SEMIFREE INVOLUTIONS ON SPHERE KNOTS

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If m is a positive integer, an m-*knot* (S^{m+2}, Σ^m) consists of an $(m + 2)$-homotopy sphere S^{m+2} and an m-homotopy sphere Σ^m differentiably embedded in it. A $(2n - 1)$-knot is *simple* if the homotopy groups of its complement $S^{2n+1} - \Sigma^{2n-1}$ coincide with those of the circle in dimension less than n. From now on, we assume $n \geq 2$.

To each simple knot $(S^{2n+1}, \Sigma^{2n-1})$ there corresponds an associated Seifert matrix A (see [1], [3]) such that $A + \varepsilon A^T$ is unimodular, where $\varepsilon = (-1)^n$ and A^T is the transpose of A. The matrix A is determined by a Seifert submanifold V, a 2n-submanifold of S^{2n+1} that bounds M^{2n-1}; and V can be chosen to be $(n - 1)$-connected (see [2]). The normal bundle of Σ^{2n-1} in S^{2n+1} is trivial. Let $Y = \text{closure}(S^{2n+1} - \Sigma^{2n-1} \times D^2)$, and let Y_V be the $(n - 1)$-connected manifold obtained by cutting S^{2n+1} along V with $\partial Y_V = V_+ \cup V_-$ (two copies of V). The matrix A is the matrix for the mapping $j_+: H_n(V_+) \to H_n(Y_V)$, and the matrix $(-1)^{n+1} A^T$ is the matrix for the mapping $j_-: H_n(V_-) \to H_n(Y_V)$, with respect to the bases given by the Alexander duality (see [1], [2]).

Let T be an involution (T^2 = identity) acting differentiably on S^{2n+1} with Σ^{2n-1} as its fixed points, denoted by $(T; S^{2n+1}, \Sigma^{2n-1})$. If $(S^{2n+1}, \Sigma^{2n-1})$ is simple, then the orbit space S^{2n+1}/T is easily seen to be a homotopy sphere, and $(S^{2n+1}/T, \Sigma^{2n-1})$ is again a simple knot. The purpose of this note is to determine which simple knot can be realized as the orbit space of such an involution.

PROPOSITION. *A simple knot* $(S^{2n+1}, \Sigma^{2n-1})$ *is the orbit space of a* $(T; S_1^{2n+1}, \Sigma^{2n-1})$ *if and only if both* $A + A^T$ *and* $A - A^T$ *are unimodular, where* A *is its Seifert matrix determined by some* $(n - 1)$*-connected submanifold* V^{2n} *in* S^{2n+1}.

Proof. Let $\{W_i, V_{+i}, V_{-i}\}$ $(i = 1, 2)$ be two copies of $\{Y_V, V_+, V_-\}$. Construct a manifold X by joining W_1 and W_2 by gluing V_{+1} to V_{-1} and V_{+2} to V_{-1}. From the Mayer-Vietoris sequence

$$\to H_{q+1}(X) \to H_q(V_{+1}) \oplus H_q(V_{-1}) \xrightarrow{\lambda_q} H_q(W_1) \oplus H_q(W_2) \to H_q(X) \to$$

we see that

$$H_q(X) = \begin{cases} Z & \text{if } q = 0 \text{ or } q = 1, \\ 0 & \text{if } q \neq n \text{ and } q \neq n+1, \end{cases}$$

and $H_n(X) = H_{n+1}(X) = 0$ if and only if λ_n is unimodular. But

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$$\lambda_n = \begin{pmatrix} A & (-1)^{n+1} A^T \\ (-1)^{n+1} A^T & A \end{pmatrix};$$

if we write $B = (-1)^{n+1} A^T$, then

$$\det \lambda_n = \begin{vmatrix} A & B \\ B & A \end{vmatrix} = \begin{vmatrix} A+B & A \\ B+A & A \end{vmatrix} = \begin{vmatrix} A+B & B \\ 0 & A-B \end{vmatrix} = |A+B|\,|A-B|.$$

Thus λ_n is unimodular if and only if both $A + A^T$ and $A - A^T$ are unimodular.

