GOLDIE CENTRALIZERS OF SEPARABLE SUBALGEBRAS

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1. INTRODUCTION

In this paper we shall discuss some relations between the structure of an algebra R and a certain subalgebra of R.

Before stating our results, let us recall some of the definitions and known results. Let R be a ring. The center of R is denoted Z(R). The ring R is *semiprime* if R has no nontrivial nilpotent ideals; it is *prime* if the relation $AB = 0$ for ideals A and B implies $A = 0$ or $B = 0$. The ring R is a (right) *Goldie* ring provided R satisfies the maximum condition on right annihilators and R contains no infinite direct sum of right ideals.

For x in R, we set

$$r(x) = \{y \in R \mid xy = 0\} \qquad \text{and} \qquad \ell(x) = \{y \in R \mid yx = 0\}.$$

An element x of R is *regular* if $r(x) = \ell(x) = 0$.

If A is a subset of R, the *centralizer* of A in R is

$$C_R(A) = \{x \in R \mid xa = ax \text{ for all } a \in A\}.$$

In [8], S. Montgomery explored some relations between the structure of an algebra R over a field and the structure of the centralizer $C_R(A)$ of a finite-dimensional separable subalgebra A. In [2] we showed that if R is a semiprime n-torsion-free ring, a is an element of R such that $a^n \in Z(R)$, and $C_R(A)$ is a semiprime (prime) Goldie ring, then R is a semiprime (prime) Goldie ring.

In this paper we shall show that if R is a semiprime algebra over a field and A is a finite-dimensional separable subalgebra, then $C_R(A)$ is a Goldie ring if and only if R is a Goldie ring.

As a consequence we can extend the results of [2] to the case where Z(R) is a field and a is a zero of a separable polynomial over Z(R).

We shall use localizations later. Let T be a nonempty set of regular elements of R. Then T is a *right denominator set* if T is closed under multiplication and if $xT \cap tR \neq 0$ for each nonzero x in R and each t in T. P. M. Cohn [3, p. 21], has shown that the *localization* of R by T,

$$R_T = \{xt^{-1} \mid x \in R, t \in T\}$$

exists for such a T and $R \subset R_T$.

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A ring $Q(R) \supseteq R$ is a *right quotient ring* of R. If Q(R) has an identity, each regular element of R is invertible in Q(R), and for each $q \in Q(R)$ there exist $a, b \in R$ with b regular and $q = ab^{-1}$.

If Q(R) satisfies the conditions above, R is said to be a *right order* in Q(R).

As usual, R_n denotes the ring of all n-by-n matrices over R. If R has a unit element 1, then $R \hookrightarrow R_n$ via $r \to rI$, where e_{ij} denotes the n-by-n matrix with 1 in the ij-th place and 0 elsewhere, and $I = \sum_{i=1}^{n} e_{ii}$ denotes the identity matrix.

A right ideal E of a ring R is said to be an *essential* right ideal if E intersects every nonzero right ideal of R.

2. PRELIMINARY LEMMAS

We start with a necessary and sufficient condition for a ring to be a semiprime Goldie ring.

LEMMA 1. *A ring* R *is a semiprime Goldie ring if and only if* R *contains a multiplicative subsemigroup* T *satisfying the conditions*

(1) *the elements of* T *are regular in* R,

(2) *every essential right ideal of* R *contains an element of* T,

(3) *for each* $t \in T$, *the right ideal* tR *is essential.*

Proof. If R is a semiprime Goldie ring, then the set of all regular elements of R satisfies the conditions above [4, pp. 174-175].

Now assume that T exists and satisfies (1) to (3). For $t \in T$ and $x \in R$ $(x \neq 0)$, let $W = \{r \in R \mid xr \in tR\}$. By (3), tR is an essential right ideal; hence W is essential, and by (2), $W \cap T \neq \emptyset$. Thus $tR \cap xT \neq 0$ whenever $t \in T$ and $0 \neq x \in R$. Therefore T is a right denominator set for R, and R_T exists.

If E is any essential right ideal of R_T, then $E \cap R$ is an essential right ideal of R. Hence, by (2), $E \cap R$ contains an element of T that is invertible in R_T, and therefore $E = R_T$. Since R_T has no proper essential right ideals, R_T is a semisimple Artinian ring [7, p. 61]. In an Artinian ring with an identity, every right-regular element is invertible. Since every regular element of R is right-regular in R_T, we see as a consequence that every regular element of R is invertible in R_T. Therefore R is an order in a semisimple Artinian ring, and it is thus a semiprime Goldie ring [4, p. 177].

LEMMA 2. *Let* R *be a ring with an identity. If* R *is a semiprime Goldie ring, then each essential right ideal of* R_n *contains a regular element of* R.

