A CLASSIFICATION OF UNSPLITTABLE-LINK COMPLEMENTS

Wilbur Whitten

Several persons have recently given algebraic characterizations of tame knots and links in S^3: (1) J. Simon characterized the type of a knot by the free product of two cable-knot groups [6]; (2) J. H. Conway and C. McA. Gordon characterized the type of an oriented knot by the free product of the associated group and the group

$$|\mu, \lambda, a, b : a^5 = 1,\ \lambda^{-1} a\lambda = a^2,\ b^7 = 1,\ \mu^{-1} b\mu = b^2,\ [a, b] = [\mu, \lambda] = 1|$$

amalgamated along the peripheral subgroup $|\mu, \lambda : [\mu, \lambda] = 1|$ (μ is a meridian of the knot; λ, a longitude) [1]; (3) I characterized the (ambient) isotopy type of a link by the group of the link's "double" ([9], [10], [11]). Like mine, Simon's characterizing groups obviously have other geometric interpretations, but the proofs that these groups actually characterize knots and links are long and difficult, requiring the heaviest machinery in the theory of three-manifolds. On the other hand, though the Conway-Gordon groups have no *prima-facie* geometric interpretations apart from their knot-classifying properties, the proof that they characterize oriented-knot types is rather simple and is all algebraic until the end, when basic results of C. D. Papakyriakopoulos [5, p. 19, Theorem (28.1)(i)] and of F. Waldhausen [8, p. 80, Corollary 6.5] change algebra into geometry.

In this paper, roughly speaking, we adjoin copies of $Z \times Z \times Z_2$ along the peripheral structure of an *unsplittable* link's group to obtain a group characterizing the topological type of the link's complement. My proof, like Conway and Gordon's, is chiefly algebraic.

First, some preliminaries. All knots and links are oriented, and they are tamely imbedded in S^3; all links are unrestricted as to splittability, unless specified otherwise; all mappings are piecewise linear; and all regular neighborhoods, at least second-regular. Let L denote the link $K_1 \cup \cdots \cup K_m$, let V_i be a closed regular neighborhood of K_i ($i = 1, \cdots, m$), and assume that $V_i \cap V_j = \emptyset$, when $i \neq j$. Set $C(L) = S^3 - \mathrm{Int}(V_1 \cup \cdots \cup V_m)$. For each of $i = 1, \cdots, m$, the inclusion map $\tau_i\colon \partial V_i \to C(L)$ induces a homomorphism $\tau_{i_*}\colon \pi_1(\partial V_i) \to \pi_1(C(L))$ determined up to an inner automorphism of $\pi_1(C(L))$, and $\tau_{i_*}(\pi_1(\partial V_i))$ is determined up to conjugacy in $\pi_1(C(L))$; of course, we are assuming a fixed basepoint for each of $\pi_1(C(L))$ and $\pi_1(\partial V_i)$.

We define a special group $G^*(L)$ as follows. Let p be the basepoint of $\pi_1(C(L))$, and let p_i be the basepoint of $\pi_1(\partial V_i)$. To fix τ_{i_*}, we choose a path γ_i from p to p_i. Let $\{\mu_i, \lambda_i\}$ be a set of generators (in $\pi_1(C(L), p)$) of the free abelian group $\tau_{i_*}(\pi_1(\partial V_i, p_i))$, which is, of course, either of rank one or of rank two. If $\pi_1(C(L), p) = |X : R|$, set

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$$G^*(L) = |X, a_1, \cdots, a_m : R, a_i^2 = [\mu_i, a_i] = [\lambda_i, a_i] = 1 \ (i = 1, \cdots, m)|.$$

PROPOSITION. *If* L *is a link in* S^3, *then* $G^*(L)$ *is independent, up to isomorphism, of the choice of basepoints, connecting paths, and generating sets* $\{\mu_i, \lambda_i\}$ $(i = 1, \cdots, m)$.

