STEENROD SQUARES AND REDUCTION OF STRUCTURE GROUP FOR FIBRATIONS

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INTRODUCTION

All spaces considered in this paper will be assumed to have the homotopy types of regular CW-complexes with integral homology of finite type. Z_2 coefficients will be used for homology and cohomology, and A will denote the mod-2 Steenrod algebra. We shall regard two principal fibrations $G \to E \to B$ and $G' \to E' \to B'$ as fiber-homotopically equivalent as principal fibrations, if there is a homotopy equivalence $G \to G'$ that is a homomorphism and is compatible with a fiber-homotopy equivalence of $E \to B$ with $E' \to B'$. We shall be concerned with properties of principal fibrations that are invariant under modifications of the type just described.

The diagram $G \to E \xrightarrow{\pi} B$ will therefore be used to denote any suitably chosen representative of such a fiber-homotopy equivalence class of principal fibrations with simply connected bases. For a representative that is a fiber bundle, the structure group will be regarded as acting on the right on E. For a subgroup H of G, $H \backslash G$ will denote the space of right cosets of H, and G/H the space of left cosets.

$E(\pi)$ will denote the Z_2 Serre cohomology spectral sequence of the fibration π, with $E_2^{p\,q} = H^p(B) \otimes H^q(G)$. In [1], [8, Section 4], [9], and [13], natural operations

$$S^i\colon E_r^{p\,q} \to E_{2r-2}^{p+i-q\;2q} \qquad (i \geq q,\; 2 \leq r \leq \infty)$$

have been defined. These are compatible with the differentials, and they are compatible on E_∞ with the action of the Steenrod algebra A on $H^*(E)$. If $b \otimes g \in E_2^{p\,q}$, then $S^i(b \otimes g) = Sq^{i-q}b \otimes g^2$. Of course, for any fibration, the action of A on the cohomology of the total space, the base space, and the fiber must be compatible with the homomorphisms induced by the projection and the inclusion of the fiber in the total space. Because of the properties of the S^i just mentioned, the S^i clearly impose additional restrictions upon the choices of the three spaces for a fibration.

In this paper, we consider two cases in which, for the fiber bundle $G \to E \xrightarrow{\pi} B$, the assumption that the structure group G is reducible to a subgroup H leads to a factorization of π of the form $E \xrightarrow{\bar{\pi}} B \times (H \backslash G) \to B$, where $\bar{\pi}$ is also a fiber bundle. In the first case, H is normal; in the second case, $G \to H \backslash G$ is itself a fiber bundle. The above-mentioned properties of the S^i can therefore be employed with some choices of π to show that such a reduction of the structure group is impossible.

For principal bundles over spheres, both types of reduction of the structure group are worthy of consideration. As we shall see, the high connectivity of the sphere leads to a reduction to an advanced stage in the Postnikov resolution of the

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path-space fibration over G, which is a normal subgroup of G. It is also possible to determine the reducibility to a subgroup H of the second type by looking at the exact homotopy sequence for the fibration $H \to G \to H \backslash G$, since for a bundle $G \to E \to S^n$ the structure group can be reduced to H if and only if the homotopy class of the classifying map $S^{n-1} \to G$ is in the image of $\pi_{n-1}(H) \to \pi_{n-1}(G)$.

We shall apply these principles to the calculation of the cohomology rings as algebras over A of some principal bundles over spheres with compact Lie groups as fibers. In Section 2, we also work in the other direction, using the known action of the Steenrod algebra on $H^*(K)$, for compact Lie groups K, to show that for some projections on left coset spaces $K \to K/G$, the structure group cannot be reduced to specific Lie subgroups H of G.

This material comprises part of the author's doctoral dissertation at Princeton University.

1. REDUCTION OF STRUCTURE GROUP

For the fibration $G \xrightarrow{i} E \xrightarrow{\pi} B$ we have seen that, in $E_2(\pi)$,

$$S^i(b \otimes g) = Sq^{i-q} b \otimes g^2 .$$

Thus we lose all but one of the terms of

$$Sq^i(b \otimes g) = \sum_{j+k=1} Sq^j b \otimes Sq^k g$$

that the Cartan formula would give if the fibration were trivial and $H^*(E)$ were simply $H^*(B) \otimes H^*(G)$. We shall discuss some partial remedies for this loss in the case where π is partially trivial, that is, the structure group can be reduced to some proper subgroup.