Proof. Let E be an essential right ideal of R_n. Let $E_i = \{r \in R \mid re_{ii} \in E\}$, and let J be a nonzero right ideal of R. Since $\left(\sum_{j=1}^{n} Je_{ij}\right) \cap E \neq 0$, we see that

$$\left(\sum_{j=1}^{n} Je_{ij}\right) e_{m_i} \cap E \neq 0 \qquad \text{for some } m;$$

therefore $E_i \cap J \neq 0$. We have shown that E_i is an essential right ideal of R for each $i = 1, \cdots, n$. Since $\bigcap_{i=1}^{n} E_i$ is an essential right ideal of the semiprime

Goldie ring R, it contains a regular element t of R, and $te_{ii} \in E$ for each $i = 1, \cdots, n$. Consequently $t = tI \in E$.

LEMMA 3. *Let* R *be a ring with an identity. Then* R *is a semiprime (prime) Goldie ring if and only if the matrix ring* R_n *is a semiprime (prime) Goldie ring.*

Proof. By [9, p. 606], R_n is a semiprime Goldie ring whenever R is a semiprime Goldie ring. The converse is also true, since R is a subring of R_n and hence inherits the maximum condition on right annihilators. Also, since every direct sum of right ideals of R gives rise to a direct sum of right ideals in R_n, the ring R contains no infinite direct sums of right ideals.

The fact that R is semiprime (prime) if and only if R_n is semiprime (prime) stems from the well-known result that every ideal of R_n is of the form J_n, where J is an ideal of R.

LEMMA 4. *Let* R *be an algebra over a field* F, *and let* K *be a finite separable extension of* F. *Then* R *is a semiprime Goldie ring if and only if* $R \otimes_F K$ *is a semiprime Goldie ring.*

Proof. If R is a semiprime Goldie ring and Q(R) its ring of quotients, then Q(R) is semisimple Artinian. The tensor product $Q(R) \otimes_F K$ is semisimple Artinian, since K is a finite separable extension of F, and by [6, p. 116 and p. 252]. Hence every regular element of $R \otimes_F K$ is invertible in $Q(R) \otimes_F K$. Also, by the common-multiple property of R, every element of $Q(R) \otimes_F K$ is of the form ab^{-1}, where $a \in R \otimes_F K$ and b is a regular element of R. Hence $R \otimes_F K$ is an order in $Q(R) \otimes_F K$ that is a semisimple Artinian ring, and thus $R \otimes_F K$ is a semiprime Goldie ring.

The converse can easily be verified.

LEMMA 5. *Let* R *be a semiprime (prime) Goldie ring, let* M *be a right ideal of* R, *and let* $L = M \cap \ell(M)$. *Then* M/L *is a semiprime (prime) Goldie ring. In particular, if* $e \in R$ *is a nonzero idempotent, then for* M = eR *we have* eRe, *which is isomorphic to* M/L, *a semiprime (prime) Goldie ring.*

Proof. Each nonzero right ideal of M/L is the image $\overline{V}$ of some right ideal V of M, where VM is a nonzero right ideal of R. Hence M/L is a semiprime (prime) ring. Now, since L is a two-sided ideal of M and in addition is a left annihilator in M, there exists an order-preserving correspondence between the left annihilators of M/L and certain left annihilators of M [5, p. 74]. Since the maximum condition on right annihilators is equivalent to the minimum condition on left annihilators, M as a subring of R satisfies this minimum condition. By the argument above, M/L satisfies the maximum condition on right annihilators.

If $\overline{V}_1 \oplus \overline{V}_2 \oplus \cdots$ is a direct sum of right ideals of M/L, then $V_1 M + V_2 M + \cdots$ is a direct sum of right ideals of R. To prove this we must show that the right ideal $W_j = \left(\sum_{i \neq j} V_i M\right) \cap V_j M$ is 0 for each j. Now,

$$\overline{W_j} \subset \left(\sum_{i \neq j} \overline{V_i}\right) \cap \overline{V_j} = 0;$$

hence $W_j^2 = 0$ because $W_j M = 0$. Semiprimeness of R now implies $W_j = 0$. Therefore M/L has no infinite direct sum of right ideals, since R does not.

3. THE MAIN THEOREM

We now turn to the situation described in [8]. In the following, R will denote an algebra over a field F, and A a finite-dimensional, separable subalgebra of R. By [1, p. 45], there exists a finite separable field extension K of F (called a *splitting field* for A) such that

$$A \otimes_F K = K_{n_1} \oplus \cdots \oplus K_{n_m},$$

where K_{n_i} denotes the n_i-by-n_i matrix ring over K. The algebra A is called *split* if it is already a direct sum of complete matrix rings over F.