Proof. After giving some preliminaries, we shall prove that

$$G^*(L, p, p_i, \gamma_i, \{\mu_i, \lambda_i\} \ (i = 1, \cdots, m)) \approx$$
$$G^*(L, p', p_i', \gamma_i', \{\mu_i', \lambda_i'\} \ (i = 1, \cdots, m)).$$

For each of $h = 1, \cdots, m$, set

$$G_h^*(L) = |X, a_1, \cdots, a_h : R, a_i^2 = [\mu_i, a_i] = [\lambda_i, a_i] = 1 \ (i = 1, \cdots, h)|,$$

set $\Gamma_i = \tau_{i_*}(\pi_1(\partial V_i, p_i))$ $(i = 1, \cdots, m)$, and set

$$X_i = |\mu_i, \lambda_i, a_i : a_i^2 = [\mu_i, a_i] = [\lambda_i, a_i] = [\mu_i, \lambda_i] = 1| \quad \text{or} \quad X_i = |t_i, a_i : a_i^2 = [t_i, a_i] = 1|,$$

depending on whether τ_{i_*} is or is not a monomorphism. If τ_{i_*} is a monomorphism, then $X_i \approx Z \times Z \times Z_2$; otherwise, $X_i \approx Z \times Z_2$, and t_i $(= \mu_i^v \lambda_i^s)$ generates the infinite cyclic group $\tau_{i_*}(\pi_1(\partial V_i, p_i))$. Evidently,

$$G_h^*(L) \approx (\cdots((\pi_1(C(L)) \underset{\Gamma_1}{*} X_1) \underset{\Gamma_2}{*} X_2)\cdots) \underset{\Gamma_h}{*} X_h.$$

Finally, with other basepoints $p', p_1', \cdots, p_m'$, with other paths $\gamma_1', \cdots, \gamma_m'$, and with other generating sets $\{\mu_i', \lambda_i'\}$ $(i = 1, \cdots, m)$, we have $\pi_1(C(L), p') = |X' : R'|$,

$$'G_h^*(L) = |X', a_1', \cdots, a_h' : R', a_i'^2 = [\mu_i', a_i'] = [\lambda_i', a_i'] = 1 \ (i = 1, \cdots, h)|,$$

and $\Gamma_i' = \tau'_{i_*}(\pi_1(\partial V_i, p_i'))$.

The proof is by induction on h. A path γ from p to p' induces an isomorphism $\alpha: \pi_1(C(L), p) \to \pi_1(C(L), p')$. Choose a path δ_1 on ∂V_1 from p_1 to p_1'. Conjugation of $\pi_1(C(L), p')$ by $[\gamma_1' \delta_1^{-1} \gamma_1^{-1} \gamma]$ yields an inner automorphism β sending $\alpha(\Gamma_1)$ onto Γ_1' and, in general, sending $\alpha(\Gamma_i)$ onto a conjugate of Γ_i' $(i = 1, \cdots, m)$. If we define $\beta\alpha(a_1) = a_1'$, then $\beta\alpha$ extends to an isomorphism of $G_1^*(L)$ onto $'G_1^*(L)$, as one can easily prove.

Suppose now that $m > 1$, that $1 \leq h < m$, and that there is an isomorphism $\alpha: G_h^*(L) \to {}'G_h^*(L)$ taking $\pi_1(C(L), p)$ onto $\pi_1(C(L), p')$ and taking Γ_i onto a conjugate of Γ_i' $(i = 1, \cdots, m)$. Clearly,

$$G_{h+1}^*(L) = G_h^*(L) \underset{\Gamma_{h+1}}{*} X_{h+1} \quad \text{and} \quad 'G_{h+1}^*(L) = {}'G_h^*(L) \underset{\Gamma_{h+1}'}{*} X_{h+1}'.$$

Conjugation of $'G_h^*(L)$ by an appropriate element of $\pi_1(C(L), p')$ yields an inner automorphism β of $'G_h^*(L)$ taking $\alpha(\Gamma_{h+1})$ onto Γ_{h+1}'. If we define $\beta\alpha(a_{h+1}) = a_{h+1}'$, then $\beta\alpha$ extends to an isomorphism of $G_{h+1}^*(L)$ onto $'G_{h+1}^*(L)$ such that

$\beta\alpha(\pi_1(C(L), p)) = \pi_1(C(L), p')$ and such that $\beta\alpha(\Gamma_i)$ is a conjugate of Γ_i' $(i = 1, \cdots, m)$.

Induction now implies that $G_m^*(L) \approx {}'G_m^*(L)$. But

$$G_m^*(L) = G^*(L, p, p_i, \gamma_i, \{\mu_i, \lambda_i\} \ (i = 1, \cdots, m))$$

and

$$'G_m^*(L) = G^*(L, p', p_i', \gamma_i', \{\mu_i', \lambda_i'\} \ (i = 1, \cdots, m)),$$

and so the proposition is proved.

