QUADRANTS OF RIEMANNIAN METRICS

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We shall study the set of Riemannian metrics $\mathscr{R}$ by considering 2 dimensional quadrants in $C^\infty(M, S^2M)$ where M is a paracompact manifold and S^2M is the 2 symmetric tensor bundle over M. In particular, we are interested in quadrants defined by two Riemannian metrics; that is, if $g_0, g_1 \in \mathscr{R}$ we shall consider the quadrant $\{sg_0 + tg_1: s > 0 \text{ and } t \geq 0\}$. These quadrants are of interest in studying complete metrics because if g_0 is complete then so are all metrics in the quadrant (Lemma 2). Using these ideas we prove that, whereas the set of complete metrics has empty interior (Proposition 5), the set of incomplete metrics must lie on the edge of the set of all Riemannian metrics. See Theorem 6 for a precise statement. Our interest in the study of the set of complete metrics started with the result of Nomizu and Ozeki [5] that if g is an arbitrary Riemannian metric then there is a complete Riemannian metric which is conformal to g. In particular, the set of complete Riemannian metrics, $\mathscr{C}$, is non-empty. (Of course, if M is compact then $\mathscr{C} = \mathscr{R}$.) We investigate the relationship of $\mathscr{C}$ to $\mathscr{R}$ deriving as a corollary J. Morrow's result [4] that $\mathscr{C}$ is dense in $\mathscr{R}$. We also prove that the incomplete Riemannian metrics are dense in $\mathscr{R}$ if M is non-compact (Proposition 5).

Let S^2M be the vector bundle of 2 symmetric tensors on M then $C^\infty(M, S^2M)$, the set of C^∞ sections of S^2M, is a topological space with the C^∞ topology (topology of uniform convergence of the sections and their derivatives on compact sets, see [1]). The set of all Riemannian metrics $\mathscr{R} = \mathscr{R}(M)$ is given a topology as a subspace of $C^\infty(M, S^2M)$. We let $\mathscr{C} = \mathscr{C}(M) \subset \mathscr{R}(M)$ be the subspace of all complete Riemannian metrics on M.

PROPOSITION 1. *If* $g_0, g_1 \in \mathscr{R}(M)$, $g_0 \in \mathscr{C}(M)$ *and* $f: M \to \mathbb{R}$ *is a positive* C^∞ *function which is bounded away from zero then* $g = fg_0 + g_1 \in \mathscr{C}(M)$.

Proof. Let d_0 (resp. d) be the metric on M induced by g_0 (resp. g). Let $L > 0$ be such that $f(x) \geq L$ for all $x \in M$ and $\{p_n\}$ a Cauchy sequence in the d metric. We will show that $\{p_n\}$ is Cauchy in the d_0 metric. Since d_0 is complete, there will thus be a point $p \in M$ such that $p_n \to p$ in the d_0 metric, hence in the manifold topology [2, page 70].

For $\varepsilon > 0$ there is an N such that for all $m, n \geq N$, $d(p_m, p_n) < \sqrt{L}\,\varepsilon$. Thus for $m, n \geq N$

$$\sqrt{L}\,\varepsilon > \inf_\alpha \int \sqrt{fg_0(\dot\alpha, \dot\alpha) + g_1(\dot\alpha, \dot\alpha)} \geq \sqrt{L} \inf_\alpha \int \sqrt{g_0(\dot\alpha, \dot\alpha)} = \sqrt{L}\, d_0(p_m, p_n).$$

Thus $d_0(p_m, p_n) < \varepsilon$ if $m, n \geq N$ and so $\{p_n\}$ is Cauchy in the d_0 metric.

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This proposition is false if we omit the hypothesis that f is bounded away from zero. For a counterexample take $M = \mathbb{R}^1$, g_0 the usual Euclidean metric, $f(x) = e^{-x}$ and g_1 any incomplete metric. That $fg_0 + g_1$ is not complete will be obvious from Proposition 8. The form of the above proposition which is most useful is:

LEMMA 2. *If* $g_0, g_1 \in \mathscr{R}(M)$ *and* $g_0 \in \mathscr{C}(M)$ *then*

$$g_{s,t} = sg_0 + tg_1 \in \mathscr{C}(M) \qquad \text{for all } s > 0 \text{ and } t \geq 0.$$

COROLLARY 3. *The set of complete Riemannian metrics on* M *is a dense, convex subspace of the space of Riemannian metrics.*

Proof. If g_0 and g_1 are complete then the straight line of Riemannian metric $g_{s,1-s}$ is a path of complete Riemannian metrics (by the lemma) joining g_0 to g_1 so that $\mathscr{C}(M)$ is convex.

If $g_1 \in \mathscr{R}(M)$, let g_0 be an arbitrary element of $\mathscr{C}(M)$. By the lemma,

$$g_n = \frac{1}{n} g_0 + g_1 \in \mathscr{C}(M).$$

Since $g_n \to g_1$ as $n \to \infty$, $\mathscr{C}(M)$ is dense in $\mathscr{R}(M)$.

