SIMPLY CONNECTED SURGERY OVER A RING

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1. INTRODUCTION

In [1] it is shown that the surgery obstructions for a simply connected problem are given by the signature, Kervaire invariant or invariants β_p lying in a certain 2-torsion group determined in [2]. The first two listed have been treated extensively in the literature. It is the purpose of this paper to compute the β_p-invariants of a normal map $f: M \to X$ in terms of M,X and the degree of f (section 3). We also deduce a product formula. Applications are given to Poincare complexes, homology spheres, singular manifolds and involutions.

2. SURGERY OBSTRUCTION GROUPS

Let R be a principal ideal domain. A map $f: X \to Y$ between path-connected, simply connected spaces is an R-*homotopy equivalence* if

$$f_{\#} \otimes 1: \pi_i(X) \otimes R \cong \pi_i(Y) \otimes R \qquad \text{for all } i.$$

Suppose $\pi_1 X = 0$ and $(X, \partial X)$ satisfies Poincare duality with coefficients in R, given by cap product with $[X,\partial X] \in H_n(X,\partial X)$. Let $f: (M,\partial M) \to (X,\partial X)$ be a map so that

(i) $(M,\partial M)$ is a compact n-manifold,

(ii) $f_*[M,\partial M]$ is a unit in $H_n(X,\partial X;R) \cong R$,

(iii) there is a bundle ξ over X and a bundle map $b: \nu_M \to \xi$ covering f,

and

(iv) $f|\partial M$ is an R-homotopy equivalence.

In [1] we construct a cobordism group $L_n(1;R)$ so that if $n \geq 5$, f is normally cobordant to an R-homotopy equivalence if and only if an obstruction in $L_n(1;R)$ vanishes. Let K be the set of primes p so that $R \otimes \mathbb{Z}/p = 0$; then $L_n(1;R) \cong L_n(\mathbb{Z}_K)$ where $\mathbb{Z}_K = \mathbb{Z}[1/p: p \in K]$, and $L_n(\mathbb{Z}_K)$ is K-theoretic group of [10]. The following is proved in [1]:

THEOREM 2.1. (i) $L_{2n+1}(\mathbb{Z}_K) = 0$

(ii) $L_{4n+2}(\mathbb{Z}_K) \cong \mathbb{Z}/2 \otimes \mathbb{Z}_K$

(iii) $L_{4n}(\mathbb{Z}_K) \cong W(\mathbb{Z}_K)$.

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Here $W(\mathbb{Z}_K)$ denotes the Witt-Wall ring of non-singular even quadratic forms over $\mathbb{Z}_K$, modulo kernels, and is computed in [2]:

$$W(\mathbb{Z}_K) \subseteq a_K\mathbb{Z} \oplus \bigoplus_{p\in K} W(\mathbb{F}_p)$$

where

$$a_K = \begin{cases} 1 & K \equiv 0 \bmod (2) \\ 2 & K \equiv 3 \bmod (4),\ K \not\equiv 0 \bmod (2) \\ 4 & K \not\equiv 3 \bmod (4),\ K \not\equiv 0 \bmod (2),\ K \equiv 1 \bmod (4) \\ 8 & \text{otherwise.} \end{cases}$$

We write $K \equiv a \bmod (b)$ if $p \equiv a \bmod (b)$ for some $p \in K$ and

$$W(\mathbb{F}_p) = \begin{cases} \mathbb{Z}/2 & p = 2 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & p \equiv 1 \bmod (4) \\ \mathbb{Z}/4 & p \equiv 3 \bmod (4). \end{cases}$$

The map $W(\mathbb{Z}_K) \to a_K\mathbb{Z}$ is given by the signature, and $\beta_p\colon W(\mathbb{Z}_K) \to W(\mathbb{F}_p)$ is defined as follows: Let q be a non-singular quadratic form over $\mathbb{Z}_K$. Diagonalize q (over $\mathbb{R}$) as $\langle a_1, \ldots, a_k\rangle$, where $a_i = b_i/c_i^2$, $b_i \in \mathbb{Z}$. Then for $p \neq 2$, if n_i is the greatest power of p that divides b_i, we let $\beta_p(q) \in W(\mathbb{F}_p)$ be the form $\bigoplus_{n_i \text{odd}} \langle p^{-n_i}.b_i\rangle$. We also define $\alpha_p(q) = \beta_p(pq)$ $(p \neq 2)$, and

$$\alpha_2(q) = \dim(q) \bmod (2), \quad \beta_2(q) = m \bmod (2),$$

where m is the largest power of 2 that divides det (q).

