THE CURVATURE OF $\alpha \mathrm{I} + \beta \mathrm{II} + \gamma \mathrm{III}$ ON A SURFACE IN A 3-MANIFOLD OF CONSTANT CURVATURE

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1. INTRODUCTION

There has been growing interest during recent years in the differential-geometric properties of nonstandard metrics on immersed surfaces. This interest has centered on the study of the geometry of the second and third fundamental forms II and III on compact surfaces in Euclidean 3-space E^3. (See [10] for an extensive review, or [12], [17], [18], and [21].) However, in [3] and [4], N. V. Efimov obtained impressive results about open complete surfaces in E^3 by studying the properties of the metric $|K|\,\mathrm{I}$, where I is the first fundamental form and K is Gauss curvature.

In this paper, we consider a surface immersed in a Riemannian 3-manifold of constant curvature, and for arbitrary constants α, β, and γ, we compute the curvature of the (not necessarily Riemannian) metric $\Lambda = \alpha \mathrm{I} + \beta \mathrm{II} + \gamma \mathrm{III}$ wherever Λ is nondegenerate. Our formula extends work due to N. Hicks [5] and J. A. Wolf [23], and it yields as a minor byproduct the fact that the curvature K(III) on such a surface is just the ratio of intrinsic to extrinsic curvature. It is remarkable that this simple formula (which can easily be verified directly) seems to have first appeared in the literature just recently as a special case of a more general result due to B. Wegner [22].

The applications included in this paper are fairly pedestrian. Perhaps others will find more significant uses for our formulas. But we hope this article will encourage the study of metrics other than I, II, and III that are nonetheless determined by the immersion of a surface in some Riemannian 3-manifold. (See [13], [14], and [15].) The goal of such efforts should be the accumulation of information useful in solving problems in-the-large.

2. THE BASIC COMPUTATION

Suppose $X: S \to \mathcal{M}$ is a C^3 immersion of a surface S in some Riemannian 3-manifold $\mathcal{M}$, and that ν is a unit normal vector field on the immersed surface. Let D denote the covariant differential in $\mathcal{M}$ [6, p. 56] and let $\cdot$ denote the inner product provided by the Riemannian metric on $\mathcal{M}$ [6, p. 21]. Then there are two restrictions [16, p. 527] on the fundamental forms

$$\mathrm{I} = DX \cdot DX = E\,dx^2 + 2F\,dx\,dy + G\,dy^2,$$

$$\mathrm{II} = -DX \cdot D\nu = L\,dx^2 + 2M\,dx\,dy + N\,dy^2$$

of the immersion. First, the intrinsic curvature K(I) [6, p. 29] of the official Riemannian metric I on S is related to the extrinsic curvature

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$$K = (LN - M^2)/(EG - F^2)$$

by the Gauss equation

$$K(I) = \mathscr{K} + K, \tag{1}$$

where $\mathscr{K}$ is the sectional curvature of $\mathscr{M}$ associated with the tangent plane to the immersed surface at any point [6, p. 73]. Second, the Codazzi-Mainardi equations must be satisfied. If $\mathscr{M}$ has constant sectional curvature $\mathscr{K}$, then the Codazzi-Mainardi equations are the same [11, p. 144] as they are in the classical case when $\mathscr{M} = E^3$. One can verify, using the Weingarten equations [16, p. 524] that the familiar relation

$$III = 2H\,II - K\,I \tag{2}$$

still holds, where $III = D\nu \cdot D\nu$ is the third fundamental form and H is mean curvature on S, given by

$$2H = (EN + GL - 2FM)/(EG - F^2).$$

If k_1 and k_2 are the principal curvatures on S, then $K = k_1 k_2$ and $2H = k_1 + k_2$. We shall use the notation $2H' = |k_2 - k_1|$ and $2H'' = k_2 - k_1$, so that

$$H' = \sqrt{H^2 - K} = |H''|.$$

Umbilics occur on S wherever H' (or H'') vanishes.