J. Morrow [4] has proved that $\mathscr{C}$ is dense in $\mathscr{R}$. He proved this fact by showing that given any Riemannian metric g_1 and compact set K there is $g_0 \in \mathscr{C}$ such that $g_0 = g_1$ on K. In some sense, our approximation of g_1 is more uniform. In particular, note that if we took g_0 to be conformally related to g_1 (as we may by the result of Nomizu-Ozeki) then each metric of the sequence g_n would be conformally related to g_0. In spite of the corollary, $\mathscr{C}$ is still not very thick as the next proposition shows.

LEMMA 4. *Let* M *be a non-compact manifold and* K *be a compact subset of* M. *If* $V = M - K$ *then there is a Riemannian metric* g *on* M *and a non-convergent sequence* $\{p_n\}$ *which is Cauchy in the metric induced by* g *such that* $p_n \in V$ *for all but finitely many* n.

Proof. Since there is an incomplete Riemannian metric on M, the result follows immediately from the fact that a Cauchy sequence has a convergent subsequence if and only if it converges.

If M is compact then $\mathscr{C}(M) = \mathscr{R}(M)$. In the non-compact case we have

PROPOSITION 5. *If* M *is non-compact then the interior of* $\mathscr{C}$ *(in* $\mathscr{R}$*) is empty. (In particular, the set of incomplete Riemannian metrics is dense in* $\mathscr{R}$*.)*

Proof. Let $g_0 \in \mathscr{C}$ and W(K, U) be a basic open set of $\mathscr{R}$ (so that $K \subset M$ is compact and $U \subset S^2M$ is open). We will show that there is $g \in W(K, U)$ which is not complete, hence $\mathrm{Int}_{\mathscr{R}}\mathscr{C} = \emptyset$.

Let V be an open set of M which is properly contained in $M - K$. Let h, f be non-negative C^∞ functions with $h(K) = 1$, $h(V) = 0$, $f(K) = 0$ and $f(V) = 1$. Let g_1 be the Riemannian metric given by Lemma 4 and set $g = fg_1 + hg_0$. Since $g_x = g_0(x)$ if $x \in K$, $g \in W(K, U)$. Since $g_x = g_1(x)$ if $x \in V$ and there is a non-convergent Cauchy sequence in V, $g \notin \mathscr{C}$.

We now introduce the notion of the endpoints and inner points of $\mathscr{R}$ and show that (although Int $\mathscr{C} = \emptyset$) the complete metrics are precisely the inner points. This will say that the incomplete metrics lie on the "edge" of $\mathscr{R}$. A *line* is any set L for which there are $g_1, g_2 \in \mathscr{R}$ such that

$$L = L(g_1, g_2) = \{tg_1 + (1-t)g_2 \colon t \in \mathbb{R}\} \subset C^\infty(M, S^2M).$$

The set of endpoints of the line $L = L(g_1, g_2)$ is

$$\mathscr{E}L(g_1, g_2) = \overline{L(g_1, g_2) \cap (C^\infty(M, S^2M) - \mathscr{R}(M))} \cap \mathscr{R}(M).$$

An *endpoint* of $\mathscr{R}$ is a Riemannian metric g which has the following property: if g appears on a line on which a complete Riemannian metric does, then g appears as an endpoint of the line. More precisely,

$$\mathscr{E} = \{g \in \mathscr{R} \colon g \in L \text{ and } L \cap \mathscr{C} \neq \emptyset \text{ implies } g \in \mathscr{E}L\}$$

where L denotes a line. The set of *inner points* of $\mathscr{R}$ is $\mathscr{I} = \mathscr{R} - \mathscr{E}$.

THEOREM 6. $\mathscr{I} = \mathscr{C}$.

Proof. Let $g \in \mathscr{C}$ and set $g_2 = 2g$ and $g_1 = g/2$. If

$$L = L(g_1, g_2) = \{(1-t)g_1 + tg_2\},$$

then $g \in L$ (take $t = 1/3$) and $g \in L \cap \mathscr{C}$ so $L \cap \mathscr{C} \neq \emptyset$. Since $g \notin \mathscr{E}L(g_1, g_2)$ this shows that $g \notin \mathscr{E}$ hence is an element of $\mathscr{R} - \mathscr{E}$. Therefore $\mathscr{C} \subset \mathscr{I} = \mathscr{R} - \mathscr{E}$.

Conversely, let $g \in \mathscr{I}$, then there is a line L such that there is a complete metric on L but $g \notin \mathscr{E}L$. By Corollary 3, g is complete.

