, $x \in X$, let $B_r(x) = \{y \in X, \rho(y, x) < r\}$.

LEMMA 3.1. *For fixed* $r > 0$,

$$\lim_{t \to 0} |P|(t, x, B_r^c(x)) = 0$$

uniformly in x.

Proof. Let x be fixed and let $f \in C$ be such that $f(x) = 1$, $\|f\| \le 1$ and $f = 0$ on $B_r^c(x)$. Then

$$\int_X P(0, x, dy) f(y) = f(x) = \lim_{t \to 0} S_t f(x) = \lim_{t \to 0} \int_X P(t, x, dy) f(y). \tag{3.1}$$

By assumption on f and (3.1), one has

$$\liminf_{t \to 0} |P|(t, x, B_r(x)) \ge 1. \tag{3.2}$$

Inequalities (2.4) and (3.2) imply that

$$\limsup_{t \to 0} |P|(t, x, B_r^c(x)) \le \limsup_{t \to 0} |P|(t, x, X) - \liminf_{t \to 0} |P|(t, x, B_r(x)) \le \lim_{t \to 0} e^{\alpha t} - 1 = 0.$$

Therefore $|P|(t, x, B_r^c(x))$ converges to 0 as $t \to 0$. The proof of the uniformity of convergence is similar to that of ordinary probability measures (see [4, II, p. 114, Remark]).

COROLLARY 3.2. *For each* $r > 0$,

$$\lim_{t \to 0} \inf_{x \in X} |P|(t, x, B_r(x)) = 1.$$

Proof. From the fact that $P(t, x, X) = 1$ and inequality (2.4),

$$1 \le |P|(t, x, B_r(x)) + |P|(t, x, B_r^c(x)) \le e^{\alpha t}.$$

Therefore, Corollary 3.2 follows from the above inequalities and Lemma 3.1.

For $0 \leq t \leq s \leq T$, let $\mathscr{F}_t^s = \sigma\{x(r), t \leq r \leq s\}$, $\mathscr{F}^t = \mathscr{F}_0^t$, $\mathscr{F}_t = \mathscr{F}_t^T$ and $\mathscr{F} = \mathscr{F}_0^T$. If $x \in X$, let $P_{t,x}$ denote the standard measure constructed from the transition function $P(\cdot, \cdot, dy)$ on $\mathscr{F}_t$ which is concentrated on the set $\{x(t) = x\}$. That is, if $F = \{(x(t_1), \ldots, x(t_n)) \in G\}$ with $t \leq t_1 < \ldots < t_n \leq T$, $G \in \sigma(X^n)$, then

$$P_{t,x}(F) = \int_G \cdots \int \prod_{i=1}^{n} P(t_i - t_{i-1}, y_{i-1}, dy_i) \tag{3.3}$$

where $t_0 = t$, $y_0 = x$. For $0 \leq t \leq s \leq r \leq T$, $x \in X$, let $P_{t,x}^{s,r}$ denote the restriction of $P_{t,x}$ on $\mathscr{F}_s^r$. Then, it follows from (2.4) and definition (3.3) that

$$|P_{t,x}^{s,r}|(\Omega) \leq e^{\alpha(r-t)}. \tag{3.4}$$

LEMMA 3.3. *For* $r > 0$,

$$\lim_{t \to 0} |P_{0,x}|\{x(t) \in B_r^c(x)\} = 0$$

uniformly in x.

Proof. Let $F = \{x(t) \in B_r(x)\}$. For arbitrary $\epsilon > 0$, by Corollary 3.2, there exists $\delta > 0$ such that $|P|(t, x, B_r(x)) \geq 1 - \epsilon$ for $0 \leq t \leq \delta$, $x \in X$. Let P^t denote the restriction of $P_{0,x}$ on $\sigma(x(t))$. Then

$$|P^t|(F) = |P^t|\{x(t) \in B_r(x)\} = |P|(t, x, B_r(x)).$$

Therefore, for $0 \leq t \leq \delta$, $x \in X$,

$$|P_{0,x}^{0,t}|(F) \geq |P^t|(F) \geq 1 - \epsilon.$$

The above inequalities and (3.4) implies that

$$|P_{0,x}^{0,t}|(F^c) \leq e^{\alpha t} - 1 + \epsilon \tag{3.5}$$

for $0 \leq t \leq \delta$, $x \in X$. Let x be fixed for a moment. If $D \in \mathscr{F}$ is a cylinder set, there exists a cylinder set D_t in $\mathscr{F}^t$ and an $\mathscr{F}^t$-measurable function p_t, bounded by $\exp\{\alpha(T - t)\}$, such that (see [8, Lemma 2.2])

$$P_{0,x}(DF^c) = \int_{F^c D_t} p_t \, dP_{0,x}^{0,t}.$$

Combine the above equality with (3.5), one obtains that

$$|P_{0,x}(DF^c)| \leq e^{\alpha(T-t)}(e^{\alpha t} - 1 + \epsilon)$$

for $0 \leq t \leq \delta$. Therefore, from the definition of the total variation of a measure,

$$|P_{0,x}|(F^c) \leq 2e^{\alpha(T-t)}(e^{\alpha t} - 1 + \epsilon)$$

for $0 \le t \le \delta$. The above inequality holds for all $x \in X$. Since ϵ is arbitrary, Lemma 3.3 follows by letting $t \to 0$.