For the moment, assume that S is C^∞-immersed in $\mathscr{M}$. Then in the neighborhood of each nonumbilic on S there are C^∞ line-of-curvature coordinates x, y such that

$$I = E\,dx^2 + G\,dy^2, \qquad II = k_1 E\,dx^2 + k_2 G\,dy^2, \qquad III = k_1^2 E\,dx^2 + k_2^2 G\,dy^2.$$

The classical formula [20, p. 141]

$$4(EG)^2 K(I) = G_x(EG)_x + E_y(EG)_y - 2EG(G_{xx} + E_{yy}) \tag{3}$$

gives $K(I)$. Setting $\varepsilon_1 = \alpha + \beta k_1 + \gamma k_1^2$ and $\varepsilon_2 = \alpha + \beta k_2 + \gamma k_2^2$, we obtain the formula

$$\Lambda = \alpha I + \beta II + \gamma III = \varepsilon_1 E\,dx^2 + \varepsilon_2 G\,dy^2,$$

so that the equation

$$\begin{aligned} 4(\varepsilon_1 \varepsilon_2 EG)^2 K(\Lambda) &= (\varepsilon_2 G)_x (\varepsilon_1 \varepsilon_2 EG)_x \\ &\quad + (\varepsilon_1 E)_y (\varepsilon_1 \varepsilon_2 EG)_y - 2\varepsilon_1 \varepsilon_2 EG[(\varepsilon_2 G)_{xx} + (\varepsilon_1 E)_{yy}] \end{aligned} \tag{4}$$

gives $K(\Lambda)$ wherever $\varepsilon_1 \varepsilon_2 \neq 0$. The Codazzi-Mainardi equations reduce [20, p. 149] to

$$(k_2)_x G = -H'' G_x, \qquad (k_1)_y E = H'' E_y. \tag{5}$$

If we set $\varepsilon = \alpha + \beta H + \gamma K$, then (5) gives the equations

$$(\varepsilon_2 G)_x = \varepsilon G_x, \qquad (\varepsilon_1 E)_y = \varepsilon E_y, \tag{6}$$

so that substitution of (6) in (4) yields the formula

$$\begin{aligned}4(\varepsilon_1\varepsilon_2)^2 EGK(\Lambda) &= 4\varepsilon\varepsilon_1\varepsilon_2 EGK(I)\\ &\quad + \varepsilon[G_x(\varepsilon_1\varepsilon_2)_x + E_y(\varepsilon_1\varepsilon_2)_y] - 2\varepsilon_1\varepsilon_2(\varepsilon_x G_x + \varepsilon_y E_y).\end{aligned} \tag{7}$$

Using the values for G_x and E_y provided by (5), we get

$$\begin{aligned}2(\varepsilon_1\varepsilon_2)^2 K(\Lambda) &= 2\varepsilon\varepsilon_1\varepsilon_2 K(I)\\ &\quad + (\beta^2 - 4\alpha\gamma)\left\{\frac{dk_2}{ds_1}\left[(\beta+\gamma H)K\frac{d}{ds_1}\log\left|\frac{k_2}{k_1}\right| - \gamma H''\frac{dK}{ds_1} + 2\alpha\frac{dH''}{ds_1}\right]\right.\\ &\quad \left. + \frac{dk_1}{ds_2}\left[(\beta+\gamma H)K\frac{d}{ds_2}\log\left|\frac{k_1}{k_2}\right| + \gamma H''\frac{dK}{ds_2} - 2\alpha\frac{dH''}{ds_2}\right]\right\},\end{aligned} \tag{8}$$

where $d/ds_1 = (1/\sqrt{E})\partial/\partial x$ and $d/ds_2 = (1/\sqrt{G})\partial/\partial y$ are the directional derivatives in the principal directions $dy \equiv 0$, $dx > 0$ and $dx \equiv 0$, $dy > 0$, respectively.