3. COMPUTATION OF THE SURGERY OBSTRUCTIONS

Let M^{4k} be a closed oriented rational Poincare complex and $A = H^{2k}(M;\mathbb{Q})$. Then the pairing

$$B(x,y) = \langle x \cup y,\ [M]\rangle \qquad x,y \in A$$

is a non-singular symmetric bilinear form on A, and thus defines a quadratic form q on A by $q(x) = B(x,x)$. Define the *Hasse-Minkowski invariants* of M by

$$\alpha_p(M) = \alpha_p(q) \quad \text{and} \quad \beta_p(M) = \beta_p(q).$$

By definition, $\alpha_p(M) = \beta_p(M) = 0$ if $\dim(M) \not\equiv 0 \bmod (4)$.

If M has non-empty boundary, we can define a bilinear form B' on

$$A' = H^{2k}(M,\partial M;\mathbb{Q})$$

by the composition

$$H^{2k}(M,\partial M;\mathbb{Q}) \xrightarrow{i^*} H^{2k}(M;\mathbb{Q}) = \mathrm{Hom}\,(H_{2k}(M;\mathbb{Q}),\mathbb{Q})$$

$$= \mathrm{Hom}\,(H^{2k}(M;\partial M;\mathbb{Q}),\mathbb{Q}).$$

This form induces a non-singular quadratic form q on $A = A'/\ker(i^*)$. Define $\alpha_p(M) = \alpha_p(q)$, $\beta_p(M) = \beta_p(q)$.

LEMMA 3.1. *If* $M = \partial N$, *then* $\alpha_p(M) = \beta_p(M) = 0$.

Proof. Consider the following diagram (with rational coefficients throughout):

$$\begin{array}{ccccc} H^{2k}(N) & \xrightarrow{j^*} & H^{2k}(M) & \rightarrow & H^{2k+1}(N,M) \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ H_{2k+1}(N,M) & \rightarrow & H_{2k}(M) & \rightarrow & H_{2k}(N), \end{array}$$

where the vertical maps are given by Poincare duality. Then $\mathrm{Im}\,(j^*) \cong \ker\,(j^*)$ by exactness and $\mathrm{Im}\,(j^*) \cong H_{2k}(M)/\ker\,(j^*)$ under the isomorphism

$$H^{2k}(M) \rightarrow H_{2k}(M).$$

Thus $\dim\,(\mathrm{Im}\,(j^*)) = (1/2)\dim\,(H_{2k}(M))$, and for $x = j^*(y) \in \mathrm{Im}\,(j^*)$,

$$\begin{aligned} \langle x \cup x, [M] \rangle &= \langle j^*y \cup j^*y, [M] \rangle \\ &= \langle y \cup y, j_*[M] \rangle = 0. \end{aligned}$$

By Lemma 5.3 of [10], the quadratic form q corresponding to M is a kernel, and so is a kernel over $\mathbb{Q}_p$. Thus by Sylvester's Theorem, [6],

$$\alpha_p(M) = \beta_p(M) = 0 \qquad \text{for } p \neq 2.$$

For p = 2, we have $\alpha_2(M) = \dim\,(H^{2k}(M)) \bmod (2) = 0$, and $\beta_2(M) = 0$ since q is a kernel.

LEMMA 3.2.

$$\beta_p(M \times N) = \beta_p(M)\alpha_p(N) + \beta_p(N)\alpha_p(M)$$

$$\alpha_p(M \times N) = \begin{cases} \alpha_2(M)\,\alpha_2(N) & p = 2 \\ \alpha_p(M)\,\alpha_p(N) + \beta_p(M)\beta_p(N) & p \neq 2. \end{cases}$$

The proof follows from Proposition 3.1 of [2] and the proof of the product formula for the signature, [8].

