ON FRAMED BORDISM

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1. INTRODUCTION

Let S^n denote the unit n-sphere with its standard differentiable structure in (n + 1)-dimensional Euclidean space. In [6], Novikov showed that if $(m, n - m) \neq (1, 1), (3, 3), 7, 7)$, then any framing of the stable tangent bundle of the product of spheres $S^m \times S^{n-m}$ determines, by the Pontrjagin-Thom construction, an element in the image of the stable J-homomorphism, $J: \pi_n(SO_k) \to \pi_{n+k}(S^k)$ for $k > n + 1$. Our purpose here is to give a simple geometric proof of a generalization of Novikov's result; this generalization is stated in Theorem 1. We shall also obtain a result for the exceptional dimensions $(m, n - m) = (1, 1), (3, 3)$, and $(7, 7)$. For example, we shall see that any framing of any connected sum of 14-dimensional products of standard spheres determines either 0 or Toda's element σ^2 in the stable group $\pi_{14+k}(S^k)$. It is known that the parallelization of S^7 gives rise to a framing of $S^7 \times S^7$ that determines σ^2. For information about representing nontrivial elements in the homotopy groups of spheres by framings of Lie groups, we refer the reader to Atiyah, Smith [1], [7], Gershenson [4], Steer [8], and Wood [10].

In this paper all differentiable manifolds, with or without boundary, are compact, oriented, and of class C^∞. We denote the connected sum of r products of spheres of positive dimension by T_r^n; thus $T_r^n = (S^{m_1} \times S^{n-m_1}) \# \cdots \# (S^{m_r} \times S^{n-m_r})$, where # denotes the operation of connected sum. We shall prove the following theorem.

THEOREM 1. *Let* N^n *be a connected, differentiable* n-*manifold without boundary, and let* f *be a framing of the stable tangent bundle of the connected sum* $T_r^n \# N^n$. *If* $n = 2, 6, 14$ *and the integral homology group* $H_{n/2}(T_r^n)$ *is not zero, then make the following assumptions: for* $n = 6, 14$ *assume* $H_{n/2}(N^n) = 0$ *and* $(T_r^n \# N^n, f)$ *has Kervaire invariant zero; for* $n = 2$, *if* $H_1(N^2) = 0$, *assume* $(T_r^2 \# N^2, f)$ *has Kervaire invariant zero.*

Then there exists a framing g *of the stable tangent bundle of* N^n *such that* $(T_r^n \# N^n, f)$ *and* (N^n, g) *are framed-cobordant. Furthermore, if* $n \neq 2$ *then* g *may be chosen such that the restrictions* $f \mid N^n$-disk *and* $g \mid N^n$-disk *are equal; hence* $f \mid T_r^n$-disk *extends to a framing* f_1 *of* T_r^n.

To see the need for the assumptions made in dimensions 2, 6, and 14 in this theorem, notice that if N^n is a homotopy sphere and g is any framing of its stable tangent bundle, then (N^n, g) has Kervaire invariant zero. Inasmuch as framed-cobordant manifolds have the same Kervaire invariant, it follows that (N^n, g) cannot be framed-cobordant to a framed manifold $(T_r^n \# N^n, f)$ with Kervaire invariant 1. In each of these dimensions, $n = 2m = 2, 6$, and 14, there is a framing f of

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$T_1^{2m} = S^m \times S^m$ such that $(S^m \times S^m, f)$ has Kervaire invariant 1; this framing is obtained from parallelization of S^m for $m = 1, 3,$ and 7.

Theorem 1 has two corollaries; the first corollary is obtained by choosing $N^n = S^n$ in Theorem 1.

COROLLARY 1. *Let* f *be a framing of the stable tangent bundle of*

$$T_r^n = (S^{m_1} \times S^{n-m_1}) \# \cdots \# (S^{m_r} \times S^{n-m_r}).$$

In dimensions $n = 2, 6,$ *and* 14, *if some* $m_i = n/2$, *assume that* (T_r^n, f) *has Kervaire invariant zero. Then the framed manifold* (T_r^n, f) *represents an element in the image of the stable* J*-homomorphism.* $J: \pi_n(SO_k) \to \pi_{n+k}(S^k)$ *for* $k > n+1$.