Suppose now that S is only C^4-immersed in $\mathcal{M}$. Then $K(\Lambda)$ is defined and continuous wherever Λ is nondegenerate. Note that Λ is degenerate if and only if $\varepsilon_1\varepsilon_2 = 0$. At an umbilic, $\varepsilon = \varepsilon_1 = \varepsilon_2$, so that Λ is degenerate if and only if $\varepsilon = 0$. Of course, we cannot automatically use (8) to compute $K(\Lambda)$ at nonumbilics where $\varepsilon_1\varepsilon_2 \neq 0$, since line-of-curvature coordinates need only be C^2-smooth.

On a C^4-immersed S, it will be convenient to distinguish as *regular* any umbilic in whose neighborhood there exist C^3 coordinates orthogonal with respect to *both* I and II. (Every interior point of the umbilic set on S is regular, for example.) The set containing all nonumbilics and all regular umbilics will be called the *regular set*. Note that the regular set is open and dense on S.

3. RESULTS

Throughout this section, S denotes a surface C^4-immersed in a C^∞ Riemannian 3-manifold $\mathcal{M}$ of constant curvature $\mathcal{K}$. As in Section 2, we set $\Lambda = \alpha I + \beta II + \gamma III$ for constants α, β, and γ, with

$$\varepsilon = \alpha + \beta H + \gamma K, \qquad \varepsilon_1 = \alpha + \beta k_1 + \gamma k_1^2, \qquad \varepsilon_2 = \alpha + \beta k_2 + \gamma k_2^2.$$

In the neighborhood of each nonumbilic on S, d/ds_1 and d/ds_2 are unit tangent vector fields in the appropriately oriented principal directions associated with the principal curvatures k_1 and k_2, respectively. The following theorem is the main result of this paper. (A generalization of this theorem for hypersurfaces of arbitrary dimension in Riemannian manifolds of constant sectional curvature has recently been obtained by J. D'Atri in [2].)

THEOREM. *At a nonumbilic where* $\varepsilon_1\varepsilon_2 \neq 0$, *the curvature* $K(\Lambda)$ *is given by* (8). *At regular umbilics where* $\varepsilon \neq 0$,

$$K(\Lambda) = K(I)/\varepsilon. \tag{9}$$

Proof. Let p be a nonumbilic point on S where $\varepsilon_1\varepsilon_2 \neq 0$. Consider a C^∞-immersion $\hat{X}$: $U \to \mathcal{M}$, where U is some neighborhood of p on S. Choose $\hat{X}$ so that the nth partial derivatives of X and $\hat{X}$ at p (with respect to local coordinates near p) agree for $n = 0, 1, 2, 3$, and 4. It follows that at the one point p, the forms I, II, and III as well as the curvatures H, K, H', H'', k_1, and k_2 must coincide for the two immersions X and $\hat{X}$. In particular, p is nonumbilic for $\hat{X}$ with $\varepsilon_1 \varepsilon_2 \neq 0$. Moreover, $K(\Lambda)$ exists and must be the same for both X and $\hat{X}$ at p. Finally, everything on the right side of (8) makes sense for X as well as for $\hat{X}$ near p, and the right side of (8) must be the same when computed for the two immersions at p. But our previous computations show that (8) gives $K(\Lambda)$ at p for $\hat{X}$. Thus (8) also gives $K(\Lambda)$ at p for X.

If $\varepsilon \neq 0$ at a regular umbilic that is not an interior point of the umbilic set, then (8) holds at a sequence of nonumbilics converging to p. By continuity, (8) holds also at p. Since $H'' = 0$ at p, (5) shows that $dk_2/ds_1 = dk_1/ds_2 = 0$ at p, so that (9) is valid there.

If p is an interior umbilic on S, use C^3 coordinates x, y in some neighborhood of p that are orthogonal with respect to both I and II. Then (5) and (6) are valid with $\varepsilon_1 \equiv \varepsilon_2 \equiv \varepsilon$. Hence (7) easily reduces to (9). (For a discussion of all-umbilic portions of S, see [19].)

Remark 1. If $\varepsilon \neq 0$ at a nonregular umbilic p that is a limit point of regular umbilics on S, then (9) still holds at p, by continuity. To find $K(\Lambda)$ at any other nonregular umbilic where $\varepsilon \neq 0$, choose a sequence of nonumbilic points converging to p, and take the limit of the values $K(\Lambda)$ provided by (8) at the points of the sequence. This process will not always yield the value given by (9). (See Remark 10, or Corollary 11 below.)

