REDUCTIVE OPERATORS THAT COMMUTE WITH A COMPACT OPERATOR

Robert L. Moore

A bounded operator T on a Hilbert space $\mathscr{H}$ is *reductive* if every invariant subspace of T reduces T. It is well known that every reductive operator is normal if and only if every operator has a nontrivial invariant subspace [4]. In 1963, T. Andô [1] showed that every compact reductive operator is normal, and in 1968 P. Rosenthal [10] was able to extend this result by showing that every polynomially compact reductive operator is normal. In this paper we use the work of V. I. Lomonosov [7] to generalize these results; the principal theorem is that a reductive operator that commutes with an injective compact operator must be normal.

Rosenthal [11] has recently shown that if an injective compact operator is contained in the commutant of a reductive algebra, then the reductive algebra must be self-adjoint. In addition, recent papers by E. Azoff [2] and A. I. Loginov and V. S. Šul'man [6] contain generalizations of Rosenthal's result. Rosenthal's theorem is stronger than our Theorem 1; however, the techniques used herein are quite different from Rosenthal's, and several of the intermediate results are of interest in themselves. The proof of the first proposition is essentially in [1] and [10]; we include it here for completeness.

PROPOSITION 1. *Let* C *be a nonzero compact operator. Let* $\mathscr{G}$ *be a family of subspaces with the following properties:*

(i) $\mathscr{G}$ *is totally ordered by reverse inclusion;*

(ii) *each subspace* $\mathscr{M}$ *in* $\mathscr{G}$ *reduces* C;

(iii) *for each* $\mathscr{M}$ *in* $\mathscr{G}$, $\|C \mid \mathscr{M}\| = \|C\|$.

Then the intersection $\mathscr{M}_0 = \bigcap \mathscr{G}$ *is nonzero and* $\|C \mid \mathscr{M}_0\| = \|C\|$.

Proof. For each $\mathscr{M} \in \mathscr{G}$, $C \mid \mathscr{M}$ is a compact operator, and since a compact operator achieves its norm, there is a unit vector $f_{\mathscr{M}} \in \mathscr{M}$ such that $\|Cf_{\mathscr{M}}\| = \|C \mid \mathscr{M}\| = \|C\|$. Because the $f_{\mathscr{M}}$ all lie in the unit ball of the Hilbert space and the unit ball is weakly compact, there is a weak cluster point f_0 of the set $\{f_{\mathscr{M}}\}$ in the unit ball. We consider $\{f_{\mathscr{M}}\}$ as a net, indexed by the totally ordered set $\mathscr{G}$; some subnet of $\{f_{\mathscr{M}}\}$ converges to f_0, and we assume without losing generality that the full net $\{f_{\mathscr{M}}\}$ converges to f_0. Since C is compact, $Cf_{\mathscr{M}} \to Cf_0$ in norm, whence $\|Cf_0\| = \|C\|$; because C is nonzero, f_0 is nonzero. Moreover, for each $\mathscr{M}'$ in $\mathscr{G}$, the tail of the net $\{f_{\mathscr{M}}\}$ lies in $\mathscr{M}'$ (since $\mathscr{G}$ is ordered by reverse inclusion), so that f_0 lies in $\mathscr{M}_0$. Thus $\mathscr{M}_0$ is nonzero and $\|C \mid \mathscr{M}_0\| = \|C\|$.

Received April 7, 1975.

This paper is a portion of the author's thesis, written at Indiana University under the direction of P. R. Halmos, to whom the author is grateful for his guidance and encouragement. Thanks are also due the referee for several important suggestions.

Michigan Math. J. 22 (1975).

We shall call a pair of operators $\{B_1, B_2\}$ *completely reducible* if, whenever $\mathcal{M}$ reduces both B_1 and B_2, and $\dim \mathcal{M} \geq 2$, then $\mathcal{M}$ properly contains a nonzero subspace that reduces B_1 and B_2.