THEOREM 3.3. *Let* X *be an oriented Poincare complex. Then*

$$\alpha_p(X) = \mathrm{Sign}\,(X) \cdot 1 \quad \textit{and} \quad \beta_p(X) = 0.$$

Proof. Notice that the quadratic form q associated to X is unimodular over the integers, so $\beta_p(X) = 0$ by Lemma 2.2 of [2]. Define

$$q' = q \oplus |\mathrm{Sign}\,(X)|\langle \pm 1\rangle,$$

where we take + or − according to whether or not Sign (X) < 0. Then

$$\begin{aligned} \beta_p(q') &= \beta_p(q) + |\mathrm{Sign}\,(X)|\,\beta_p\langle \pm 1\rangle = \beta_p(q) = 0 \\ \text{and}\quad \sigma(q') &= \sigma(q) - \mathrm{Sign}\,(X) = 0, \end{aligned}$$

so q' is a kernel over $\mathbb{Q}$. Thus

$$0 = \alpha_p(q') = \alpha_p(X) - \mathrm{Sign}\,(X)\langle 1\rangle.$$

More generally, if X is a Poincare complex over $\mathbb{Z}_K$, then $\beta_p(X) = 0$ for $p \notin K$. However, $\alpha_p(X)$ need not equal Sign (X) · 1.

For X a smooth manifold, Theorem 3.3 follows independently of the Hasse-Minkowski principal. First of all, dimensional considerations show that α_p and β_p vanish on tor (Ω_*^{SO}). Let Ω_*^{CP} be the subring of Ω_*^{SO} generated by $\mathbb{C}P^{2n}$, $n = 0,1,\ldots$. By [8], $\Omega_*^{CP} \to \Omega_*^{SO}$ has cokernel an odd torsion group, and thus α_p and β_p are determined by their values on complex projective spaces.

In contrast to this we have the following result:

THEOREM 3.4. (1) *If* p *is an odd prime, then there exist smooth manifolds* $(M,\partial M)$ *with* $\alpha_p(M) \neq \mathrm{Sign}\,(M)\cdot 1$ *and* $\beta_p(M) \neq 0$.

(2) *Let* p *be a set of primes and* $n \in \mathbb{Z}$, $x_p \in W(\mathbb{F}_p)$ *so that* $(n,x_p)_{p\in P}$ *is in* $W(\mathbb{Z}_K)$. *Then there exists a closed, oriented,* $\mathbb{Z}_K$*-homology manifold* M^{4k}, $k > 0$, *so that* Sign (M) = n *and* $\beta_p(M) = x_p$.

Proof. (2) follows immediately from the plumbing theorem, coning over the boundary; see [1], [4] or [10].

(1) also follows from plumbing using the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 2k \end{pmatrix}$$

if p = 4k − 1, and the matrix

$$\begin{pmatrix} -2 & 1 \\ 1 & 2k \end{pmatrix}$$

if p = 4k + 1. In the first case, $\alpha_p = \langle 2\rangle$, Sign = 2, $\beta_p = \langle 2\rangle$ (and so $\alpha_p \equiv \pm 1 \bmod (4)$) and in the second case, $\alpha_p = \langle 2\rangle$, Sign = 0, $\beta_p = \langle 2\rangle$.

Remarks. (1) $\alpha_2(M) \equiv \mathrm{Sign}\,(M) \bmod (2)$ is always true.

(2) The form $\langle 2\rangle$ over $\mathbb{Q}$ has $\beta_2 \neq 0$, and so there exists a smooth manifold with boundary, $(M,\partial M)$, so that $\beta_2(M) \neq 0$.

(3) By coning over the boundary, the manifolds in (1) allow us to construct closed oriented $\mathbb{Z}_K$-homology manifolds M, with $\alpha_p(M) \neq \mathrm{Sign}(M) \cdot 1$.

THEOREM 3.5. (*Novikov Additivity*). *Let* M *and* N *be oriented manifolds and* f: $\partial M \to \partial N$ *an orientation reversing homeomorphism. Then*

$$\alpha_p(M \cup_f N) = \alpha_p(M) + \alpha_p(N)$$
$$\beta_p(M \cup_f N) = \beta_p(M) + \beta_p(N).$$

The proof is similar to the signature case; see [3]. This also generalizes to partial unions as in [9].