Remark 2. If H is constant on S, computation reduces (8) to the equation

$$(\varepsilon_1 \varepsilon_2)^2 K(\Lambda) = \varepsilon\varepsilon_1 \varepsilon_2 K(I) + (\beta^2 - 4\alpha\gamma)(2\gamma H^2 + \beta H - \gamma K + \alpha)\, |\mathrm{grad}\, k_i|^2 \qquad (i = 1, 2).$$

(See Corollaries 2 and 11.) Simplifications of (8) also occur if $\beta^2 = 4\alpha\gamma$, if K, H'', k_1, k_2, or k_1/k_2 is constant, or if S is a surface of revolution in a complete, simply connected $\mathcal{M}$. (See Corollaries 4, 10, 12, and 13.)

The following extends a result due to Wolf [23].

COROLLARY 1. *If* $\varepsilon \equiv 0$ *on* S, *then* $K(\Lambda) \equiv 0$ *wherever* $\varepsilon_1 \varepsilon_2 \neq 0$.

Proof. Here Λ is degenerate at every umbilic, since $\varepsilon = 0$. Thus (8) applies wherever $\varepsilon_1 \varepsilon_2 \neq 0$, and further computation yields $K(\Lambda) \equiv 0$. (If the immersion is smoother, $K(\Lambda) \equiv 0$ follows easily from (7).)

Corollaries 2 and 3 extend results due to Hicks [5] and Wolf [23].

COROLLARY 2. *If* H *is constant, and if* S *is* C^2*-immersed in* $\mathcal{M}$, *the metric* $\Lambda = -HI + II$ *is flat and indefinite. If* $H \equiv 0$ (*so that* S *is minimal*), II *is flat and indefinite wherever* $K \neq 0$.

Proof. Because H is constant, one can choose coordinates on S so as to make its immersion C^∞. (We thank Joel Spruck for this fact, and for the reference [7].) Apply Corollary 1 with $\alpha = -H$, $\beta = 1$, and $\gamma = 0$. One easily verifies that Λ is indefinite at nonumbilics and degenerate at umbilics. If $H \equiv 0$, then $\Lambda = II$ and $K \leq 0$, with umbilics characterized by $K = 0$.

Remark 3. In Corollary 2, Λ is not the flat metric H'I studied, for example, in [11] and [24]. For suitable coordinates x, y near any nonumbilic, $\Lambda = dx^2 - dy^2$ while $H'I = dx^2 + dy^2$. (See [11].)

COROLLARY 3. *If* K *is a nonzero constant on* S, *then away from umbilics, the metric* $\Lambda = -\mathrm{KI}+\mathrm{III}$ *is flat, definite where* $K<0$, *and indefinite where* $K>0$.

Proof. Apply Corollary 1 with $\alpha=-K$, $\beta=0$, and $\gamma=1$. Simple arithmetic based on (2) shows that Λ is degenerate at umbilics, definite where $K<0$, and indefinite where $K>0$.

COROLLARY 4. *If* α *and* γ *are nonnegative constants, then wherever* $\varepsilon = \alpha \pm 2\sqrt{\alpha\gamma}H + \gamma K \neq 0$ *on* S, *the curvature of* $\Lambda = \alpha\mathrm{I} \pm 2\sqrt{\alpha\gamma}\mathrm{II} + \gamma\mathrm{III}$ *is given by the equation*

$$K(\Lambda) = K(\mathrm{I})/\varepsilon. \tag{10}$$

Proof. Apply the Theorem with $\beta^2 = 4\alpha\gamma$. Here $\varepsilon_1 = (\sqrt{\alpha} \pm \sqrt{\gamma}k_1)^2$ and $\varepsilon_2 = (\sqrt{\alpha} \pm \sqrt{\gamma}k_2)^2$, so that $\varepsilon_1\varepsilon_2 = \varepsilon^2$. If $\varepsilon \neq 0$ at a nonumbilic, then $\varepsilon_1\varepsilon_2 \neq 0$, and (8) reduces to (10). At regular umbilics, $\varepsilon_1=\varepsilon_2=\varepsilon$, so that (10) simply restates (9) if $\varepsilon \neq 0$. Because the regular set is dense on S, (10) holds wherever $\varepsilon_1\varepsilon_2 \neq 0$, by continuity.