COROLLARY 2. *If* Σ^{2m} *is a homotopy sphere of dimension* $2m > 4$ *such that for some integer* r *the connected sum* $T_r^{2m} \# \Sigma^{2m}$ *is the boundary of a parallelizable manifold, then* Σ^{2m} *is diffeomorphic to the standard sphere* S^{2m}.

To obtain this corollary let W^{2m+1} be a parallelizable manifold bounded by $T_r^{2m} \# \Sigma^{2m}$, and let F be a framing of the tangent bundle of W^{2m+1}. Thus $f = F \mid \partial W^{2m+1}$ is a framing of the stable tangent bundle of $T_r^{2m} \# \Sigma^{2m}$ such that $(T_r^{2m} \# \Sigma^{2m}, f)$ has Kervaire invariant zero; hence, according to Theorem 1, $(T_r^{2m} \# \Sigma^{2m}, f)$ is framed-cobordant to (Σ^{2m}, g). We can attach this framed cobordism to (W^{2m+1}, F) along $(T_r^{2m} \# \Sigma^{2m}, f)$ by the identity map to obtain a framed manifold bounded by (Σ^{2m}, g). Thus, according to Kervaire and Milnor [5, Theorem 5.1], Σ^{2m} is h-cobordant to the standard 2m-sphere; inasmuch as $2m > 4$, it follows from Smale's h-cobordism theorem that Σ^{2m} is diffeomorphic to S^{2m}.

In an early paper [3], we used a special case of Corollary 2 to prove the following theorem.

THEOREM 2. *An* $(m-1)$*-connected, differentiable manifold* M^{2m} *of dimension* $2m > 4$ *that bounds a parallelizable manifold is diffeomorphic to the connected sum of* r *copies of* $S^m \times S^m$, *where* 2r *is the* m*-th Betti number of* M^{2m}.

This theorem is obtained from Corollary 2 by showing that M^{2m} is diffeomorphic to a connected sum

$$T_r^{2m} \# \Sigma^{2m} = (S^m \times S^m) \# \cdots \# (S^m \times S^m) \# \Sigma^{2m}$$

for some homotopy sphere Σ^{2m}, and this we did in [3]. In that paper, we began a proof of Corollary 2 in the special case where T_r^{2m} is a connected sum of r copies of $S^m \times S^m$ as follows. Let W^{2m+1} denote a parallelizable manifold whose boundary is $\partial W^{2m+1} = T_r^{2m} \# \Sigma^{2m}$, and let F denote a framing of the tangent bundle of W^{2m+1}. The restriction $f = F \mid \partial W^{2m+1}$ is a framing of the stable tangent bundle of $T_r^{2m} \# \Sigma^{2m}$. At this point we claimed it is not hard to show that, by a sequence of framed spherical modifications, the framed manifold $(T_r^{2m} \# \Sigma^{2m}, f)$ can be reduced to (Σ^{2m}, g), where g is some framing of the stable tangent bundle of Σ^{2m} (compare this statement with Theorem 1). We did not give a proof of this claim, but instead we referred the reader to [5, Lemma 6.2]. It turns out that the proof I had in mind is not at all obvious from this reference; in fact, the proof breaks down in dimension $2m = 14$ because the homomorphism $\pi_m(SO_m) \to \pi_m(SO_{2m+1})$ is not surjective for $m = 7$, whereas it is surjective for $m \neq 1, 3, 7$.

The same problem appears in the proof of Theorem 1 in dimensions 2, 6, and 14, and a special argument is given in Section 3 for each of these dimensions. These arguments use the representation of the stable homotopy group of spheres $\pi_{2m+k}(S^k)$ as the framed bordism group of framed 2m-dimensional manifolds. The proof of Theorem 1 in dimensions $n \neq 2, 6, 14$ is given in Section 2, which proof does not make use of the structure of the homotopy groups of spheres.