Let $(M,\partial M)$ be an oriented manifold and $(X,\partial X)$ a simply connected Poincare pair over $\mathbb{Z}_K$, both of dimension $4k \geq 8$. Suppose $f: (M,\partial M) \to (X,\partial X)$ is a normal map of degree 1, so that $f|\partial M$ is a $\mathbb{Z}_K$-homotopy equivalence. Do surgery on M, relative to ∂M, so that $f_*: H_i(M) \to H_i(X)$ is an isomorphism for $i < 2k$, and

$$A = \ker (f_*: H_{2k}(M;\mathbb{Z}_K) \to H_{2k}(X;\mathbb{Z}_K))$$

is a free $\mathbb{Z}_K$-module. Self-intersections of 2k-spheres in M that are null-homotopic in X define a non-singular even quadratic form q over $\mathbb{Z}_K$. Define

$$\alpha_p(f) = \alpha_p(q)$$
$$\beta_p(f) = \beta_p(q)$$
$$\mathrm{Sign}(f) = \sigma(q).$$

By 2.1, Sign (f) and $\beta_p(f)$, $p \in K$, are the surgery obstructions for f.

THEOREM 3.6.
$$\alpha_p(f) = \alpha_p(M) - \alpha_p(X)$$
$$\beta_p(f) = \beta_p(M) - \beta_p(X)$$
$$\mathrm{Sign}(f) = \mathrm{Sign}(M) - \mathrm{Sign}(X).$$

Proof. Since α_p, β_p and Sign are cobordism invariants, it follows from [4], Theorem V.1.3, that $\alpha_p(f) = \alpha_p(q^*)$, $\beta_p(f) = \beta_p(q^*)$ and $\mathrm{Sign}(f) = \mathrm{Sign}(q^*)$, where q^* is defined by the pairing

$$B^*: A^* \times A^* \to \mathbb{Q}, \quad B^*(x,y) = \langle x \cup y, [M] \rangle$$

and $A^* = \mathrm{coker}(f^*: H^{2k}(X,\partial X;\mathbb{Q}) \to H^{2k}(M,\partial M;\mathbb{Q}))$.

We have $H^{2k}(M,\partial M;\mathbb{Q}) = A^* \oplus f^* H^{2k}(X,\partial X;\mathbb{Q})$, and furthermore, this is a splitting of the bilinear form B on $H^{2k}(M,\partial M;\mathbb{Q})$, since for $x \in A^*, y \in H^{2k}(X, X;\mathbb{Q})$,

$$\langle x \cup f^*y, [M] \rangle = (f^*y \cap (x \cap [M])) = y \cap f_*(x \cap [M])$$
$$= y \cap (f_*x \cap [X]) = 0.$$

So $\alpha_p(M) = \alpha_p(q^*) + \alpha_p(X)$, etc., since the form on $f^*H^{2k}(X,\partial X;\mathbb{Q})$ is the same as the form on $H^{2k}(X,\partial X;\mathbb{Q})$.

Note that this result applies only to degree 1 maps. For arbitrary degree, we use the following notation: If p is a prime number and n is an integer, let $d_p(n)$ be the largest integer $m \geq 0$ so that p^m divides n; let $e_p(n) = p^{-d_p(n)}n$.

COROLLARY 3.7. *Let* $f\colon (M,\partial M) \to (X,\partial X)$ *be a degree* n *normal map as above. Then*

$$\operatorname{Sign}(f) = \operatorname{Sign}(M) \pm \operatorname{Sign}(X) \qquad (+ \text{ if } n < 0,\ - \text{ if } n > 0)$$

$$\alpha_2(f) = \alpha_2(M) + \alpha_2(X)$$

$$\beta_2(f) = \beta_2(M) + d_2(n)\alpha_2(X) + \beta_2(X)$$

$$\alpha_p(f) = \begin{cases} \alpha_p(M) - \langle e_p(n)\rangle \alpha_p(X) & \text{if } d_p(n) \equiv 0 \bmod (2),\ p \neq 2 \\ \alpha_p(M) - \langle e_p(n)\rangle \beta_p(X) & \text{if } d_p(n) \equiv 1 \bmod (2),\ p \neq 2 \end{cases}$$

$$\beta_p(f) = \begin{cases} \beta_p(M) - \langle e_p(n)\rangle \beta_p(X) & \text{if } d_p(n) \equiv 0 \bmod (2),\ p \neq 2 \\ \beta_p(M) - \langle e_p(n)\rangle \alpha_p(X) & \text{if } d_p(n) \equiv 1 \bmod (2),\ p \neq 2. \end{cases}$$

Proof. Let $(Y,\partial Y)$ be the Poincare pair over $\mathbb{Z}_K[1/n]$ with underlying space $(X,\partial X)$ and fundamental class $n[X,\partial X]$. Then f induces a degree 1 map $f'\colon (M,\partial M) \to (Y,\partial Y)$ and $\alpha_p(f) = \alpha_p(f')$, etc. If q, q′ are the quadratic forms corresponding to X, Y, then $nq(x) = q'(x)$. Thus if q has a diagonalization $\langle a_1, \ldots, a_k\rangle$, q′ has a diagonalization $\langle na_1, \ldots, na_k\rangle$. The result now follows from the theorem.