PROPOSITION 2. *Let* C *be a nonzero compact operator. If the set* $\{B, C\}$ *is completely reducible, then* B *and* C *have a common reducing eigenvector.*

Proof. Let $\mathcal{G}' = \{\mathcal{M}: \mathcal{M}$ reduces B and C and $\|C \mid \mathcal{M}\| = \|C\|\}$, partially ordered by reverse inclusion. By the Hausdorff Maximality Principle, there is a maximal totally ordered subset of $\mathcal{G}'$, which we call $\mathcal{G}$. Let $\mathcal{M}_0 = \bigcap \mathcal{G}$. Then $\mathcal{M}_0$ reduces B and C, and according to the previous proposition, $\mathcal{M}_0$ is nonzero and $\|C \mid \mathcal{M}_0\| = \|C\|$. If $\dim \mathcal{M}_0 \geq 2$, complete reducibility gives a proper subspace $\mathcal{M}'$ of $\mathcal{M}_0$ that reduces B and C. Since $\|C\| = \|C \mid \mathcal{M}_0\|$ is the larger of $\|C \mid \mathcal{M}'\|$ and $\|C \mid \mathcal{M}_0 \ominus \mathcal{M}'\|$, either $\mathcal{M}'$ or $\mathcal{M}_0 \ominus \mathcal{M}'$ lies in $\mathcal{G}'$ and is strictly smaller than $\mathcal{M}_0$, and this contradicts the construction of $\mathcal{G}$ and $\mathcal{M}_0$. Thus $\dim \mathcal{M}_0 \leq 1$, and since $\mathcal{M}_0$ is nonzero, the dimension must be 1. Hence each unit vector in $\mathcal{M}_0$ must be a common reducing eigenvector for B and C.

LEMMA 1. *Suppose that* R, S, *and* X *are operators on* $\mathcal{H}$ *for which* $R \oplus S$ *is reductive and* $RX = XS$. *Then* $R^*X = XS^*$ *as well (that is, if* X *intertwines* R *and* S, *it also intertwines* R^* *and* S^*).

Proof. The set $\mathcal{M} = \{\langle Xf, f\rangle : f \in \mathcal{H}\}$ is a subspace of $\mathcal{H} \oplus \mathcal{H}$. It is invariant under $R \oplus S$, because

$$\begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} Xf \\ f \end{pmatrix} = \begin{pmatrix} RXf \\ Sf \end{pmatrix} = \begin{pmatrix} XSf \\ Sf \end{pmatrix}.$$

Since $R \oplus S$ is reductive, $\mathcal{M}$ is invariant under $(R \oplus S)^*$. Thus, for each $f \in \mathcal{H}$, the vector $\begin{pmatrix} R^*Xf \\ S^*f \end{pmatrix}$ must lie in $\mathcal{M}$; it follows that $R^*Xf = XS^*f$ for all f.

A subspace $\mathcal{M}$ is *hyperinvariant* for an operator A if $\mathcal{M}$ is invariant for every operator in the commutant of A. If $\mathcal{M}$ reduces every operator in the commutant of A, we call $\mathcal{M}$ *hyperreducing* for A.

PROPOSITION 3. *If* A *is reductive, then every hyperinvariant subspace of* A *is hyperreducing.*

Proof. Suppose that $\mathcal{M}$ is hyperinvariant for A, and suppose that B commutes with A. Then $\mathcal{M}$ is invariant under B, and with respect to the decomposition $\mathcal{M} \oplus \mathcal{M}^\perp$ we can write A and B as operator matrices as follows:

$$A = \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} E & F \\ 0 & G \end{pmatrix}.$$

Since $AB = BA$, it is true that $RF = FS$, and by Lemma 1, $R^*F = FS^*$ as well. The last equation is the same as $F^*R = SF^*$, and this means that A commutes with the operator

$$D = \begin{pmatrix} 0 & 0 \\ F^* & 0 \end{pmatrix}.$$

But $\mathcal{M}$ is hyperinvariant for A, and hence is invariant under D. Thus $F^* = 0$, or $F = 0$ and $\mathcal{M}$ reduces B.