2. DIMENSION $n \neq 2, 6, 14$

We shall prove Theorem 1 by induction on the number r of summands of T^n_r. For $r = 1$ we have $T^n_1 = S^m \times S^{n-m}$ where $2m \leq n$. The sphere S^{n-m} may be written as the union of two diffeomorphic copies of the unit $(n - m)$-disk D^{n-m}, the upper hemisphere D^{n-m}_+ and the lower hemisphere D^{n-m}_-, identified along the boundary S^{n-m-1} by the identity map. Thus $S^m \times S^{n-m}$ may be written as the disjoint union of $S^m \times D^{n-m}_+$ and $S^m \times D^{n-m}_-$ with points along the boundary $S^m \times S^{n-m-1}$ identified by the identity map id; that is,

$$S^m \times S^{n-m} = (S^m \times D^{n-m}_+) \cup_{id} (S^m \times D^{n-m}_-).$$

Let $p: D^{n-m}_+ \to D^{n-m}$ be the diffeomorphism defined by projection on the first $n - m$ coordinates. Let $\phi: S^m \times D^{n-m} \to S^m \times S^{n-m}$ denote the differentiable embedding defined as follows: $\phi(u, v) = (u, p^{-1}(v)) \in S^m \times D^{n-m}_+$ for each $(u, v) \in S^m \times D^{n-m}$.

Now suppose that N^n is a connected differentiable n-manifold without boundary and let f be a framing of the stable tangent bundle of $(S^m \times S^{n-m}) \# N^n$. This connected sum is made in the interior of $S^m \times D^{n-m}_-$ in $S^m \times S^{n-m}$. Thus from the differentiable embedding ϕ we obtain a differentiable embedding of $S^m \times D^{n-m}$ in $(S^m \times S^{n-m}) \# N^n$, which embedding we denote also by ϕ, that is,

$$\phi: S^m \times D^{n-m} \to (S^m \times S^{n-m}) \# N^n.$$

If $\alpha: S^m \to SO_{n-m}$ is a differentiable map, we define a new embedding

$$\phi_\alpha: S^m \times D^{n-m} \to (S^m \times S^{n-m}) \# N^n$$

by the equation $\phi_\alpha(u, v) = \phi(u, \alpha(u) \cdot v)$, where $\alpha(u) \cdot v$ denotes the standard action of $\alpha(u) \in SO_{n-m}$ on $v \in D^{n-m}$. We shall perform the spherical modification of $M^n = (S^m \times S^{n-m}) \# N^n$ that removes the embedded m-sphere $\phi_\alpha(S^m \times \{0\})$ with product structure $\phi_\alpha(S^m \times D^{n-m})$, where 0 denotes the center of the disk D^{n-m}. The result of this modification is the differentiable n-manifold M^n_α obtained from the disjoint union $(D^{m+1} \times S^{n-m-1}) \cup (M^n - \phi_\alpha(S^m \times \{0\}))$ by identifying (tu, v) with $\phi_\alpha(u, tv)$ for each $(u, v) \in S^m \times S^{n-m-1}$ and $0 < t \leq 1$. The manifolds $M^n = (S^m \times S^{n-m}) \# N^n$ and M^n_α together bound a manifold

$$W^{n+1}_\alpha = (D^{m+1} \times D^{n-m}) \cup_{\phi_\alpha} (M^n \times [0, 1]),$$