Observation. Suppose the group of the fiber bundle π can be reduced to a normal subgroup H of G. Then there exists a principal bundle $H \to E \xrightarrow{\bar{\pi}} B \times (H \backslash G)$ such that π factors as $E \xrightarrow{\bar{\pi}} B \times (H \backslash G) \to B$. This follows from the fact that the group of π can be reduced to H if and only if the associated bundle

$$H \backslash G \to E \times_G (G/H) = E \times_G (H \backslash G) \to B$$

has a cross-section; this in turn is the case if and only if this associated bundle is trivial. In this case the bundle $H \to E \to E \times_G (H \backslash G)$ can be viewed as having base $B \times (H \backslash G)$.

THEOREM 1. *Suppose the group of* π *can be reduced to a normal subgroup* H *of* G. *Suppose* $\bar{g} \in H^q(H \backslash G)$ *and* $\rho^*(\bar{g}) = g$, *where* ρ *is the projection* $G \to H \backslash G$. *Then for each* $b \in H^p(B)$, *the element* $b \otimes g \in E_2^{p\,q}(\pi)$ *survives to* $E_\infty^{p\,q}$ *and is the coset of* $\bar{\pi}^*(b \otimes \bar{g})$.

Proof. Since the ring structure of $E_\infty(\pi)$ is compatible with that of E_2 and that of $H^*(E)$, it suffices to prove the theorem for $\bar{g} = 1$ or $b = 1$. Suppose that $\bar{g} = 1$. Then $E \xrightarrow{\pi} B$ factors as $E \xrightarrow{\bar{\pi}} B \times (H \backslash G) \to B$, and we see that $\bar{\pi}^*(b \otimes 1) = \pi^*(b)$;

observe that $\pi^*(b)$ is in the coset of $E_\infty^{p\,0}(\pi)$ determined by $b \otimes 1 \in E_2^{p\,0}(\pi)$. Now suppose $b = 1$. The map $G \to H \setminus G$ has the factorization

$$G \to E \to E \times_G (H \setminus G) = B \times (H \setminus G) \to H \setminus G,$$

so that $i^* \bar{\pi}^*(1 \otimes \bar{g}) = g$, showing that $1 \otimes g$ survives to $E_\infty^{0\,q}$ and contains $\bar{\pi}^*(1 \otimes \bar{g})$. ■

We shall denote $\bar{\pi}^*(b \otimes \bar{g})$ by $b * g$.

COROLLARY 1. *If π satisfies the hypothesis of Theorem 1, and if in addition $E(\pi)$ collapses, then the A-subalgebra $\bar{\pi}^*(H^*(B \times (H \setminus G)))$ of $H^*(E)$ is isomorphic to $H^*(B) \otimes \mathrm{Im}\, \rho^*$ with the Cartan formula action of A. The correspondence is $b * g \to b \otimes g$, that is, it sends each element to its coset in $E_\infty^{p\,q}(\pi)$. If ρ^* is surjective, then this subalgebra is all of $H^*(E)$.*

Proof. Since $E_\infty^{p\,q} = E_2^{p\,q}$, the correspondence is one-to-one. The action of A commutes with $\bar{\pi}^*$ and with $(1 \times \rho)^*: H^*(B \times (H \setminus G)) \to H^*(B \times G)$. Since A acts on $H^*(B \times G)$ by the Cartan formula, it follows that if we regard A as having the Cartan formula action on $H^*(B) \otimes \mathrm{Im}\, \rho^*$, the above-mentioned correspondence will be a map of A-algebras. If $\mathrm{Im}\, \rho^* = H^*(G)$, then

$$H^*(B) \otimes \mathrm{Im}\, \rho^* = H^*(B) \otimes H^*(G) = E_2(\pi) = E_\infty(\pi).$$

Because E_∞ is the graded module associated with a composition series for $H^*(E)$, which is of finite type, we conclude that $\bar{\pi}^*$ is surjective. ■

Remark. (See [7].) Suppose G is simply connected. Then we can modify G within its homotopy type, by a modification of the type described in the Introduction, and then choose Postnikov resolutions $\{G^k\}$ $(k \geq 0)$ for G and $\{G_k\}$ $(k \geq 0)$ for the path-space fibration over G, with the following properties.