COROLLARY 3.8. *Let* $f\colon M \to N$ *be a degree* n *normal map between closed oriented manifolds. Then for* $p \neq 2$

$$\alpha_p(f) = \begin{cases} \operatorname{Sign}(M)\langle 1\rangle - \operatorname{Sign}(N)\langle e_p(n)\rangle & \text{if } d_p(n) \equiv 0 \bmod (2) \\ \operatorname{Sign}(M)\langle 1\rangle & \text{if } d_p(n) \equiv 1 \bmod (2) \end{cases}$$

$$\beta_p(f) = \begin{cases} 0 & \text{if } d_p(n) \equiv 0 \bmod (2) \\ -\operatorname{Sign}(N)\langle e_p(n)\rangle & \text{if } d_p(n) \equiv 1 \bmod (2). \end{cases}$$

As a typical application, we have

THEOREM 3.9. *Let* $f\colon M \to N$ *be a normal map of degree* $n > 0$ *between simply connected closed manifolds of dimension* 4k, *where* $\operatorname{Sign}(N) \equiv 0 \bmod (4)$. *Then* f *is normally cobordant to a* $\mathbb{Z}_K$*-homotopy equivalence,* $K = \{p\colon d_p(n) > 0\}$, *if and only if* $\operatorname{Sign}(M) = \operatorname{Sign}(N)$.

Proof. Since $4W(\mathbb{F}_p) = 0$, Corollary 2.8 shows that each $\beta_p(f) = 0$. By Corollary 3.7, the surgery obstruction is $\operatorname{Sign}(f) = \operatorname{Sign}(M) - \operatorname{Sign}(N)$.

Remarks. (1) If no $p \in K$ is 3 mod (4), or $2\langle e_p(n)\rangle = 0$, for all p, then we can relax the hypothesis to $\operatorname{Sign}(N) \equiv 0 \bmod (2)$.

(2) If n is a square, then we need no condition on the signature.

THEOREM 3.10. *Let* f *be as in* 3.7, N *a manifold, and* $f'\colon M\#N \to X$ *induced from* f. *Then* $\beta_p(f') = \beta_p(f) + \beta_p(N)$.

Proof. By Theorem 3.5, $\beta_p(M\#N) = \beta_p(M) + \beta_p(N)$. Thus by Corollary 3.7, the result holds.

4. PRODUCT FORMULAS

In this section we prove the product formulas for α_p and β_p; *i.e.,* we determine the pairing $L_*(1;\mathbb{Z}_K) \times L_*(1;\mathbb{Z}_K) \to L_{4*}(1;\mathbb{Z}_K)$.

THEOREM 4.1. *Let* $f: (M,\partial M) \to (X,\partial X)$, $g: (N,\partial N) \to (Y,\partial Y)$ *be degree* 1 *normal maps as above with* $\dim(M \times N) = 4k$. *Then*

$$\begin{aligned}
\alpha_2(f \times g) &= \alpha_2(f)\alpha_2(g) + \alpha_2(f)\alpha_2(Y) + \alpha_2(g)\alpha_2(X) \\
\alpha_p(f \times g) &= \alpha_p(f)\alpha_p(g) + \beta_p(f)\beta_p(g) + \alpha_p(f)\alpha_p(Y) + \beta_p(f)\beta_p(Y) \\
&\quad + \alpha_p(g)\alpha_p(X) + \beta_p(g)\beta_p(X) \quad \text{if } p \neq 2 \\
\beta_p(f \times g) &= \beta_p(f)\alpha_p(g) + \beta_p(g)\alpha_p(f) + \beta_p(f)\alpha_p(Y) + \beta_p(Y)\alpha_p(f) \\
&\quad + \beta_p(g)\alpha_p(X) + \beta_p(X)\alpha_p(g).
\end{aligned}$$

Proof. We prove the second assertion only; the others are similar. First assume the dimensions of M and N are not 0 mod (4). Then both sides of the equation are 0. To see this, it suffices to show the form associated to $M \times N$ is a kernel.