where the handle $D^{m+1} \times D^{n-m}$ is attached to $M^n \times \{1\}$ along $S^m \times D^{n-m}$ by the embedding ϕ_α of $S^m \times D^{n-m}$ into $M^n = M^n \times \{1\}$. The differentiable structure of W^{n+1}_α is obtained by smoothing along the corner $S^m \times S^{n-m-1}$ (see [5, page 519] for this smoothing procedure). The only obstruction to extending the given framing f of $M^n = M^n \times \{0\}$ to a framing of the tangent bundle of W_α^{n+1} is an element $\gamma(\phi_\alpha) \in \pi_m(SO_{n+1})$ (see [5, Lemma 6.1]). It was shown in [5] that if the homomorphism $s_*: \pi_m(SO_{n-m}) \to \pi_m(SO_{n+1})$ is surjective, then a differentiable map $\alpha: S^m \to SO_{n-m}$ may be chosen such that the obstruction $\gamma(\phi_\alpha) = 0$. But $m \leq n - m$ and $n \neq 2, 6, 14$, and in these dimensions s_* is surjective. (Notice that s_* is surjective for $n = 2, 6, 14$ if $m < n - m$.) Thus it follows that there is a differentiable map $\alpha: S^m \to SO_{n-m}$ such that the framing f of $M^n = M^n \times \{0\}$ can be extended to a framing F of the tangent bundle of W_α^{n+1}; in particular, (W_α^{n+1}, F) is a framed cobordism between (M^n, f) and $(M_\alpha^n, F \mid M_\alpha^n)$.

We shall show that M_α^n is diffeomorphic to N^n. First of all, notice that the manifold M_α^n is the connected sum of N^n and the homotopy n-sphere S_α^n obtained from the disjoint union $(D^{m+1} \times S^{n-m-1}) \cup (S^m \times S^{n-m} - \phi_\alpha(S^m \times \{0\}))$ by identifying (tu, v) with $\phi_\alpha(u, tv) = \phi(u, t\alpha(u) \cdot v) \in S^m \times D_+^{n-m}$ for each

$$(u, v) \in S^{m+1} \times S^{n-m-1}$$

and $0 < t \leq 1$; this is so because the connected sum $(S^m \times S^{n-m}) \# N^n$ is made on a disk in the complement of $\phi_\alpha(S^m \times D^{n-m}) = S^m \times D_+^{n-m}$. Furthermore, the restrictions of f and $g = F \mid M_\alpha$ to N^n-disk are equal. To complete the proof for $r = 1$ we need only show that S_α^n is diffeomorphic to the standard n-sphere S^n. The sphere S^n is diffeomorphic to the manifold obtained from the disjoint union

$$(D^{m+1} \times S^{n-m-1}) \cup (S^m \times D^{n-m})$$

with points along the boundary $S^m \times S^{m-n-1}$ identified by the identity map. Thus we may define a diffeomorphism ψ from this representation of S^n onto S_α^n as follows:

$$\psi(u, v) = \begin{cases} (u, v) \in D^{m+1} \times S^{n-m-1} & \text{if } (u, v) \in D^{m+1} \times S^{n-m-1} \\ (u, q^{-1}[\alpha(u) \cdot v]) \in S^m \times D_-^{n-m} & \text{if } (u, v) \in S^m \times D^{n-m} \end{cases} ;$$

here $q: D_-^{n-m} \to D^{n-m}$ is the diffeomorphism defined by projection on the first $n - m$ coordinates. It follows that $N^n = S^n \# N^n$ is diffeomorphic to $M_\alpha^n = S_\alpha^n \# N^n$.

Let us make the induction hypothesis that the theorem is true for manifolds $T_r^n \# N^n$ such that $r < s$ and $n \neq 2, 6, 14$. If T_s^n is a connected sum of s products of spheres, then write $T_s^n = T_1^n \# T_{s-1}^n$, where $T_1^n = S^{m_1} \times S^{n-m_1}$ and T_{s-1}^n is a connected sum of $s - 1$ products of spheres. Let f denote any framing of the connected sum $T_s^n \# N^n = T_1^n \# N_1^n$. We have shown that the framed manifold $(T_1^n \# N_1^n, f)$ is framed-cobordant to (N_1^n, g_1), where g_1 is some framing of $N_1^n = T_{s-1}^n \# N^n$ such that $f \mid N_1^n$ - disk $= g_1 \mid N_1^n$ - disk. By the induction hypothesis, $(T_{s-1}^n \# N^n, g_1)$ is framed-cobordant to (N^n, g) for some framing g such that $g_1 \mid N^n$ - disk $= g \mid N^n$ - disk. If we combine the two framed cobordisms we obtain a framed cobordism from $(T_s^n \# N^n, f)$ to (N^n, g), such that

$$f \mid N^n \text{ - disk} = g \mid N^n \text{ - disk}.$$

Inasmuch as f is an arbitrary framing, the proof of Theorem 1 is complete in dimensions $n \neq 2, 6, 14$. Notice that we have also proved Theorem 1 in dimensions $n = 2, 6, 14$ provided that T_r^n has no homology in its middle dimension.

