EXTRINSIC SPHERES IN IRREDUCIBLE HERMITIAN SYMMETRIC SPACES

Chorng-Shi Houh

1. INTRODUCTION

Let M be an irreducible Hermitian symmetric space. Then $\tilde{M}$ is simply connected and its canonical Hermitian structure is Kählerian. Let 2m be the real dimension of $\tilde{M}$. An n-dimensional submanifold N of $\tilde{M}$ is called an extrinsic sphere if it is umbilical and has parallel, nonzero mean curvature vector. In Remark 2 of [2], Chen indicated that if the rank of $\tilde{M}$ is ℓ, then $\tilde{M}$ admits extrinsic spheres of dimensions $\leq \ell - 1$ with flat normal connections; namely, extrinsic spheres of maximal flat totally geodesic submanifolds of $\tilde{M}$. In this paper we investigate extrinsic spheres with flat normal connections in irreducible Hermitian symmetric spaces and shall prove the following.

THEOREM. *If* N *is an* n-*dimensional* $(n \geq 2)$ *complete, simply connected extrinsic sphere with flat normal connection in an irreducible Hermitian symmetric space* $\tilde{M}$, *then* $n \leq \operatorname{rank} \tilde{M}$ *and* N *is isometric to a standard* n-*sphere.*

2. PRELIMINARIES

$\tilde{M}$ is always assumed to be an irreducible Hermitian symmetric space of real dimension 2m $(m > 1)$. Let J and g be the complex structure and Kähler metric of $\tilde{M}$, let N be an n-dimensional submanifold of $\tilde{M}$, and let $\tilde{\nabla}$ and ∇ be the covariant differentiations on $\tilde{M}$ and N, respectively. The second fundamental form h of N in $\tilde{M}$ is defined by $h(X, Y) = \tilde{\nabla}_X Y - \nabla_X Y$, where X, Y are vector fields tangent to N. Then h is symmetric, with values in the normal bundle. For a vector field ξ normal to N we write $\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi$, where $-A_\xi X$ and $D_X \xi$ denote the tangential and normal components of $\tilde{\nabla}_X \xi$. If $D_X \xi = 0$, ξ is said to be *parallel*. If $h(X, Y) = g(X, Y)H$, where $H = (\operatorname{trace} h)/n$ is the mean curvature vector of N in $\tilde{M}$, N is said to be *umbilical*. Let $\tilde{R}$, R, and $R^\perp$ be the curvature tensors associated with $\tilde{\nabla}$, ∇, and D, respectively. Let

$$(\overline{\nabla}_X h)(Y, Z) = D_X(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$$

for X, Y, Z tangent to N. Then the equations of Codazzi and Ricci are

$$(\tilde{R}(X, Y)Z)^\perp = (\overline{\nabla}_X h)(Y, Z) - (\overline{\nabla}_Y h)(X, Z);$$

$$\tilde{R}(X, Y; \xi, \eta) = R^\perp(X, Y, \xi, \eta) - g([A_\xi, A_\eta]X, Y),$$

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where X, Y, Z are tangent to N, ξ and η are normal to N, and $(\tilde{R}(X, Y)Z)^{\perp}$ denotes the normal component of $\tilde{R}(X, Y)Z$. N is said to have *flat normal connection* in $\tilde{M}$ if $R^{\perp}(X, Y) = 0$. If N is simply connected and has flat normal connection, then there exist 2m - n mutually orthogonal parallel unit normal vector fields

$$\xi_1, \xi_2, \cdots, \xi_{2m-n}$$

along N. Furthermore, if N has parallel mean curvature vector H, we may assume that $H = \alpha\xi_1$, where α is a nonzero constant. Hence if N is an extrinsic sphere with flat normal connection in $\tilde{M}$, then $h(X, Y) = \alpha g(X, Y)\xi_1$, $\overline{\nabla}_X h = 0$, $R^{\perp}(X, Y) = 0$. The Codazzi and Ricci equations on N become