If $\dim(M) = 4\ell + 1$, $\dim(N) = 4h - 1$, $k = \ell + h$, then, assuming $\partial M = \partial N = \emptyset$ for clarity,

$$H^{2k}(M \times N;\mathbb{Q}) = \bigoplus_{i=2\ell+1}^{k} H^i(M;\mathbb{Q}) \otimes H^{2k-i}(N;\mathbb{Q}) \oplus \bigoplus_{j=2h+1}^{k} H^{2k-j}(M;\mathbb{Q}) \otimes H^j(N;\mathbb{Q})$$

$$\text{and} \quad q_{M\times N}(x \otimes y) = q_M(x)q_N(y) = 0 \qquad \text{for } x \in H^i(M;\mathbb{Q}),\ i \geq 2\ell + 1.$$

Thus the form $q_{M\times N}$ on $H^{2k}(M \times N;\mathbb{Q})$ is a kernel.

If $\dim(M) = 4\ell + 2$, $\dim(N) = 4h - 2$, then B_{q_M} is skew-symmetric on $H^{2\ell+1}(M;\mathbb{Q})$ and so we find a symplectic basis, $x_1, \dots, x_r, y_1, \dots, y_r$, that is, a basis with $B_{q_M}(x_i,x_j) = B_{q_M}(y_i,y_j) = 0$, and $B_{q_M}(x_i,x_j) = \delta_{ij}$. Let S be the subspace spanned by $x_1, \dots, x_r$. Then $q_{M\times N}$ is 0 on $S \otimes H^{2h-1}(N;\mathbb{Q})$, and $H^{2\ell+1}(M;\mathbb{Q}) \otimes H^{2h-1}(N;\mathbb{Q})$ is a kernel. But

$$H^{2k}(M \times N;\mathbb{Q}) = H^{2\ell+1}(M;\mathbb{Q}) \otimes H^{2h-1}(N;\mathbb{Q}) \oplus \bigoplus_{i\neq 2\ell+1} H^i(M;\mathbb{Q}) \otimes H^{2k-i}(N;\mathbb{Q})$$

and the argument above shows the second summand is a kernel.

Now, if $\dim(M)$, $\dim(N) \equiv 0 \bmod (4)$, then we have

$$\begin{aligned}
\alpha_p(f \times g) &= \alpha_p(M \times N) - \alpha_p(X \times Y) \qquad \text{by Theorem 3.6} \\
&= \alpha_p(M)\alpha_p(N) + \beta_p(M)\beta_p(N) - (\alpha_p(X)\alpha_p(Y) + \beta_p(X)\beta_p(Y)) \\
&= (\alpha_p(f) + \alpha_p(X))(\alpha_p(g) + \alpha_p(Y)) + (\beta_p(f) + \beta_p(X))(\beta_p(g)) \\
&\quad + \beta_p(Y)) - (\alpha_p(X)\alpha_p(Y) + \beta_p(X)\beta_p(Y)) \\
&= \alpha_p(f)\alpha_p(g) + \beta_p(f)\beta_p(g) + \alpha_p(f)\alpha_p(Y) + \beta_p(f)\beta_p(Y) \\
&\quad + \alpha_p(g)\alpha_p(X) + \beta_p(g)\beta_p(X).
\end{aligned}$$

A similar result holds for arbitrary degree.

COROLLARY 4.2. *Let* f: (M,∂M) → (X,∂X) *be a degree* n *normal map and* (N,∂N) *a manifold. Then*

$$\alpha_2(f \times 1_N) = \alpha_2(f)\alpha_2(N)$$
$$\alpha_p(f \times 1_N) = \alpha_p(f)\alpha_p(N) + \beta_p(f)\beta_p(N) \qquad \text{if } p \neq 2$$
$$\beta_p(f \times 1_N) = \beta_p(f)\alpha_p(N) + \beta_p(N)\alpha_p(f).$$

COROLLARY 4.3. *If* N *is closed, then*

$$\alpha_p(f \times 1_N) = \alpha_p(f)\alpha_p(N)$$
$$\beta_p(f \times 1_N) = \beta_p(f)\alpha_p(N).$$

5. APPLICATIONS

In this section we give a number of applications of the previous sections.

a. *Structures on Poincare Complexes.*

Let (X,∂X) be a Poincare pair over $\mathbb{Z}_K$ of dimension $n \geq 5$, $\partial X \neq \emptyset$, $\pi_1(X) = 0$, ∂X a manifold. Suppose the Spivak normal fibration of X is $\mathbb{Z}_K$-fiber homotopy equivalent to a topological bundle, rel ∂X.