THEOREM. *If* L *is an unsplittable link, then the isomorphism class of the group* $G^*(L)$ *characterizes the topological type of* $C(L)$; *that is, if* L *and* L' *are links and if* L *is unsplittable, then* $C(L) \cong C(L')$ *if and only if* $G^*(L) \approx G^*(L')$.

Proof. Suppose that $\phi\colon C(L) \to C(L')$ is a homeomorphism. Then

$$H_1(C(L)) \approx H_1(C(L')),$$

and, because the number of components in a link is equal to the rank of the link-complement's first homology group, the links L and L' have the same number m of components. Clearly,

$$G^*(L, p, p_i, \gamma_i, \{\mu_i, \lambda_i\} \ (i = 1, \cdots, m)) \approx$$
$$G^*(L', \phi(p), \phi(p_i), \phi(\gamma_i), \{\phi_*(\mu_i), \phi_*(\lambda_i)\} \ (i = 1, \cdots, m)) ;$$

hence, by the proposition, $G^*(L) \approx G^*(L')$, and the necessity is established without, we note, requiring the unsplittability of L.

Now, let $G = \pi_1(C(L))$, and consider $G_h^*(L)$ $(h = 1, \cdots, m)$. Because $G_1^*(L) = G \underset{\Gamma_1}{*} X_1$, elements of finite order in $G_1^*(L)$ are conjugates of elements belonging either to G or to X_1 [3, p. 208, Corollary 4.4.5]. Because G is torsion free [5, p. 23, Corollary (31.9)] and because $X_1 \approx Z \times Z \times Z_2$ or $X_1 \approx Z \times Z_2$, the nontrivial elements of finite order in $G_1^*(L)$ are precisely the conjugates of a_1.

Suppose that $m > 1$, that $1 \le h < m$, and that the nontrivial elements of finite order in $G_h^*(L)$ are the conjugates of $a_1, \cdots, a_h$. Then the nontrivial elements of finite order in $G_{h+1}^*(L)$ are just the conjugates of $a_1, \cdots, a_{h+1}$, because $G_{h+1}^*(L) = G_h^*(L) \underset{\Gamma_{h+1}}{*} X_{h+1}$ and because each element of finite order in $G_{h+1}^*(L)$ is in a conjugate of $G_h^*(L)$ or of X_{h+1}. Thus, induction implies that the nontrivial elements of finite order in $G^*(L)$ $(= G_m^*(L))$ are just the conjugates of $a_1, \cdots, a_m$.

Remark 1. If L is not a trivial knot, then, because L is unsplittable, each homomorphism $\tau_{i_*}\colon \pi_1(\partial V_i) \to \pi_1(C(L))$ $(i = 1, \cdots, m)$ is a monomorphism, whence each $X_i \approx Z \times Z \times Z_2$. To prove this, one can construct an argument involving the combined Dehn lemma and loop theorem [7, p. 5, Theorem 1.B.2].

Before proving the sufficiency, we note one other property of $G^*(L)$: the subgroup X_i is the normalizer N_i of $|a_i : a_i^2 = 1|$ in $G^*(L)$. Denote $|a_i : a_i^2 = 1|$ by A_i. Clearly, $X_i \subseteq N_i$. Now, if $\langle a_i \rangle$ denotes the consequence of $\{a_i\}$ in $G^*(L)$, then $G^*(L) = (G^*(L)/\langle a_i \rangle) \underset{\Gamma_i}{*} X_i$, and one can write each element of $G^*(L)$ uniquely in the form $gc_1 \cdots c_d$, in which $g \in \Gamma_i$, each c_k is a coset representative of Γ_i in $G^*(L)/\langle a_i \rangle$ or in X_i, no $c_k = 1$, and c_k and c_{k+1} are neither both in $G^*(L)/\langle a_i \rangle$ nor both in X_i [3, p. 205, Corollary 4.4.1]. Hence, if N_i contains an element $f \notin X_i$, then there are nontrivial coset representatives $h_1, \cdots, h_n$ of Γ_i in $G^*(L)/\langle a_i \rangle$ and there is an element $g \in \Gamma_i$ such that f can be written uniquely in exactly one of the following forms: $gh_1 a_i h_2 \cdots a_i h_n$; $gh_1 a_i \cdots h_n a_i$; $ga_i h_1 \cdots a_i h_n$; $ga_i h_1 a_i \cdots h_n a_i$. Because of the normal form's uniqueness, it is easy to see that $fA_i \neq A_i f$. Consequently, $N_i = X_i$.