Remark 4. If $K \equiv 0$ in Corollary 4, then $\Lambda = \alpha\mathrm{I} + 2(\gamma H \pm \sqrt{\alpha\gamma})\mathrm{II}$ and $\varepsilon = \alpha \pm 2\sqrt{\alpha\gamma}H$. If H is bounded, one can take α/γ so large that (10) holds everywhere. If $K(\Lambda)$ is constant and $K \equiv 0$, then either $K(\mathrm{I}) = \mathscr{K}$ is zero by (1), or else H is constant. The first case occurs for every developable surface in E^3. The second case (H constant, $K\equiv 0$, and $\mathscr{K}\neq 0$) can occur only for totally geodesic surfaces, so that $H \equiv K \equiv 0$.

Remark 5. If $H \equiv 0$ in Corollary 4, then $\Lambda = (\alpha - \gamma K)\mathrm{I} \pm 2\sqrt{\alpha\gamma}\mathrm{II}$ and $\varepsilon = \alpha + \gamma K$. If K is bounded, one can take α/γ so large that (10) holds everywhere.

We thank J. D'Atri for pointing out that the following is a special case of a result in [22].

COROLLARY 5. *Wherever* $K \neq 0$, $K(\mathrm{III}) = K(\mathrm{I})/K$.

Proof. Apply Corollary 4 with $\alpha = 0$ and $\gamma = 1$.

Remark 6. Using (1) and Corollary 5, we see that $K(\mathrm{III}) = 1 + (\mathscr{K}/K)$. Thus $K(\mathrm{III}) \equiv 1$ if and only if $\mathscr{K} \equiv 0$. (For a surface in E^3, III is the pull-back of the metric on the unit 2-sphere under the Gauss spherical-image map.)

Remark 7. If S is immersed in the unit sphere $\mathscr{M} = \mathscr{S}^3 \subset E^4$, the unit normal to S describes a surface $\hat{S}$ in $\mathscr{S}^3$ called the polar surface of S. Wherever $K \neq 0$ on S, $\hat{S}$ is nondegenerate and its first fundamental form pulls back to III on S. (We thank S. S. Chern for reminding us of this.) Note that S and $\hat{S}$ have the same intrinsic curvature, wherever $K = 1$ on S.

Remark 8. If $K \neq 0$, then III is flat if and only if I is flat, and $K(\mathrm{III})$ is constant if and only if $\mathscr{K}/K$ is constant.

COROLLARY 6. *Suppose* III *is complete, and* $\mathscr{K}/K \geq c - 1$, *where* $c > 0$ *is constant and* $K \neq 0$. *Then* S *is compact, while* K, K(I), *and the Euler characteristic* χ *of* S *are all strictly positive.*

Proof. Here (1) and Corollary 5 yield the equation

$$K(\mathrm{III}) = (\mathscr{K}/K) + 1 \geq c > 0. \tag{11}$$

Since III is complete, the Bonnet-Hopf-Rinow theorem [9] implies that S is compact. By (11), the total curvature of III on S must be positive. Thus $\chi > 0$, by the

Gauss-Bonnet theorem, and the total curvature of I on S must be positive as well. Since K never changes sign, while $K(III) = K(I)/K > 0$, it follows that $K(I)$ never changes sign. Thus $K(I)$ is positive, and so is K.