LEMMA 6. *If* $A = F_{n_1} \oplus \cdots \oplus F_{n_m}$, *then*

(1) $C = C_R(A) = e_1 C + \cdots + e_m C + \left(1 - \sum_{i=1}^m e_i\right) C$, *where* e_i *is the* n_i-by-n_i *identity matrix, and* $\left(1 - \sum_{i=1}^m e_i\right) C = \left\{x - \sum_{i=1}^m e_i x \mid x \in C\right\}$,

(2) $e_i R e_i = (e_i C)_{n_i}$ *for* $i = 1, \cdots, m$,

(3) $\left(1 - \sum_{i=1}^m e_i\right) R \left(1 - \sum_{i=1}^m e_i\right) = \left(1 - \sum_{i=1}^m e_i\right) C$.

Proof. Since $e_i \in Z(A)$, we see that $e_i \in Z(C)$, and (1) follows.

Next, by [4, p. 112] we have the isomorphism

$$e_i R e_i \cong e_i A e_i \otimes C_{e_i R e_i}(e_i A e_i);$$

but $e_i A e_i = F_{n_i}$ and $C_{e_i R e_i}(e_i A e_i) = e_i C e_i = e_i C$, hence

$$e_i R e_i \cong F_{n_i} \otimes e_i C \cong (e_i C)_{n_i}.$$

Since the isomorphisms are natural, we may identify $e_i R e_i$ with $(e_i C)_{n_i}$, and (2) follows. Finally, since

$$\left(1 - \sum_{i=1}^{m} e_i\right) R \left(1 - \sum_{i=1}^{m} e_i\right) \subset C,$$

the rest follows.

We are ready to prove the main result of this paper.

THEOREM. *Let* R *be a semiprime algebra over a field* F, *and let* A *be a separable, finite-dimensional subalgebra of* R. *Then* R *is a Goldie ring if and only if* $C_R(A)$ *is a Goldie ring.*

Proof. First, let us show that we can assume that A is split. If K is a splitting field of A, then $R \otimes_F K$ is semiprime if and only if R is semiprime [8, p. 16]. By Lemma 4, $R \otimes_F K$ is a semiprime Goldie ring if and only if the same is true of R. Since

$$C_{R \otimes_F K}(A \otimes_F K) = C_R(A) \otimes_F K,$$

we see by Lemma 4 that $C_{R\otimes_F K}(A \otimes_F K)$ is a semiprime Goldie ring if and only if $C_R(A)$ is.

Second, by [8, p. 19], it is not necessary to specify that $C_R(A)$ is semiprime, for semiprimeness of R implies that of $C_R(A)$.

Therefore, let $A = F_{n_1} \oplus \cdots \oplus F_{n_m}$, and assume R is a Goldie ring. Denote $C_R(A)$ by C, and $1 - \sum_{i=1}^{m} e_i$ by e_{m+1}, and let $n_{m+1} = 1$. Then by (2) and (3) of Lemma 6,

$$e_i R e_i = (e_i C)_{n_i} \qquad (i = 1, \cdots, m+1).$$

Since e_{m+1} acts as an idempotent and since the e_i's are idempotents, it follows from Lemma 5 that $e_i R e_i$ is a semiprime Goldie ring for $i = 1, \cdots, m+1$. Thus, by Lemma 3, $e_i C$ is a semiprime Goldie ring for $i = 1, \cdots, m+1$. Since every right ideal M of C is a direct sum of M_i, where M_i is a right ideal of $e_i C$ $(i = 1, \cdots, m+1)$, and since these are Goldie rings, C is a Goldie ring.

Conversely, assume that C is a Goldie ring. As we noted earlier, C is actually a semiprime Goldie ring. Let T denote the set of all regular elements of C. We shall show that T satisfies conditions (1) to (3) of Lemma 1 and thus prove that R is a Goldie ring.

Condition (1) was established in [8, p. 22]. Here we shall give a shortened proof. Since $\sum_{i=1}^{m+1} e_i x = x$ for each $x \in R$, we see that if J is a nonzero right (left) ideal of R, then there exists an i for which $e_i J \neq 0$. Since R is semiprime, $e_i J e_i \neq 0$. Now, if $t \in T$, then

$$e_i t = t e_i \in (e_i C)_{n_i} = e_i R e_i.$$

Since t is regular in C and $e_i \in Z(C)$, we see that $e_i t$ is regular in $e_i C$; hence $e_i t$ is regular in $(e_i C)_{n_i} = e_i R e_i$. Therefore, if $J = r(t)$ is nonzero, then by the argument above, $e_i r(t) e_i \neq 0$ for some i. But $e_i t(e_i r(t) e_i) = e_i t r(t) e_i = 0$; hence by the regularity of $e_i t$ in $e_i R e_i$ we see that $e_i r(t) e_i = 0$, a contradiction. Therefore $r(t) = 0$, and similarly $\ell(t) = 0$.