Using the Van Kampen theorem repeatedly, we see that $\pi_1(\partial X) = \pi_1(X) = Z$. By means of the Mayer-Vietoris sequence, we can show as above that $H_1(X, \partial X) = 0$. The mapping $\pi_1(\partial X) \to \pi_1(X)$ is surjective, and $\partial X = \Sigma^{2n-1} \times S^1$. Construct S_1^{2n+1} by gluing $\Sigma^{2n-1} \times D^2$ and X along their boundary. Define an involution T on S_1^{2n+1} by combining the covering transformation on X and the action on $\Sigma^{2n-1} \times D^2$ defined by $(y, re^{i\theta}) \to (y, re^{2i\theta})$. If λ_n is unimodular, then S_1^{2n+1} is a homotopy sphere, and the involution $(T; S_1^{2n+1}, \Sigma^{2n-1})$ has $(S^{2n+1}, \Sigma^{2n-1})$ as the orbit.

Conversely, given an involution $(T; S_1^{2n+1}, \Sigma^{2n-1})$, where $(S_1^{2n+1}, \Sigma^{2n-1})$ is simple, we may apply equivariant surgery as in [4] to obtain two (n - 1)-connected Seifert submanifold V_1 and V_2 such that $TV_1 = V_2$. The set $V_1 \cup V_2$ divides S_1 into two parts W_1' and W_2' with $TW_1' = W_2'$. Thus $X = \overline{S_1 - \Sigma \times D^2}$ can be expressed as a union of the sets $\{W_i, V_+, V_-\}$ (i = 1 or 2) as above. Then $S^{2n+1} = S_1/T = \Sigma \times D^2 \cup Y$, where Y is obtained from W_1 by gluing V_+ and V_- together. In the orbit space, $V_+ = V_- = V$ is a Seifert submanifold for $(S^{2n+1}, \Sigma^{2n-1})$. By the first part of the proof, we see that the Seifert matrix A determined by V satisfies the equations $\det(A + A^T) = \pm 1 = \det(A - A^T)$. ∎

Remark. Let Z_p denote the cyclic group of order p. Then the argument above also works for other Z_p semifree actions, where

$$\lambda_n = \begin{pmatrix} A & B & & \\ & A & \cdot & \\ & & \cdot & B \\ B & & & A \end{pmatrix} \quad \text{(P copies of A on the diagonal)}.$$

In [3], J. Levine introduced some cobordism groups of matrices G_+, G_-, and he showed that for $n \geq 3$, there is an isomorphism ϕ_n from the knot cobordism group C_{2n-1} to G_{ε_n} $(\varepsilon_n = (-1)^n)$. A matrix A has the property ε $(\varepsilon = \pm 1)$ if $A + \varepsilon A^T$ is unimodular. G_ε is the group of cobordism classes of matrices with property ε (see [3, p. 231]). Let D_{2n-1} be the subgroup of C_{2n-1} consisting of the cobordism classes having a representative $(S^{2n+1}, \Sigma^{2n-1})$, a simple knot, that is the orbit space of some $(T, S_1^{2n+1}, \Sigma^{2n-1})$. Since every cobordism class contains a simple knot [3], we have the following result.

COROLLARY. *For* $n \geq 3$, *under the isomorphism* $\phi_n: C_{2n-1} \to G_{\varepsilon_n}$ $(\varepsilon_n = (-1)^n)$, D_n *is in one-to-one correspondence with the subgroup* $G_+ \cap G_-$.

Levine [3, p. 243] showed that $G_+ \cap G_-$ is an infinitely generated subgroup of G_+ or G_- such that

$$A_k = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & -k & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (k = 1, 2, \cdots);$$

here $A_k \in G_+ \cap G_-$, and the A_k represent linearly independent elements of G_+.

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