Now, the sufficiency. Suppose that $\pi_1(C(L')) = |Y : Q|$ and that

$$G^*(L') = |Y, b_1, \cdots, b_{m'} : Q, b_j^2 = [\mu_j', b_j] = [\lambda_j', b_j] = 1 \quad (j = 1, \cdots, m')| .$$

We assume the existence of an isomorphism $\psi_*: G^*(L) \to G^*(L')$.

For each of $i = 1, \cdots, m$, there exists g_i in $G^*(L')$ such that $\psi_*(a_i) = g_i b_{j_i} g_i^{-1}$ for some j_i $(1 \leq j_i \leq m')$. Furthermore, because no two distinct a_i's are conjugate [3, p. 212, Theorem 4.6(ii)] and because ψ_* is an isomorphism, ψ_* maps distinct a_i's onto conjugates of distinct b_j's; hence, $m \leq m'$. Similarly, $m' \leq m$, whence $m = m'$ and $\langle b_1, \cdots, b_{m'} \rangle = \langle b_1, \cdots, b_m \rangle$. Moreover,

$$\langle \psi_*(a_1), \cdots, \psi_*(a_m) \rangle = \langle b_1, \cdots, b_m \rangle,$$

and, because $\psi_* \langle a_1, \cdots, a_m \rangle = \langle \psi_*(a_1), \cdots, \psi_*(a_m) \rangle$ [2, p. 39, Theorem (1.1)], we have $\psi_* \langle a_1, \cdots, a_m \rangle = \langle b_1, \cdots, b_m \rangle$. Therefore, ψ_* induces a unique isomorphism

$$\Psi_*: G^*(L)/\langle a_1, \cdots, a_m \rangle \to G^*(L')/\langle b_1, \cdots, b_m \rangle .$$

But $G^*(L)/\langle a_1, \cdots, a_m \rangle = \pi_1(C(L))$ and $G^*(L')/\langle b_1, \cdots, b_m \rangle = \pi_1(C(L'))$; thus, we have the consistent diagram

$$\begin{tikzcd}
G^*(L) \arrow[r, "\psi_*"] \arrow[d, "\eta_1"'] & G^*(L') \arrow[d, "\eta_2"] \\
\pi_1(C(L)) \arrow[r, "\Psi_*"'] & \pi_1(C(L'))
\end{tikzcd}$$

in which η_1 and η_2 are the natural homomorphisms.

We know that each of L and L' has m components. If L is a trivial knot, then $\pi_1(C(L)) \approx Z$; hence, the diagram in the foregoing paragraph implies that $\pi_1(C(L'))$ is also isomorphic to Z. Consequently, L' is also a trivial knot [5, p. 19, Theorem (28.1)]; thus, $C(L') \cong C(L)$.

We now assume that L is not a trivial knot; clearly then, L' is also not a trivial knot. Because L is unsplittable, $\pi_1(C(L))$ is indecomposable (as a free product of nontrivial groups) [5, p. 19, Theorem (27.1)]. Because $\pi_1(C(L')) \approx \pi_1(C(L))$, the group $\pi_1(C(L'))$ is also indecomposable; thus, L' as well as L is unsplittable [5, (*loc. cit.*)].

Let $K_1', \cdots, K_m'$ denote the components of L', let V_j' be a closed regular neighborhood of K_j' ($j = 1, \cdots, m$), and suppose that $V_i' \cap V_j' = \emptyset$, when $i \neq j$. We can assume that $C(L') = S^3 - \mathrm{Int}(V_1' \cup \cdots \cup V_m')$. The inclusion map $\sigma_j \colon \partial V_j' \to C(L')$ induces a monomorphism $\sigma_{j_*} \colon \pi_1(\partial V_j') \to \pi_1(C(L'))$, because L' is unsplittable; *cf.* Remark 1. Set $\Omega_j = \sigma_{j_*}(\pi_1(\partial V_j'))$, and set

$$Y_j = |\mu_j', \lambda_j', b_j : b_j^2 = [\mu_j', b_j] = [\lambda_j', b_j] = [\mu_j', \lambda_j'] = 1| \qquad (j = 1, \cdots, m);$$

we are assuming, of course, that we have chosen basepoints p' and p_j', connecting path γ_j', and generating set $\{\mu_j', \lambda_j'\}$.