$$(1) \qquad (\tilde{R}(X, Y)Z)^{\perp} = 0;$$

$$(2) \qquad \tilde{R}(X, Y; \xi, \eta) = -g([A_\xi, A_\eta]X, Y).$$

For a Hermitian symmetric space $\tilde{M}$ there is a triple (G, H, σ) consisting of a connected Lie group G, a closed subgroup H of G, and an involutive automorphism σ of G such that $\tilde{M} = G/H$. Let $\mathfrak{G}$ and $\mathfrak{H}$ be the Lie algebras of G and H, and let σ be the automorphism of $\mathfrak{G}$ which is induced by σ of G. $\mathfrak{H}$ is the eigenspace of σ for the eigenvalue 1. Let $\mathfrak{M}$ be the eigenspace of σ for the eigenvalue -1. Then we have the canonical decomposition $\mathfrak{G} = \mathfrak{H} + \mathfrak{M}$ of $(\mathfrak{G}, \mathfrak{H}, \sigma)$ so that $[\mathfrak{H}, \mathfrak{H}] \subset \mathfrak{H}$, $[\mathfrak{H}, \mathfrak{M}] \subset \mathfrak{M}$, $[\mathfrak{M}, \mathfrak{M}] \subset \mathfrak{H}$. We may identify the tangent space $T_0(\tilde{M})$ of $\tilde{M}$ at the origin 0 of $\tilde{M}$ with $\mathfrak{M}$. The curvature tensor $\tilde{R}$ of $\tilde{M}$ is then given by

$$(3) \qquad \tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = -[[\tilde{X}, \tilde{Y}], \tilde{Z}], \qquad \tilde{X}, \tilde{Y}, \tilde{Z} \in T_0(\tilde{M}).$$

For the Kähler structure J on $\tilde{M}$, we have $\tilde{R}(J\tilde{X}, J\tilde{Y}) = \tilde{R}(\tilde{X}, \tilde{Y})$. Hence we may use the relation $[J\tilde{X}, J\tilde{Y}] = [\tilde{X}, \tilde{Y}]$ for $\tilde{X}, \tilde{Y} \in T_0(\tilde{M})$.

3. PROOF OF THE THEOREM

Let $\tilde{M}$ be an irreducible Hermitian symmetric space of real dimension 2m $(m > 1)$. It is clear that $\tilde{M}$ is either of compact or of noncompact type. Let N be a complete, simply connected extrinsic sphere with flat normal connection in $\tilde{M}$. The dimension of N is n and we assume that $n \geq 2$.

Let $\tilde{H}(\tilde{X})$ be the holomorphic sectional curvature of $\tilde{M}$ determined by the vector $\tilde{X}$. Then the holomorphic pinching of $\tilde{H}$ can be found in [2]. We shall utilize the results in [2] in the following form.

LEMMA 1. *If* $\tilde{M}$ *is of compact type, then* $\tilde{H} > 0$. *If* $\tilde{M}$ *is of noncompact type, then* $\tilde{H} < 0$.

We shall prove several other lemmas.

LEMMA 2. *Let* N *be a complete simply connected extrinsic sphere with flat normal connection in* $\tilde{M}$. *Then either* N *is isometric to a standard sphere and* $\tilde{R}(X, Y) = 0$ *for all* X, Y *tangent to* N, *or* $J\xi_1$ *is tangent to* N, *where* ξ_1 *is the direction of the mean curvature vector of* N.

Proof. If the dimension of N is even, Chen proved in [1] that N is isometric to a standard sphere of radius $1/\alpha$ and that $\tilde{R}(X, Y) = 0$ for all X, Y tangent to N.

Here α is the length of the mean curvature vector of N. Let the dimension of N be odd, say $2k+1$. We may choose parallel orthonormal normal vectors $\xi_1, \xi_2, \cdots, \xi_{2m-2k-1}$ so that $A_{\xi_1} = \alpha I$ and $A_{\xi_r} = 0$ for $r \geq 2$. We then have

(4) $$\tilde{\nabla}_X \xi_r = -A_{\xi_r} X + D_X \xi_r = 0, \quad r \geq 2.$$

Define the functions ϕ_r on N by $\phi_r = g(J\xi_1, \xi_r)$, $r \geq 2$. As in the proof given in [1] for the even-dimensional case, ϕ_r satisfy the differential equations for all vectors X tangent to N:

(5) $$\nabla_X d\phi_r = -\alpha^2 \phi_r X, \quad r \geq 2.$$

If there exists a nonconstant function ϕ_r defined on N satisfying (5), then by a result of Obata [5], N is isometric to the standard sphere of radius $1/\alpha$. We can also prove $\tilde{R}(X, Y) = 0$ for all X, Y tangent to N, as done in [1] for the case where N has even dimension.