3. THE EXCEPTIONAL DIMENSIONS: 2, 6, AND 14

It remains to prove Theorem 1 in the case where $H_m(T_r^{2m}) \neq 0$ in dimensions $2m = 2, 6, 14$. Let Π_n denote the stable homotopy group $\pi_{n+k}(S^k)$, where $k > n + 1$. We shall interpret Π_n as the framed bordism group of framed n-manifolds, and use the fact that the Kervaire invariant is a framed bordism invariant. For $m = 1, 3, 7$, let h denote the framing of S^m obtained from parallelization by left translation; the framed manifold $(S^m, h) \times (S^m, h)$ has Kervaire invariant 1 and represents a generator of Π_{2m}. The zero element of Π_{2m} is represented by (S^{2m}, j), which has Kervaire invariant zero; up to homotopy, j is the unique framing of S^{2m} because $\pi_{2m}(SO_k) = 0$ for $2m = 2, 6, 14$, and $k > 2m + 1$.

Consider $2m = 14$. The group Π_{14} is isomorphic to the direct sum of two copies of the group of order 2. Thus, in Toda's notation [9], $\Pi_{14} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ where the first summand is generated by a composition $\sigma \circ \sigma = \sigma^2$ represented by $(S^7, h) \times (S^7, h)$, and the second summand is generated by κ. B. Steer [8] and R. Wood [10] have shown that there is a framing w of the exceptional 14-dimensional Lie group G_2 such that (G_2, w) represents κ. According to Borel [2, Théorème 17.2], $H_7(G_2) = 0$. Thus, by framed spherical modifications, the framed manifold (G_2, w) may be reduced to a framed homotopy sphere; it is only necessary to check that the result of each framed modification has seventh homology group equal to zero. It follows that (G_2, w) is framed-cobordant to (Σ^{14}, j') where Σ^{14} is (up to diffeomorphism) the unique exotic 14-sphere; in particular, (G_2, w) has Kervaire invariant zero. Now, given any framing f of $T_r^{14} \# N^{14}$, there are framings f_1 and g of T_r^{14} and N^{14}, respectively, such that $(T_r^{14} \# N^{14}, f) = (T_r^{14}, f_1) \# (N^{14}, g)$. In fact, the framing $f \mid T_r^{14} - \{\text{point}\}$ may be extended to a framing f_1 of T_r^{14} because $\pi_{13}(SO) = 0$, and this extension is essentially unique because $\pi_{14}(SO) = 0$. Similarly, $f \mid N^{14} - \{\text{point}\}$ has a unique extension to a framing g of N^n. (These conclusions are also valid for manifolds of dimension $n \equiv 5, 6 \pmod 8$, as then

$$\pi_{n-1}(SO) = \pi_n(SO) = 0.)$$

LEMMA. *For any framing* f_1 *of* T_r^{14}, *the framed manifold* (T_r^{14}, f_1) *represents either* 0 *or* σ^2 *in* Π_{14}.

Proof. The manifold T_r^{14} is a connected sum of r products of the form $S^m \times S^{14-m}$, where $m \leq 7$. By a sequence of framed spherical modifications, (T_r^{14}, f_1) may be reduced to $(s(S^7 \times S^7), f_2)$, where $s(S^7 \times S^7)$ denotes a connected sum of s copies of $S^7 \times S^7$, and $s \leq r$. Inasmuch as $\pi_{13}(SO) = 0$, it follows that $(s(S^7 \times S^7), f_2)$ is a connected sum of s framed manifolds of the form $(S^7 \times S^7, f_3)$. We shall complete the proof of the lemma by showing that *for any framing* f' of $S^7 \times S^7$, *the framed manifold* $(S^7 \times S^7, f')$ *represents a multiple of* σ^2. Let us recall the framing h of the normal bundle of S^7 in Euclidean space $\mathbb{R}^{16}$, which framing is obtained from parallelization of S^7 and which differs from the natural framing (that extends over the disk D^8) by a map $\alpha: S^7 \to SO_9$ that represents a generator of the infinite cyclic group $\pi_7(SO_9)$. That is, for each $u \in S^7$,