We have $\eta_1(X_i) = \Gamma_i$ and $\eta_2(Y_{j_i}) = \Omega_{j_i}$. Furthermore, the normalizer of $|b_{j_i} : b_{j_i}^2 = 1|$ in $G^*(L')$ is clearly Y_{j_i}. Thus, $\psi_*(X_i) = g_i Y_{j_i} g_i^{-1}$, because $\psi_*(a_i) = g_i b_{j_i} g_i^{-1}$. Consequently, $\Psi_*(\Gamma_i) = \eta_2(g_i)\Omega_{j_i}\eta_2(g_i^{-1})$, and so the isomorphism Ψ_* preserves the peripheral group system. Therefore, the hypotheses of Waldhausen's result—Corollary 6.5, p. 80 of [8]—hold, and we have $C(L) \cong C(L')$, completing the theorem's proof.

Remark 2. If L is unsplittable, then the isomorphism class of $G^*(L)$ characterizes the topological type of $S^3 - L$ as well as the topological type of $C(L)$, because $S^3 - L \cong S^3 - L'$ if and only if $C(L) \cong C(L')$.

In conclusion, we note that $G^*(L)$ does not necessarily characterize $C(L)$, when L is splittable; that is, there are splittable links L and L' for which $G^*(L) \approx G^*(L')$ but for which $C(L) \not\cong C(L')$. For example, let K_1 and K_2 be trefoil knots, and let K_i^* be K_i's mirror image. Suppose that $L = K_1 \cup K_2$, that $L' = K_1 \cup K_2^*$, and that each of L and L' is splittable. Evidently, $G^*(L) = G^*(K_1) * G^*(K_2)$ and $G^*(L') = G^*(K_1) * G^*(K_2^*)$; furthermore, our theorem implies that $G^*(K_2) \approx G^*(K_2^*)$. Hence, $G^*(L) \approx G^*(L')$.

On the other hand, $C(L) = C(K_1) \# C(K_2)$ and $C(L') = C(K_1) \# C(K_2^*)$, and each of the knot manifolds, $C(K_i)$ and $C(K_i^*)$ ($i = 1, 2$), is prime, because it is irreducible. Moreover, because K_i ($i = 1, 2$) is not amphicheiral and because the topological type of K_i's complement determines the knot type of K_i, the knot manifolds $C(K_i)$ and $C(K_i^*)$ are not isomorphic; that is, there is no orientation-preserving homeomorphism of $C(K_i)$ onto $C(K_i^*)$. Thus, $C(L)$ is not isomorphic to $C(L')$ [4, p. 5, Generalization 1]. If there were an orientation-reversing homeomorphism of $C(L)$ onto $C(L')$, then $C(L)$ would be isomorphic to $C(K_1^*) \# C(K_2)$. Evidently, this is not true (see [4, *loc. cit.*]); hence, $C(L) \not\cong C(L')$.

REFERENCES

1. J. H. Conway and C. McA. Gordon, *A group to classify knots*. Bull. London Math. Soc. 7 (1975), 84-86.

2. R. H. Crowell and R. H. Fox, *Introduction to knot theory*. Ginn and Co., Boston, 1963.

3. W. Magnus, A. Karrass, and D. Solitar, *Combinatorial group theory: presentation of groups in terms of generators and relations*. Pure and Applied Mathematics, Vol. 13, Interscience Publishers (John Wiley and Sons, Inc.), New York, 1966.

4. J. W. Milnor, *A unique decomposition theorem for* 3-*manifolds*. Amer. J. Math. 84 (1962), 1-7.

5. C. D. Papakyriakopoulos, *On Dehn's lemma and the asphericity of knots*. Ann. of Math. (2) 66 (1957), 1-26.

6. J. Simon, *An algebraic classification of knots in* S^3. Ann. of Math. (2) 97 (1973), 1-13.

7. J. Stallings, *Group theory and three-dimensional manifolds*. Yale Mathematical Monographs 4, Yale University Press, New Haven, 1971.

8. F. Waldhausen, *On irreducible* 3-*manifolds which are sufficiently large*. Ann. of Math. (2) 87 (1968), 56-88.

9. W. Whitten, *Algebraic and geometric characterizations of knots*. Invent. Math. 26 (1974), 259-270.

10. ———, *Characterizations of knots and links*. Bull. Amer. Math. Soc. 80 (1974), 1265-1270.

11. ———, *Groups and manifolds characterizing links*. Knots, groups and 3-manifolds. Annals of Mathematics Studies 84, Princeton University Press, Princeton, N.J., 1975, 63-84.

Department of Mathematics
University of Southwestern Louisiana
Lafayette, Louisiana 70504