Now suppose that all ϕ_r are constants. By (4) we have for all X tangent to N,

$$0 = X\phi_r = g(J\tilde{\nabla}_X \xi_1, \xi_r) = -g(JA_{\xi_1} X, \xi_r) = \alpha g(X, J\xi_r), \quad r \geq 2.$$

Hence $\{\xi_2, \cdots, \xi_{2m-2k-1}, J\xi_2, \cdots, J\xi_{2m-2k-1}\}$ is a subspace of the normal space ν to N. $\dim \{\xi_r, J\xi_r : 2 \leq r \leq 2m-2k-1\}$ is even and greater than or equal to $2m-2k-2$, since it is invariant by J. $\dim \nu = 2m-2k-1$ implies that

$$\dim \{\xi_r, J\xi_r : 2 \leq r \leq 2m-2k-1\} = 2m-2k-2.$$

This implies that $J\xi_1$ is tangent to N.

LEMMA 3. *If* N *is isometric to a sphere with* $\tilde{R}(X, Y) = 0$ *for all* X, Y *tangent to* N, *then* $\dim N \leq \operatorname{rank} \tilde{M}$.

Proof. Since $\tilde{M}$ is symmetric, every point x in $\tilde{M}$ can be regarded as the origin 0. Let $T_0 N$ denote the tangent space of N at 0. $\tilde{R}(X, Y) = 0$ implies that $T_0 N$ is contained in a maximal abelian subspace of $T_0\tilde{M}$ which is identified with $\mathfrak{M}$. This gives $\dim N \leq \operatorname{rank} \tilde{M}$.

Now assume that N is not isometric to a sphere. Then $J\xi_1 \in T_0(N)$. We shall prove two more lemmas and then arrive at a contradiction. In the proof of Lemma 2 we have shown for this case that the dimension of N is odd, say $2k+1$, and that $\nu_0(N) = \{\xi_1, \xi_2, \cdots, \xi_{m-k}, J\xi_2, \cdots, J\xi_{m-k}\}$. We may choose $e_1, \cdots, e_k \in T_0(N)$, so that $T_0(N) = \{e_1, \cdots, e_k, Je_1, \cdots, Je_k, J\xi_1\}$, $k \geq 1$, since we have assumed $\dim N \geq 2$.

Let

$$\mathfrak{M}_1 = T_0(N) \cap JT_0(N) = \{e_1, \cdots, e_k, Je_1, \cdots, Je_k\};$$

$$\mathfrak{M}_2 = \nu_0 \oplus J\nu_0 = \{\xi_1, \xi_2, \cdots, \xi_{m-k}, J\xi_1, \cdots, J\xi_{m-k}\}.$$

It is clear that $\mathfrak{M} = \mathfrak{M}_1 \oplus \mathfrak{M}_2$ (direct sum) and that both $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are J-invariant.

LEMMA 4. $[\mathfrak{M}_1, \mathfrak{M}_2] = 0$.

Proof. Let $X \in \mathfrak{M}_1$; then $JX \in \mathfrak{M}_1$. By (1) we have $\langle \tilde{R}(X, JX)J\xi_1, \xi_1 \rangle = 0$. This and the Bianchi identity imply that

$$\langle \tilde{R}(X, J\xi_1)\xi_1, JX \rangle + \langle \tilde{R}(X, \xi_1)JX, J\xi_1 \rangle = 0.$$

In a Hermitian space $\tilde{M}$ for $\tilde{X}, \tilde{Y}, \tilde{Z} \in T(\tilde{M})$ we have $\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z} = -[[\tilde{X}, \tilde{Y}], \tilde{Z}]$. Thus we have

$$\langle [X, J\xi_1], [\xi_1, JX] \rangle + \langle [X, \xi_1], [JX, J\xi_1] \rangle = 0.$$

Using $[J\tilde{X}, J\tilde{Y}] = [\tilde{X}, \tilde{Y}]$, we obtain for all $X \in \mathfrak{M}_1$