An equivalent statement of Proposition 3 is that the commutant of a reductive operator is a reductive algebra. In [6], Loginov and Šul'man announce the following theorem: *The commutant of a commutative reductive set of operators is reductive.* The latter result can be proved by means of essentially the technique that is employed in the proof of Proposition 3.

We can now prove the central result:

THEOREM 1. *If* A *is reductive and* C *is an injective compact operator that commutes with* A, *then* A *is diagonal, and hence normal.*

Proof. Let $\mathcal{E}$ be the subspace spanned by all the eigenvectors of A. Then $\mathcal{E}$ is invariant under C, and since each eigenvector of A is an eigenvector of A^*, the subspace $\mathcal{E}$ reduces C. Let A_1 and C_1 be the restrictions of A and C to $\mathcal{E}^\perp$. Then A_1 and C_1 satisfy the hypotheses of the theorem, and A_1 has no eigenvalues. Assertion: The pair $\{A_1, C_1\}$ is completely reducible. Reason: Let $\mathcal{M}$ be a subspace of $\mathcal{E}^\perp$, with dimension no less than two, that reduces A_1 and C_1, and let A_2 and C_2 be the restrictions of A_1 and C_1 to $\mathcal{M}$. Then A_2 is nonscalar (since A_1 has no eigenvalues) and C_2 is a nonzero compact operator that commutes with A_2. Lomonosov's result [7] therefore implies that A_2 has a hyperinvariant subspace, which by the preceding proposition is hyperreducing and therefore reduces A_1 and C_1. Thus $\{A_1, C_1\}$ is completely reducible. But then, by Proposition 2, A_1 and C_1 have a common reducing eigenvector. This last statement contradicts the construction of A_1. We deduce that $\mathcal{E}^\perp = 0$, and therefore that A is diagonal.

We point out that in order to prove Theorem 1, we need some restriction on the kernel of the compact operator. Simply requiring C to be nonzero is not sufficient; for instance, if C has the form $\begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}$ and if A is $\begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix}$, then the fact that $AC = CA$ yields no information about S at all.

The previously mentioned theorem of Rosenthal follows easily from Theorem 1:

COROLLARY 1. *If* A *is reductive and polynomially compact, then* A *is normal.*

Proof. Let $p(A) = C$ be compact, and let $\mathcal{K} = \ker C$. Then, since $AC = CA$, we see that $\mathcal{K}$ reduces A, and therefore $\mathcal{K}$ reduces C. The operator $A \mid \mathcal{K}^\perp$ commutes with the injective compact operator $C \mid \mathcal{K}^\perp$, and thus $A \mid \mathcal{K}^\perp$ is normal, by Theorem 1. On the other hand, $p(A \mid \mathcal{K}) = C \mid \mathcal{K} = 0$; that is, $A \mid \mathcal{K}$ is algebraic, so that $A \mid \mathcal{K}$ is normal, by Lemma 9.3 of [9]. Hence A itself is normal.

It is obvious from Theorem 1 that if A is reductive and commutes with an injective compact C, then A^* commutes with C. In fact, for this result it is possible to dispense with the hypothesis of injectivity.

THEOREM 2. *If* A *is reductive and commutes with a compact operator* C, *then* A^* *commutes with* C.

Proof. Let $\mathcal{M}$ be the largest subspace of the kernel of $A^*C - CA^*$ that is invariant under A and C. We shall show that $\mathcal{M}^\perp$ is the zero subspace. Using the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, we write A and C as operator matrices:

$$A = \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} \quad C = \begin{pmatrix} E & F \\ 0 & G \end{pmatrix}.$$