COROLLARY 7. *If* α *and* γ *are nonnegative constants, while* $\Lambda = \alpha I \pm 2\sqrt{\alpha\gamma} II + \gamma III$ *and* $\varepsilon = \alpha \pm 2\sqrt{\alpha\gamma} H + \gamma K \neq 0$ *on* S, *then*

$$\int\int K(\Lambda)\, dA_\Lambda = (\text{sign } \varepsilon) \int\int K(I)\, dA_I.$$

In particular, if $K \neq 0$ *on* S, *then*

$$\int\int K(III)\, dA_{III} = (\text{sign } K) \int\int K(I)\, dA_I.$$

Proof. The element of area associated with Λ is the square root of the absolute value of its determinant. Thus it is easy to verify that $dA_\Lambda = |\varepsilon|\, dA_I$. Using Corollary 4, we see that

$$\int\int K(\Lambda)\, dA_\Lambda = \int\int K(I) \frac{|\varepsilon|}{\varepsilon}\, dA_I = (\text{sign } \varepsilon) \int\int K(I)\, dA_I.$$

In particular, when $\Lambda = III$, then $\varepsilon = K$.

COROLLARY 8. *If* $K < 0$ *and* S *is compact, then the Euler characteristic* χ *of* S *is zero, and* $\mathcal{K} > 0$.

Proof. By the Gauss-Bonnet theorem, the total curvatures $T(I)$ of I and $T(III)$ of III both equal $2\Pi\chi$. By Corollary 7, $T(I) = -T(III)$, so that $T(I) = T(III) = \chi = 0$. (One can also deduce that $\chi = 0$ from the fact that S is umbilic-free. See [20, p. 244].) If $\mathcal{K} \leq 0$, (1) would imply that $K(I) < 0$, yielding the contradiction $T(I) < 0$.

Remark 9. The hypotheses of Corollary 8 are satisfied, for example, by a flat torus in the 3-sphere.

COROLLARY 9. *At nonumbilics where* $K \neq 0$,

$$K(II) = (H/K)\, K(I) + (1/4K)\left\{\frac{dk_2}{ds_1}\left(\frac{d}{ds_1}\log\left|\frac{k_2}{k_1}\right|\right) + \frac{dk_1}{ds_2}\left(\frac{d}{ds_2}\log\left|\frac{k_1}{k_2}\right|\right)\right\}. \tag{12}$$

At regular umbilics where $K \neq 0$,

$$K(II) = (H/K)\, K(I) = HK(III). \tag{13}$$

Proof. Use the theorem with $\alpha = \gamma = 0$ and $\beta = 1$, and for (13), apply Corollary 5.

Remark 10. By the procedure indicated in Remark 1, $K(II)$ can be found at all nonregular umbilics where $K \neq 0$. Note that (13) need not be valid at all such points. If the equation $z = f(x, y)$ describes a surface in E^3 with $f = f_x = f_y = f_{xy} = 0$ and $f_{xx} = f_{yy}$ at $x = y = 0$, then $K(II)$ is defined at the umbilic $x = y = z = 0$ and given by (13) if and only if $H \neq 0$ and $f_{xxy}(f_{yyy} - f_{xxy}) = f_{yyx}(f_{yyx} - f_{xxx})$. In particular, (13) fails at the umbilic $x = y = z = 0$ on the surface described by

$$z = x^2 + x^2 y + y^2 + (y^3/3).$$

Remark 11. In [5], Hicks obtained (12) with $K = K(I)$ at nonumbilic points of a surface C^∞-immersed in E^3. (Note that Hicks takes $H = k_1 + k_2$, which is twice our H.) In obtaining (8), we were guided by Hicks's formula.

COROLLARY 10. *If* $k_1 = ck_2$ *for a constant* $c \neq 0$, *then* K(II) *is given by* (13) *wherever* $K \neq 0$.

Proof. If $c \neq 1$, there are no umbilics, and where $K \neq 0$, (12) applies, reducing to (13) since k_1/k_2 and k_2/k_1 are constant. If $c = 1$, S is totally umbilic, and (13) holds wherever $K \neq 0$.