$$h(u) = \{\text{natural frame}\} \cdot \alpha(u)$$

where the dot denotes the principal action of $\alpha(u) \in SO_9$ on the frame bundle. Thus the framing given by $(S^7, h) \times (S^7, h)$ differs from the natural framing of $S^7 \times S^7 \subset \mathbb{R}^{16} \times \mathbb{R}^{16}$ by the map $\alpha \times \alpha: S^7 \times S^7 \to SO_9 \times SO_9$. Now consider an arbitrary framing f' of $S^7 \times S^7$; this framing differs from the natural framing by some map $\phi: S^7 \times S^7 \to SO_{18}$, and we shall show that ϕ is homotopic to a map of the form $\alpha^p \times \alpha^q: S^7 \times S^7 \to SO_9 \times SO_9$, where α^p and α^q represent the integral multiples $p[\alpha]$ and $q[\alpha]$, respectively, of the generator $[\alpha]$ of $\pi_7(SO_9)$. First notice that the wedge $S^7 \vee S^7$ is a deformation retract of $S^7 \times S^7 - \{\text{point}\}$. The map ϕ restricted to the first factor of the wedge defines an element $p[\alpha] \in \pi_7(SO_9) \cong \pi_7(SO_{18})$ for some integer p; that is, $[\phi \mid S^7 \vee \text{point}] = p[\alpha] = [\alpha^p]$. It follows that there are integers p and q such that $\phi \mid S^7 \vee S^7$ is homotopic to the map

$$\alpha^p \vee \alpha^q: S^7 \vee S^7 \to SO_9 \vee SO_9 \subset SO_9 \times SO_9 .$$

Thus, when restricted to $S^7 \times S^7 - \{\text{point}\}$, the maps ϕ and $\alpha^p \times \alpha^q$ are homotopic because their restrictions to the deformation retract $S^7 \vee S^7$ are homotopic. Inasmuch as $\pi_{14}(SO_{18}) = 0$, it follows that the maps ϕ and $\alpha^p \times \alpha^q$ are homotopic as maps from $S^7 \times S^7$ into SO_{18}. Thus, up to homotopy, the framing f' differs from the natural framing of $S^7 \times S^7$ by the map $\alpha^p \times \alpha^q: S^7 \times S^7 \to SO_9 \times SO_9$; hence $(S^7 \times S^7, f')$ and $(S^7, h^p) \times (S^7, h^q)$ represent the same element in Π_{14}, where h^p denotes the framing given by $h^p(u) = \{\text{natural frame}\} \cdot \alpha^p(u)$ for each $u \in S^7$. But (S^7, h^p) represents p times a generator of $\Pi_7 \cong \mathbb{Z}_{240}$ and σ is equal to 15 times a generator of Π_7. It follows that $(S^7 \times S^7, f')$ represents the composition $p\sigma \circ q\sigma = pq\sigma^2$. The proof of the lemma is complete.

Let us return to the framed manifold with which we began, namely

$$(T_r^{14} \# N^{14}, f) .$$

We have shown that it is framed-cobordant to a framed manifold of the form $[(S^7, h^p) \times (S^7, h^q)] \# (N^{14}, g)$ for some pair of integers p and q (mod 2). We are given that $H_7(N^{14}) = 0$; hence (N^{14}, g) is framed-cobordant to a framed homotopy sphere. That is to say, (N^{14}, g) represents either 0 or κ in Π_{14}. Thus $(T_r^{14} \# N^{14}, f)$ represents either $pq\sigma^2$ or $pq\sigma^2 + \kappa$, and it follows that

$$(T_r^{14} \# N^{14}, f)$$

has Kervaire invariant equal to pq reduced mod 2. But we are given that

$$(T_r^{14} \# N^{14}, f)$$

has Kervaire invariant zero; hence $pq \equiv 0 \pmod 2$ and it follows that $(T_r^{14} \# N^{14}, f)$ and (N^{14}, g) represent the same element in Π_{14}, either 0 or κ. The discussion for dimension 14 is complete.