Commutativity requires that R commute with E, that S commute with G, and that RF = FS; and, by construction of $\mathcal{M}$, the operator R^* commutes with E as well. We assert that ker G = 0. To establish this, note that ker G and $\mathcal{M}$ are invariant under A (since S and G commute), and therefore $\mathcal{M} \oplus \ker G$ is invariant under A. Further, it is clear from the matrix computation that C takes every vector in ker G into $\mathcal{M}$; thus, $\mathcal{M} \oplus \ker G$ is also invariant under C. Finally, matrix computation shows that if g lies in ker G, then

$$(A^*C - CA^*)\langle 0, g \rangle = \langle (R^*F - FS^*)g, -GS^*g \rangle.$$

Since RF = FS and $R \oplus S$ is reductive, Lemma 1 shows that $R^*F = FS^*$. Moreover, S commutes with G, so that ker G is invariant under S, and hence under S^* (S being reductive), whence $GS^*g = 0$. Therefore, ker G is contained in $\ker(A^*C - CA^*)$. We have shown that $\mathcal{M} \oplus \ker G$ is invariant under A and C and that it is a subspace of $\ker(A^*C - CA^*)$; but $\mathcal{M}$ is maximal among such subspaces. Hence it follows that ker G = 0.

S is a reductive operator, and it commutes with the compact operator G, which we have shown to be injective. Theorem 1 then asserts that S is normal, and the Fuglede theorem ensures that S^* commutes with G. We already know that $R^*F = FS^*$ and that R^* commutes with E. These three facts suffice to show that A^* commutes with C on all of H. By our choice of $\mathcal{M}$, we can conclude that $\mathcal{M}^\perp = 0$.

Corollary 2 is a previously announced result of the author [8]:

COROLLARY 2. *If a reductive operator* A *is the sum of a normal operator and a commuting compact operator, then* A *is normal.*

Proof. Let A = N + C, where N is normal, C is compact, and NC = CN. Then AC = CA; thus, by Theorem 2, $A^*C = CA^*$. Furthermore, $N^*C = CN^*$, by the Fuglede theorem. It follows that $C^*C = CC^*$, so that C is normal and A, being the sum of commuting normal operators, is normal.

Two possibilities for extending Theorem 1 suggest themselves:

1. If C is an injective compact operator, if B is nonscalar, and if AB = BA and BC = CB, then Lomonosov's result ensures the existence of an invariant subspace for A. Question: If A is reductive and B has no eigenvalues of infinite multiplicity, is A normal? (The restriction on the eigenvalues of B is necessary for reasons similar to those in the paragraph following Theorem 1.)

2. Recently, H. Kim, C. Pearcy, and A. Shields [5], generalizing the work of J. Daughtry [3], have shown that if C is a nonzero compact operator and AC - CA has rank 1, then A has a hyperinvariant subspace. Question: If A is reductive and C is injective, is A normal?

REFERENCES

1. T. Andô, *Note on invariant subspaces of a compact normal operator*. Arch. Math. 14 (1963), 337-340.

2. E. A. Azoff, *Compact operators in reductive algebras*. Canad. J. Math. 27 (1975), 152-154.

3. J. Daughtry, *An invariant subspace theorem*. Proc. Amer. Math. Soc. 49 (1975), 267-268.

4. J. A. Dyer, E. A. Pedersen, and P. Porcelli, *An equivalent formulation of the invariant subspace conjecture*. Bull. Amer. Math. Soc. 78 (1972), 1020-1023.

5. H. W. Kim, C. Pearcy, and A. L. Shields, *Rank-one commutators and hyper-invariant subspaces*. Michigan Math. J. 22 (1975) (to appear).

6. A. I. Loginov and V. S. Šul'man, *On reductive operator algebras and the invariant subspace problem*. Dokl. Adad. Nauk SSSR 216 (1974), 36-38.

7. V. I. Lomonosov, *Invariant subspaces for the family of operators which commute with a completely continuous operator*. Functional Anal. Appl. 7 (1973), 213-214.

8. R. L. Moore, *Properties of reductive operators*. Notices Amer. Math. Soc. 20 (1973), A-159 #701-47-26.

9. H. Radjavi and P. Rosenthal, *Invariant subspaces*. Ergebnisse der Mathematik, Band 77. Springer-Verlag, New York, 1973.

10. P. Rosenthal, *Completely reducible operators*. Proc. Amer. Math. Soc. 19 (1968), 826-830.

11. ———, *On commutants of reductive operator algebras*. Duke Math. J. (to appear).

Bucknell University
Lewisburg, Pennsylvania 17837