THEOREM 5.1. (X,∂X) *is* $\mathbb{Z}_K$*-homotopy equivalent to a compact* n*-manifold pair.*

If $n \not\equiv 0 \bmod (4)$, the condition $\partial X \neq \emptyset$ can be dropped. This also holds for $n \equiv 0 \bmod (4)$, provided we replace "manifold" with "Z_K-homology manifold."

Proof of Theorem 5.1. Let f: (M,∂M) → (X,∂X) be a normal map as in Section 2. Assume first that $n \equiv 0 \bmod (4)$. Let x be the surgery obstruction of f and let (N,∂N) be an n-manifold pair with $\beta_p(N) = -\beta_p(x)$, Sign (N) = −Sign (x). By Theorem 3.10, the result follows.

If $n \not\equiv 0 \bmod (4)$, the usual argument shows that stronger result above. The $\partial X = \emptyset$ case follows from Theorem 3.4 (2).

b. *Homology Spheres and Manifolds.*

Let ψ_n^K denote the group (under connected sum) of H-cobordism classes, over $\mathbb{Z}_K$, of PL n-manifolds with the $\mathbb{Z}_K$-homology of S^n. Let

$$W(\mathbb{Z}_K,\mathbb{Z}) = \operatorname{coker}(W(\mathbb{Z}) \to W(\mathbb{Z}_K)).$$

THEOREM 5.2. *For* $n > 1$, *there is an injection* $W(\mathbb{Z}_K,\mathbb{Z}) \to \psi_{4n-1}^K$.

Proof. Define $\phi: W(\mathbb{Z}_K) \to \psi_{4n-1}^K$ sending x to ∂M_x^{4n}, where M_x^{4n} is obtained by plumbing with x. φ is well-defined: Suppose M_1, M_2 are stably parallelizable, with intersection pairing x. We can then do surgery on $M_1 \# (-M_2)$ to show $[\partial M_1] = [\partial M_2]$ in ψ_{4n-1}^K. φ is a homomorphism by Theorem 3.10.

Clearly $W(\mathbb{Z}) \subset \ker(\phi)$. Suppose $\phi(x) = 0$; *i.e.* $[\partial M_x] = [S^{4n-1}]$. Let W be an H-cobordism over $\mathbb{Z}_K$ between ∂M_x and S^n. Then $M_x \cup W \cup D^n$ is a closed manifold with intersection pairing x, so that $x \in W(\mathbb{Z})$. Hence $\ker(\phi) = W(\mathbb{Z})$.

In [5], it is shown that a closed homology manifold of dimension $n \geq 5$ has the simple homotopy type of a closed n-manifold. For $K \neq \phi$, this is false:

THEOREM 5.3. *Let* $K \neq \phi$. *Then there exist closed* $\mathbb{Z}_K$*-homology manifolds* M^{4k} *that are not* $\mathbb{Z}_K$*-homotopy equivalent to a closed topological* 4k*-manifold.*

Proof. First assume that there is some prime $p \equiv 3 \bmod (4)$ in K. Let M^{4k} be a $\mathbb{Z}_K$-homology manifold with $\alpha_p(M) = \beta_p(M) = \langle 2 \rangle = \pm 1 \in \mathbb{Z}/4$. (See the proof of Theorem 3.4.) Suppose $f: N \to M$ is a $\mathbb{Z}_K$-homotopy equivalence of degree n, where N is a topological manifold. Then

$$0 = \beta_p(f) = \beta_p(N) - \langle e_p(n) \rangle \beta_p(M) = - \langle e_p(n) \rangle \beta_p(M)$$

by Theorem 3.3 and Corollary 3.7. This is a contradiction since ± 1 is not a zero divisor in $W(\mathbb{F}_p)$. The same argument holds if $p \equiv 1 \bmod (4)$ since $\langle 2 \rangle$ is the generator of one of the summands of $W(\mathbb{F}_p)$, and $W(\mathbb{F}_p) \cong \mathbb{F}_2[\mathbb{Z}/2]$ as a ring.