Example. Let $M = \mathbb{R}^1$. In this case $\mathscr{R}$ is in one-to-one correspondence with $C_+^\infty(\mathbb{R}^1)$, (positive infinitely differentiable functions) where $g \in \mathscr{R}$ corresponds to $g_t(\xi, \eta) = f(t)^2 \xi\eta$ for $f \in C_+{}^\infty(\mathbb{R}^1)$. The induced metric is then

$$d(x, y) = \int_x^y f(t)\,dt.$$

LEMMA 7. *Every Riemannian metric on* $\mathbb{R}^1$ *is an endpoint of some line.*

Proof. If $g \in \mathscr{R}$, let $g_1(x) = (1 + |x|)g(x)$. Define L by

$$L = \{\lambda g_1(x) + (1-\lambda)g(x) \colon \lambda \in \mathbb{R}\}.$$

If $g_{-n} = -\dfrac{1}{n} g_1 + \left(1 + \dfrac{1}{n}\right) g$, then for $n > 0$,

$$g_{-n}(n) = -\frac{1}{n}(1+n)g(n) + \left(1 + \frac{1}{n}\right) g(n) = 0.$$

Hence $g_{-n} \notin \mathscr{R}$. However, $g_{-n} \to g$ as $n \to \infty$ so $g \in \mathscr{E}L$; that is, g is an endpoint of L.

On $\mathbb{R}^1$ there is the following characterization of complete Riemannian metrics.

PROPOSITION 8. *Let* g(x) *correspond to the positive* C^∞ *function,* f. *Then* g *is complete if and only if both of the integrals* $\int_0^\infty f(x)\,dx$ *and* $\int_{-\infty}^0 f(x)\,dx$ *diverge.*

Proof. If $\int_0^\infty f(x)\,dx < \infty$ then $\{a_n = n\}$ is a Cauchy sequence and so g is not complete.

Now assume that g is not complete. This means that there is a non-convergent sequence a_n which is Cauchy (in the metric from g). Since a non-convergent Cauchy sequence can have no convergent subsequences we may assume that $a_n \to \infty$ and for convenience assume that $a_0 > 0$.

$$\int_0^\infty f(x)\,dx = \int_0^{a_0} f(x)\,dx + \sum_{i=0}^\infty \int_{a_i}^{a_{i+1}} f(x)\,dx = \int_0^{a_0} f(x)\,dx + \sum_{i=0}^\infty d(a_i, a_{i+1}),$$

which is finite if and only if $\sum_{i=0}^\infty d(a_i, a_{i+1})$ converges. If $s_n = \sum_{i=0}^n d(a_i, a_{i+1})$ then, for $n \geq m$,

$$s_n - s_m = \sum_{i=m}^n d(a_i, a_{i+1}) = \int_{a_m}^{a_n} f(x)\,dx = d(a_m, a_n)$$

and so s_n is Cauchy (in the usual metric on $\mathbb{R}^1$). Thus s_n converges and so

$$\int_0^\infty f(x)\,dx < \infty.$$

From Theorem 6 we may prove the following well-known result (without having to use the Hopf-Rinow Theorem as is usual [3]).

COROLLARY 9. *If* M *is compact then every Riemannian metric on* M *is complete.*

Proof. Let $\{U_i\}_{i=1}^p$ be a finite cover of M by precompact sets such that there exists a global basis $\{X_{ij}\}_{j=1}^n$ for the vector fields on U_i. Let

$$\varepsilon_{ij}: C^\infty(M, S^2M) \to C^\infty(U_i)$$

be given by $\varepsilon_{ij}(g)(x) = g_x(X_{ij}, X_{ij})$ if $g \in C^\infty(M, S^2M)$ and $x \in U_i$. Since

$$\mathscr{R}(M) = \bigcap_{i=1}^p \bigcap_{j=1}^n \varepsilon_{ij}^{-1}\{f \in C^\infty(\bar{U}_i): f(x) > 0 \text{ for all } x \in \bar{U}_i\}$$

$\mathscr{R}(M)$ is an open subset of $C^\infty(M, S^2M)$.

We therefore have the following chain of inclusions:

$$\mathscr{R} = \operatorname{Int}\mathscr{R} \subset \mathscr{I}\mathscr{R} = \mathscr{C} \subset \mathscr{R},$$

where the first is because $\mathscr{R}$ is open, the second and last ones are from the definition and the third inclusion is Theorem 6. Thus $\mathscr{R} = \mathscr{C}$.

Remark. The construction of the straight line $g_{1-s,s}$ in the case of the upper half-plane $M = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ has an amusing consequence. Let

$$g_0 = \delta_{ij} \text{ and } g_1 = \delta_{ij}/y^2$$

(the Hyperbolic metric). If $\alpha(t) = (0, 1-t)$ for $0 \le t \le 1-\varepsilon$ then the length of α in the $d_{1-s,s}$ metric is

$$L(\alpha)(s) = \int_0^{1-\varepsilon} \sqrt{(1-s) + \frac{s}{(1-t)^2}}\, dt$$
$$\ge \int_0^{1-\varepsilon} \frac{\sqrt{s}}{1-t}\, dt = \sqrt{s}\,\ell n\,(1/\varepsilon).$$

This means that $L(\alpha)(s)$ stays finite as $\varepsilon \to 0$ only when $s = 0$. This shows that if we fix the curve α and define a function from $\mathscr{R}(M) \to \mathbb{R} \cup \{\infty\}$ by "computing the length of α up to the boundary in the given Riemannian metric" this function is not continuous!

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