If H is constant on S, and $II = L\,du^2 + 2M\,du\,dv + N\,dv^2$ for isothermal coordinates u, v on S, then $\phi = L - N - 2iM$ is an analytic function of $w = u + iv$ whose zeros coincide with the umbilics on S. Thus, if S is not totally umbilic, its umbilics are isolated, and the index of an isolated umbilic p in the net of lines-of-curvature is $-n/2$, where n is the order of the zero of ϕ at p. (See [8].) This leads to the following extension of a result of Hicks [5].

COROLLARY 11. *Suppose that* H *is constant, and* S *is* C^2*-immersed in* $\mathcal{M}$. *At a nonumbilic where* $K \neq 0$,

$$K(II) = H\{K(III) + (1/2K^2)\,|\text{grad } k_i|^2\} \qquad (i = 1, 2). \tag{14}$$

At an umbilic where $K \neq 0$, K(II) *is given by* (13) *unless the umbilic is isolated and has index* -1. *Wherever* $K = 0$, $K(II) \geq HK(III)$.

Proof. Once again, we can choose coordinates on S so that its immersion is C^∞. (See [7].) At a nonumbilic, $dk_1/ds_1 = -dk_2/ds_1$ and $dk_1/ds_2 = -dk_2/ds_2$, since $k_1 + k_2$ is constant. If $K \neq 0$ at a nonumbilic, (12) reduces to (14), and $K(II) > HK(III)$. If umbilics are isolated, $K(II) \geq HK(III)$ wherever $K \neq 0$, by continuity. Otherwise, S is all-umbilic, and $K(II) \equiv HK(III)$. In some neighborhood of each isolated umbilic p, there are isothermal coordinates x, y in terms of which $\phi = 4z^n$, where $n > 0$ is an integer, and $z = x + iy$ vanishes at p. (See [1].) Since $I = \lambda(dx^2 + dy^2)$ and

$$II = (\lambda H + 2\,\Re z^n)dx^2 - 4\,\Im z^n\,dx\,dy + (\lambda H - 2\,\Re z^n)dy^2,$$

computation of K(II) at p yields the formula

$$K(II) = HK(III) + (1/\lambda H)^2(\Re z^n)_{xx},$$

if $K \neq 0$ at p. But $(\Re z^n)_{xx} = 0$ at $z = 0$ unless $n = 2$. Thus if $K \neq 0$ at p, (13) fails *if and only if* the index of p is -1.

COROLLARY 12. *If either principal curvature is a constant* c, *then wherever* $K \neq 0$, $K(II) \geq HK(III)$ *if* $c > 0$, *and* $K(II) \leq HK(III)$ *if* $c < 0$.

Proof. We may assume that $k_1 = c$. At nonumbilics where $K \neq 0$, (12) implies that

$$K(II) = HK(III) + (1/4ck_2^2)\,(dk_2/ds_1)^2.$$

Since the corollary holds on the regular set, which is dense on S, a continuity argument completes the proof.

COROLLARY 13. *Suppose that* $\mathcal{M}$ *is complete and simply connected, and that* S *is a surface of revolution. Let* k_1 *denote principal curvature in the direction of meridians. At every nonumbilic where* $K \neq 0$,

$$K(II) = HK(III) + (1/4K)\left\{\frac{dk_2}{ds_1}\left(\frac{d}{ds_2}\log\left|\frac{k_1}{k_2}\right|\right)\right\}. \tag{15}$$

At every umbilic where $K \neq 0$, $K(II)$ *is given by* (13).

Proof. Along each parallel, k_1 is constant. Thus (12) gives (15) at every non-umbilic. If $K \neq 0$ at an umbilic p off the axis of revolution, then p is regular, so that (13) holds. In case $K \neq 0$ at an umbilic p on the axis of revolution, pick a meridian ending at p, and continue along its opposite meridian leaving p. Parametrize the smooth arc μ so obtained by arc length s, with $s = 0$ at p. Then k_1 and k_2 are even, differentiable functions of s along μ, so that $dk_1/ds = dk_2/ds = 0$ at $s = 0$. But (15) holds on μ for all small values of $s = s_1 > 0$, reducing to (13) at umbilics. Computing the limit of $K(II)$ along μ as $s = s_1 > 0$ goes to zero, we get (13) at p.

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