The final case to be considered is when $K = \{2\}$. Let M^{4k} be obtained by plumbing via the matrix $\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$. Then $\beta_2(M) = 1$, $\alpha_2(M) = 0$, and if $f: N \to M$ is a $\mathbb{Z}_K$-homotopy equivalence as above, then

$$0 = \beta_2(f) = \beta_2(N) + d_2(n)\alpha_2(M) + \beta_2(M) = 1.$$

c. *Singular Manifolds*

Let M^n, N^n be $\mathbb{Z}/m$-manifolds with $\pi_1(N) = 0$, $\pi_1(\delta N) = 0$, $n \geq 5$, and let $f: M \to N$ be a normal map of degree $r \in \mathbb{Z}_K^+$. (See [7].)

THEOREM 5.4. f *is normally cobordant to a* $\mathbb{Z}_K$*-homotopy equivalence if and only if an obstruction in*

$$\begin{cases} \operatorname{tor}(W(\mathbb{Z}_K)) \otimes \mathbb{Z}/m & n \equiv 1 \bmod (4) \\ \mathbb{Z}/2 \otimes \mathbb{Z}_K \otimes \mathbb{Z}/m & n \equiv 2, 3, \bmod (4) \\ W(\mathbb{Z}_K) \otimes \mathbb{Z}/m & n \equiv 0 \bmod (4) \end{cases}$$

vanishes.

Proof. Suppose M, N are obtained from M_o, N_o by identifying m isomorphic boundary components. We regard $f: (M_o, \partial M_o) \to (N_o, \partial N_o)$. Let $\partial M_o'$, $\partial N_o'$ be corresponding boundary components.

Case 1. $n \equiv \bmod (4)$:

The obstructions to completing surgery on $f | \partial M_o': \partial M_o' \to \partial N_o'$ are

$$\operatorname{Sign}(\partial M_o') - \operatorname{Sign}(\partial N_o')$$

and
$$\begin{cases} \beta_p(\partial M_o') - \langle e_p(r) \rangle \beta_p(\partial N_o') & d_p(r) \equiv 0 \bmod (2) \\ \beta_p(\partial M_o') - \langle e_p(r) \rangle \alpha_p(\partial N_o') & d_p(r) \equiv 1 \bmod (2) \end{cases}$$

Since $m\,\partial M'_o$, $m\,\partial N'_o$ are boundaries, $\mathrm{Sign}(\partial M'_o) = \mathrm{Sign}(\partial N'_o) = 0$, and

$$m\,\beta_p(\partial M'_o) = m\,\beta_p(\partial N'_o) = m\,\alpha_p(\partial N'_o) = 0.$$

Thus there is an obstruction in $(\mathrm{tor}(W(\mathbb{Z}_K)) \otimes \mathbb{Z}/m$. If this vanishes, there is no further obstruction for f.

Case 2. $n \equiv 2 \bmod (4)$:

Do surgery on $f|\partial M'_o$. The surgery obstruction of f relative to the boundary now lies in $\mathbb{Z}/2 \otimes \mathbb{Z}_K$. The argument of [7] shows that it vanishes if m is odd.

Case 3. $n \equiv 3 \bmod (4)$: Same as Case 2.

Case 4. $n \equiv 0 \bmod (4)$.

We may do surgery on $f|\partial M'_o$ to get a $\mathbb{Z}_K$-homotopy equivalence. By Theorems 3.4 and 3.5, we may change the surgery obstruction of f by any element of $mW(\mathbb{Z}_K)$. Thus the obstruction is given as stated,

d. *Involutions.*

Let T be an involution on a $\mathbb{Z}_K$-homotopy sphere Σ^n. We say that T *desuspends* mod (K) if there is an invariant embedded $\mathbb{Z}_K$-homotopy sphere $\Sigma_o^{n-1} \subset \Sigma^n$.

THEOREM 5.5. *Let* $n \geq 6$. *Then* T *desuspends* mod (K) *if and only if an obstruction in*

$$\begin{cases} 0 & n \equiv 0 \bmod (2) \\ \mathbb{Z}/2 \otimes \mathbb{Z}_K & n \equiv w_T \bmod (4) \\ W(\mathbb{Z}_K) & n \equiv -w_T \bmod (4) \end{cases}$$

vanish, where $w_T \in \{\pm 1\}$ *is 1 if and only if* T *preserves orientation.*

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