(6) $$[X, \xi_1] = 0, \quad [X, J\xi_1] = 0.$$

By use of (2) and the fact that $A_{\xi_r} = A_{J\xi_r} = 0$ for $r \geq 2$, we have

$$\langle \tilde{R}(X, JX)\xi_r, J\xi_r \rangle = 0.$$

Using the Bianchi identity and the fact that $\tilde{R}(X, JX)J\xi_r = J\tilde{R}(X, JX)\xi_r$, we find

$$0 = \langle \tilde{R}(X, JX)\xi_r, J\xi_r \rangle = \langle \tilde{R}(X, \xi_r)X, \xi_r \rangle + \langle \tilde{R}(X, J\xi_r)X, J\xi_r \rangle.$$

We thus have $\langle [X, \xi_r], [X, \xi_r] \rangle + \langle [X, J\xi_r], [X, J\xi_r] \rangle = 0$, and hence

(7) $$[X, \xi_r] = 0, \quad [X, J\xi_r] = 0, \quad r \geq 2.$$

By (6) and (7) we have proven that $[\mathfrak{M}_1, \mathfrak{M}_2] = 0$.

LEMMA 5. $[[\mathfrak{M}_1, \mathfrak{M}_1], \mathfrak{M}_1] \subset \mathfrak{M}_1$, $[[\mathfrak{M}_2, \mathfrak{M}_2], \mathfrak{M}_2] \subset \mathfrak{M}_2$.

Proof. Let $X_1, Y_1, Z_1 \in \mathfrak{M}_1$. By (1), $\tilde{R}(X_1, Y_1)Z_1 \in T_0(N)$. There is $W_1 \in T_0(N)$ so that $Z_1 = JW_1$. $\tilde{R}(X_1, Y_1)Z_1 = J\tilde{R}(X_1, Y_1)W_1 \in JT_0(N)$. Hence we have that $\tilde{R}(X_1, Y_1)Z_1 \in \mathfrak{M}_1$. This proves that $[[\mathfrak{M}_1, \mathfrak{M}_1], \mathfrak{M}_1] \subset \mathfrak{M}_1$.

Let $X_2, Y_2, Z_2 \in \mathfrak{M}_2$ and $[[X_2, Y_2], Z_2] = W_1 + W_2$, where $W_1 \in \mathfrak{M}_1$, $W_2 \in \mathfrak{M}_2$. By Lemma 4 and the Jacobi identity, if $X_1 \in \mathfrak{M}_1$,

$$[[X_2, Y_2], X_1] = -[[Y_2, X_1], X_2] - [[X_1, X_2], Y_2] = 0.$$

Since $\mathfrak{M}_1$ is J-invariant,

$$[W_1, JW_1] = [[[X_2, Y_2], Z_2], JW_1]$$
$$= -[[Z_2, JW_1], [X_2, Y_2]] - [[JW_1, [X_2, Y_2]], Z_2] = 0.$$

If $W_1 \neq 0$, then $\langle \tilde{R}(W_1, JW_1)JW_1, W_1 \rangle = -[[[W_1, JW_1], JW_1], W_1] = 0$. Hence the holomorphic sectional curvature $\tilde{H}(W_1) = 0$. This contradicts Lemma 1. Thus we have $W_1 = 0$ and hence $[[\mathfrak{M}_2, \mathfrak{M}_2], \mathfrak{M}_2] \subset \mathfrak{M}_2$.

Lemma 5 shows that each of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ forms a Lie triple system. Then there are complete totally geodesic submanifolds M_1 and M_2 through the origin 0 such that $T_0(M_1) = \mathfrak{M}_1$, $T_0(M_2) = \mathfrak{M}_2$. For this situation, using the same argument as in [3], we can conclude that $\tilde{M} = M_1 \times M_2$. This contradicts the assumption that $\tilde{M}$ is irreducible. Hence the theorem is proved.

REFERENCES

1. B-Y. Chen, *Extrinsic spheres in Kähler manifolds*. Michigan Math. J. 23 (1976), 327-330.

2. ———, *Extrinsic spheres in Kähler manifolds, II*. Michigan Math. J. 24 (1977),

3. ———, *Holonomy groups of normal bundles*, to appear.

4. S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Vol. II.* Interscience Tracts in Pure and Applied Mathematics, No. 15, Vol. II. Interscience Publishers, New York, 1969.

5. M. Obata, *Certain conditions for a Riemannian manifold to be isometric with a sphere*. J. Math. Soc. Japan 14 (1962), 333-340.

Department of Mathematics
Wayne State University
Detroit, Michigan 48202