Let $0 \le t \le s < s + h \le T$. By the facts that the transition function of $x(t)$ is time homogeneous and $\mathscr{F}_s \subset \mathscr{F}_t$, it is easy to see that

$$|P_{s,x}|\{x(s+h) \in B_r^c(x)\} \le |P_{t,x}|\{x(t+h) \in B_r^c(x)\} \tag{3.6}$$

for $x \in X$ and $r > 0$. Let

$$a(r,h) = \sup\{|P_{s,x}|\{x(t) \in B_r^c(x)\}, 0 \le s \le t \le s + h \le T, x \in X\}.$$

Then Lemma 3.3 and (3.6) imply that

THEOREM 3.4. *For each* $r > 0$, $a(r,h) \to 0$ *as* $h \downarrow 0$.

THEOREM 3.5. *For* $r > 0$, *uniformly in* x,

$$\lim_{h \to 0} |P_{0,x}|\{\rho(x(t+h), x(t)) \ge r\} = 0. \tag{3.7}$$

Proof. Consider $0 \le t + h < t \le T$ first. Utilizing property (3.3) and inequalities (3.6), (2.4), one obtains

$$\begin{aligned} &|P_{0,x}|\{\rho(x(t+h), x(t)) \ge r\} \\ &\quad \le e^{\alpha(T-t)} \int_X |P_{0,y}|\{x(|h|) \in B_r^c(y)\}\, |P_{0,x}^{0,t+h}|\{x(t+h) \in dy\}. \end{aligned} \tag{3.8}$$

Since $|P_{0,x}^{0,t+h}|(\Omega) \le e^{\alpha t}$ for all $0 \le t + h \le t$, $x \in X$ and since the integrand in (3.8) converges uniformly to 0 as $h \to 0$ by Lemma 3.3, (3.7) follows from (3.8) by letting $h \uparrow 0$. For the case that $0 \le t < t + h \le T$, the proof is similar but simpler.

THEOREM 3.6. *The process* $x(t)$ *is stochastically equivalent to a process whose sample functions are right continuous and without the second kind of discontinuities.*

Theorem 3.6 is a consequence of Theorems 3.4 and 3.5. Since the proof is parallel to that of ordinary probability space case, the reader is referred to [4, I, pp. 180–184].

From Theorem 3.6, one can assume that $x(t)$ has right continuous sample functions which have no second kind of discontinuities. Therefore, integration of $v(x(t))$, $v \in C$, with respect to dt makes sense. This property will be used in the next section.

4. INTEGRAL REPRESENTATION

Let $u(t,x) = U(t)f(x)$ be the unique solution of (1.4). From the perturbation theorem, $U(t)f(x)$ is the unique solution of the equation

$$U(t)f = S(t)f + (S * VU)(t)f, \quad U(0)f = f, \tag{4.1}$$

where $S(t) = S_t$, V is the operator defined as $Vg(x) = v(x)g(x)$ and where "$*$" denotes the convolution of operator valued functions of t. When expressed in terms of the transition function $P(t,x,dy)$, (4.1) becomes

$$u(t,x) = \int_X P(t,x,dy)f(y) + \int_0^t ds \int_X P(x,s,dy)\,v(y)\,u(t-s,y), \quad u(0,x) = f(x). \tag{4.2}$$

The solution of this integral equation can be expressed as an infinite series whose $(n+1)$-th term is bounded by $t^n \|v\|^n e^{\alpha t}(n!)^{-1}$, $n \geq 0$.

LEMMA 4.1. *Let $w(t,x)$ denote the functional integral*

$$E_{0,x}\left\{f(x(t)) \exp\left[\int_0^t v(x(r))\,dr\right]\right\}. \tag{4.3}$$

Then $w(t,x)$ satisfies equation (4.2).

Proof. It is clear that $w(0,x) = f(x)$. By applying Fubini's theorem and taking the conditional expectation given $\mathscr{F}^s$,

$$w(t,x) - E_{0,x}\{f(x(t))\} = E_{0,x}\left\{f(x(t))\left[\exp\left(\int_0^t v(x(r))\,dr\right) - 1\right]\right\}$$
$$= E_{0,x}\left\{\int_0^t dsv(x(s))E_{s,x(s)}\left[\exp\left(\int_s^t v(x(r))\,dr\,f(x(t))\right)\right]\right\}.$$

Since the process $x(t)$ is time homogeneous,

$$E_{s,x(s)}\left\{f(x(t))\exp\left[\int_s^t v(x(r))\,dr\right]\right\} = w(t-s,x(s)).$$

Therefore,

$$w(t,x) - E_{0,x}\{f(x(t))\} = E_{0,x}\left\{\int_0^t v(x(s))\,w(t-s,x(s))\,ds\right\}$$
$$= \int_0^t ds \int_X P(s,x,dy)\,w(t-s,y)\,v(y).$$

This proves that $w(t,x)$ satisfies equation (4.2).

By uniqueness, $w(t,x) = u(t,x)$. Therefore, the solution of equation (1.4) has a functional integral representation (4.3). If $Q(t,x,F)$ is defined as in (1.6), then

(4.3) becomes (1.5). By the uniqueness of solution for (1.4), $Q(t,x,F)$ satisfies the Chapman-Kolmogorov equation. This fact can also be obtained directly by using the conditional expectation and the definition of $Q(t,x,F)$.

THEOREM 4.2. *The solution* $u(t,x)$ *of equation* (1.4) *admits an integral representation* (1.5) *where* $Q(t,x,dy)$ *is a transition function defined by* (1.6).

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