Consider $2m = 6$. The group Π_6 has order 2 with generator represented by $(S^3, h) \times (S^3, h)$. We are given that $H_3(N^6) = 0$. Thus for any framing g of N^6, the framed manifold (N^6, g) may be reduced to a homotopy sphere by a sequence of framed spherical modifications; it is only necessary to check that the result of each framed modification has third homology group equal to zero. But any homotopy 6-sphere is diffeomorphic to S^6; hence (N^6, g) represents zero in Π_6. We are also given that $(T_r^6 \# N^6, f)$ has Kervaire invariant zero; hence it also represents zero in Π_6. It follows that $(T_r^6 \# N^6, f)$ is framed-cobordant to (N^6, g) for any framing g. In particular, we can choose a framing g such that $f \mid N^6$ - disk = $g \mid N^6$ - disk because $\pi_5(SO) = 0$.

Finally, *consider* 2m = 2. The group Π_2 has order 2 with generator represented by $(S^1, h) \times (S^1, h)$. (a) If $H_1(N^2) \neq 0$ then N^2 is a connected sum of s products of circles $S^1 \times S^1$, where $s > 0$. Thus $T_r^2 \# N^2$ is a connected sum of $r + s$ products of circles. Let g_0 denote a framing of $S^1 \times S^1$ that extends to a framing of $S^1 \times D^2$. If the Kervaire invariant of $(T_r^2 \# N^2, f)$ is zero, then $(T_r^2 \# N^2, f)$ is framed-cobordant to $(N^2, g) = (S^1 \times S^1, g_0) \# \cdots \# (S^1 \times S^1, g_0)$. If the Kervaire invariant of $(T_r^2 \# N^2, f)$ is 1, then $(T_r^2 \# N^2, f)$ is framed-cobordant to

$$(N^2, g) = [(S^1, h) \times (S^1, h)] \# (S^1 \times S^1, g_0) \# \cdots \# (S^1 \times S^1, g_0),$$

which has Kervaire invariant 1. (b) If $H_1(N^2) = 0$ then N^2 is diffeomorphic to S^2; in this case we are given that $(T_r^2 \# N^2, f)$ has Kervaire invariant zero. Thus both $(T_r^2 \# N^2, f)$ and $(N^2, g) = (S^2, j)$ represent zero in Π_2; hence they are framed-cobordant.

Notice that if $H_1(N^2) \neq 0$, then $T_r^2 \# N^2 - \{\text{point}\}$ has the homotopy type of a wedge of $2r + 2s$ circles. We also have $\pi_2(SO_6) = 0$. Thus, by a proof similar to the proof of the Lemma for dimension 14, $(T_r^2 \# N^2, f)$ is (up to a homotopy of the framing) a connected sum of framed manifolds of the form $(S^1, h^p) \times (S^1, h^q)$, $p, q \in \mathbb{Z}_2$, where the symbols h^p and h^q have the same meaning as they do in that lemma; here the map $\alpha: S^1 \to SO_3$ represents the generator of $\pi_1(SO_3) \cong \mathbb{Z}_2$. In particular, the restrictions $f \mid T_r^2$ - disk and $f \mid N^2$ - disk extend to framings f_1 and g_1 of T_r^2 and N^2 respectively. Thus we see that in all dimensions the framed manifold of Theorem 1 decomposes into a connected sum of framed manifolds, $(T_r^n \# N^n, f) = (T_r^n, f_1) \# (N^n, g_1)$; however, for $n = 2$, g_1 may not be homotopic to the framing g of Theorem 1.

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