Next we establish condition (2). Let E be an essential right ideal of R. We shall show that $E \cap T \neq \emptyset$. Since E is an essential right ideal of R and $e_i R$ is nonzero, $e_i R \cap E \neq 0$. Let $E_i = e_i R \cap E$, and note that $e_i E_i = E_i \subset E$. We shall show that $E_i e_i$ is an essential right ideal of $e_i R e_i$. If J is a nonzero right ideal of $e_i R e_i$, then $J = e_i N e_i$, where N is a right ideal of $e_i R$ and $N e_i R \neq 0$ by the semiprimeness of R. The essentiality of E implies that $N e_i R \cap E \neq 0$; but $N \subset e_i R$. Hence

$$0 \neq N e_i R \cap E = N e_i R \cap e_i R \cap E = N e_i R \cap E_i.$$

By the semiprimeness of R, this implies that $e_i(N e_i R \cap E_i) e_i \neq 0$. Thus,

$$0 \neq e_i N e_i \cap e_i E_i e_i = J \cap E_i e_i,$$

and consequently $E_i e_i$ is an essential right ideal of $e_i R e_i$.

Now, since $e_i R e_i = (e_i C)_{n_i}$ $(i = 1, \cdots, m+1)$, and since $e_i C$ is by Lemma 5 a semiprime Goldie ring, we see by Lemma 2 that $E_i e_i$ contains a regular element of $e_i C$ $(i = 1, \cdots, m+1)$. Thus we have for each i an element $e_i x_i$ such that

$e_i x_i \in E_i e_i \subset E$, where $x_i \in C$ and $e_i x_i$ is a regular element of $e_i C$.

Let $t = \sum_{i=1}^{m+1} e_i x_i$, and note that $t \in E \cap C$. We claim that t is a regular element of C. To see this, assume $ty = 0$, where $y \in C$. Since $y = \sum_{i=1}^{m+1} e_i y$ we have the relations

$$0 = ty = \left(\sum_{i=1}^{m+1} e_i x_i\right)\left(\sum_{i=1}^{m+1} e_i y\right) = \sum_{i=1}^{m+1} (e_i x_i)(e_i y),$$

where the right equality holds since $e_i e_j = 0$ whenever $i \neq j$. Since the sum $\sum_{i=1}^{m+1} e_i C$ is direct, $(e_i x_i)(e_i y) = 0$ for each i; but since $e_i x_i$ is regular in $e_i C$, we see that $e_i y = 0$ $(i = 1, \cdots, m+1)$. Hence $y = 0$ and we have shown that t is right-regular in C. Similarly, t is left-regular in C; thus $t \in T$, and $E \cap T \neq \emptyset$.

Finally, we show (3). Let $t \in T$, and let $0 \neq x \in R$. We must show that $tR \cap xR \neq 0$. Since $x = e_1 x + \cdots + e_{m+1} x \neq 0$, we see that $e_i x \neq 0$ for some i; hence, by the semiprimeness of R, $e_i x R e_i \neq 0$. Let $0 \neq e_i x r_i e_i \in e_i x R e_i$; then $e_i x r_i e_i \in (e_i C)_{n_i}$. As we noted earlier, $e_i t = t e_i$ is a regular element of the semiprime Goldie ring $(e_i C)_{n_i}$. Hence there exist s_i and w_i in $(e_i C)_{n_i}$ such that $(e_i x r_i e_i) w_i = ts_i \neq 0$. Therefore,

$$0 \neq x(r_i e_i w_i) = ts_i + \sum_{j \neq i} e_j x r_i e_i w_i.$$

If $\sum_{j \neq i} e_j x r_i e_i w_i = 0$, then $xR \cap tR \neq 0$, and the proof ends. If

$$\sum_{j \neq i} e_j x r_i e_i w_i \neq 0,$$

then $e_k x r_i e_i w_i \neq 0$ for some $k \neq i$. Denote $r_i e_i w_i$ by y_i; then

$$0 \neq xy_i = ts_i + \sum_{j \neq i} e_j x y_i,$$

and $e_k x y_i \neq 0$ for some $k \neq i$.

By an argument similar to that above, there are elements r_k and w_k with $y_k = r_k e_k w_k$ such that $(e_k x y_i) y_k = ts_k \neq 0$, and

$$0 \neq x y_i y_k = t s_i y_k + t s_k + \sum_{j \neq i,k} e_j x y_i y_k.$$

Continuing this process, we deduce after at most $m + 1$ steps that $0 \neq xw = ts$, and (3) is established. The proof is now completed.

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