THE MARX CONJECTURE FOR SOME ALPHA-CONVEX FUNCTIONS

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A function $f(z) = z + a_2 z^2 + \cdots$ analytic and univalent in the unit disc U is said to be *starlike* if

$$\Re \frac{zf'(z)}{f(z)} > 0 \qquad (z \in U).$$

Let S^* denote the class of starlike functions. In 1933, A. Marx [3] conjectured that if $f \in S^*$, then $\log f'(z)$ is subordinate to $\log k'(z)$, where $k(z)$ denotes the Koebe function $z(1 - z)^{-2}$ for $|z| < 1$. (If g is subordinate to h, we write $g \prec h$.) Several authors have shown that

$$\log f'(z) \prec \log k'(z) \qquad (|z| < r)$$

for various values of r. However, in 1972, J. A. Hummel [2] showed, using a computer, that counterexamples exist for $r > 0.94$.

For each real α, let $C(\alpha)$ denote the family of functions $f(z) = z + \cdots$ analytic in U with $f'(z)f(z)/z \neq 0$ in U such that

$$\Re \{\alpha(1 + zf''(z)/f'(z)) + (1 - \alpha)zf'(z)/f(z)\} > 0 \qquad (z \in U).$$

Functions in $C(\alpha)$ are called α-convex (P. T. Mocanu [6]). The families $C(0)$ and $C(1)$ are the families S^* and C of univalent, normalized starlike and convex functions, respectively. It is known that

(i) $C(\alpha)$ consists only of univalent starlike functions,

(ii) $C(\beta) \subset C(\alpha)$ if $0 \leq \alpha < \beta$,

(iii) if $\alpha > 0$ and $f \in C(\alpha)$, then there exists a $\sigma \in S^*$ such that

$$f(z) = \left(\frac{1}{\alpha} \int_0^z \sigma^{1/\alpha}(\xi)\, \xi^{-1}\, d\xi \right)^{\alpha} \tag{1}$$

(see [5] for (i), and [4] for (ii) and (iii)).

In many instances, the function

$$f_\alpha(z) = \left(\frac{1}{\alpha} \int_0^z k(\xi)^{1/\alpha}\, \xi^{-1}\, d\xi \right)^{\alpha} \qquad (\alpha > 0)$$

is the solution to an extremal problem in $C(\alpha)$. R. Barnard and J. L. Lewis [1] have shown that if $\alpha \geq 0$ and $f \in C(\alpha)$, then

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$$f(z)/z \prec f_\alpha(z)/z \qquad (|z| < 1).$$

(For the case $\alpha = 0$, this was already proved by Marx [3].)

It is natural to consider the analogue of the Marx conjecture for $C(\alpha)$. Following the notation in [2], we define the Marx region for $C(\alpha)$ as the set

$$M_\alpha(r) = \{\log f'(r) \mid f \in C(\alpha)\}.$$

Also, we let

$$K_\alpha(r) = \{\log f_\alpha'(xr) \mid |x| = 1\}, \qquad D_\alpha(r) = \{\log f(r)/r \mid f \in C(\alpha)\}.$$

The Marx conjecture for $C(\alpha)$ is that $M_\alpha(r) = K_\alpha(r)$ when $0 \le r < 1$. Since $M_0(r) \neq K_0(r)$, it follows that the conjecture for $C(\alpha)$ is false, when α is sufficiently close to 0.

THEOREM. *Let* $f \in C(\alpha)$ $(\alpha \geq 1)$. *Then*

$$\log f'(z) \prec \log f_\alpha'(z) \qquad (|z| < 1).$$

Proof. Using (1), we compute

$$\log f'(z) = (1 - 1/\alpha) \log f(z)/z + (1/\alpha) \log \sigma(z)/z.$$

The assumption $\alpha \geq 1$ implies that $1 - 1/\alpha \geq 0$, and hence $\log f'(z)$ is a convex combination of $\log f(z)/z$ and $\log \sigma(z)/z$. Fix r $(0 < r < 1)$, and suppose $|z| = r$. If

$$\sigma(z) \not\equiv x^{-1} k(xz) \qquad (|x| = 1),$$

then [3] $\log \sigma(r)/r$ is an interior point of $D_0(r)$ and [1] $\log f(r)/r$ is an interior point of $D_\alpha(r)$. Thus there exist neighborhoods N_1 and N_2 of $\log \sigma(r)/r$ and $\log f(r)/r$, and contained in $D_0(r)$ and $D_\alpha(r)$, respectively.

Using equation (1), we may assume that each element of N_2 comes from an element of N_1. Since $1/\alpha$ and $1 - 1/\alpha$ are nonnegative, the set

$$N = (1 - 1/\alpha) N_1 + (1/\alpha) N_2$$

is a neighborhood of $\log f'(r)$, and hence boundary points of $M_\alpha(r)$ occur only if $\log \sigma(r)/r$ is a boundary point of $D_0(r)$, that is, if $f(z) = x^{-1} f_\alpha(xz)$. Therefore $M_\alpha(r) = K_\alpha(r)$, for $\alpha \geq 1$.

Note. For $\alpha = 1$, this result was obtained in [3].

The proof of Theorem 1 breaks down if $0 < \alpha < 1$, since there exist numbers w_i and r_i $(i = 1, 2)$ such that the circles C_i: $|w - w_i| < r_i$ are mapped to a set

$$A = (1 - 1/\alpha) C_1 + (1/\alpha) C_2$$

such that $(1 - 1/\alpha) w_1 + (1/\alpha) w_2$ is a boundary point.

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