1. Each G_k is a group with the homotopy type of a regular CW-complex, and the projection $j: G_k \to G$ is a monomorphism onto a normal subgroup of G.

2. Each G^{k-1} has the homotopy type of a regular CW-complex, and the map $\rho: G \to G^{k-1}$ in the Postnikov resolution is the cokernel of j.

3. In the diagram $G_k \xrightarrow{j} G \xrightarrow{\rho} G^{k-1}$ $(k \geq 1)$, the map ρ is a fibration that induces isomorphisms in homotopy (and thus in homology and cohomology) in dimensions up to $k - 1$. Also, j induces isomorphisms in homotopy for dimensions equal to or greater than k.

4. G_k is $(k - 1)$-connected, and $\pi_n(G^{k-1}) = 0$ for $n \geq k$. The spaces G_k and G^{k-1} will necessarily be simply connected and have homology of finite type.

Notice that the hypothesis of the following theorem is satisfied if B is k-connected.

THEOREM 2. *Suppose the group of π can be reduced to G_k. (Equivalently, the classifying map for π can be lifted to $B_{G_k} = (B_G)_{k+1}$.) Then $H^*(E)$ has a subalgebra over A that is isomorphic to the tensor product of $H^*(B)$ with a subalgebra of $H^*(G)$ containing all elements of dimension up to $k - 1$.*

Proof. Let $H = G_k$, so that $H \setminus G = G^{k-1}$. Apply Corollary 1, and the fact that $\mathrm{Im}\, \rho^*$ contains all elements of $H^*(G)$ of dimension less than $k - 1$. ■

THEOREM 3. *Suppose* $H^*(G)$ *is the algebra* $\Delta(x_{i_1}, \cdots, x_{i_n})$ *generated by a simple system of* n *elements that are transgressive in the universal fibration* $G \to E_G \to B_G$, *and* $H^*(B_G) = Z_2[y_{i_1+1}, \cdots, y_{i_n+1}]$ *with the Hopf algebra structure in which each* y *is primitive. Suppose* $H^*(B)$ *is a Hopf algebra. Let* $f: B \to B_G$ *be such that* $f^*: \tilde{H}^*(B_G; Z_2) \to \tilde{H}^*(B; Z_2)$ *is* 0. *Let* $G \to E \xrightarrow{\pi} B$ *denote the fibration induced by* f. *Then the* Z_2*-cohomology spectral sequence of* π *collapses.*

Proof. As in Theorem 3.2 of [10], we show that the fiber-square spectral sequence for the diagram

$$\begin{array}{ccc} E & \to & E_G \\ \downarrow & & \downarrow \\ B & \to & B_G \end{array}$$

collapses. The second-level term is

$$FSE_2 = (H^*(B)/\mathrm{Im}\, f^*) \otimes \mathrm{Tor}_{\mathrm{Ker}\, f^*}(Z_2, Z_2) = H^*(B) \otimes E[z_{i_1}, \cdots, z_{i_n}].$$

Here we regard f^* as a map of Hopf algebras and use E to denote an exterior algebra. The generators z_{i_k} are in filtration -1, while $H^*(B)$ is in filtration 0. Since each of the differentials d_r raises filtration by at least 2, we see that this spectral sequence collapses. Thus $FSE_\infty = FSE_2 \approx H^*(B) \otimes H^*(G)$ as a module over Z_2. Since FSE_∞ is the graded module associated with a composition series for $H^*(E)$, we see that $H^*(E) \approx H^*(B) \otimes H^*(G)$ as a module; in particular, the Serre spectral sequence collapses. ■

THEOREM 4. *Suppose* $H^*(G)$ *is generated as an algebra over* A *by the classes of dimension at most* n - 2, *and* $f: S^n \to B_G$, *for some* n *greater than or equal to* 2. *Suppose that either the order of* f *in* $\pi_n(B_G)$ *is odd, or that* $H^n(B_G) = 0$. *Then, for the bundle* $G \to E \xrightarrow{\pi} S^n$ *induced by* f, *the ring* $H^*(E)$ *is isomorphic to* $H^*(S^n \times G)$ *as an algebra over* A.

Proof. We shall show that $f^* = 0$ on reduced cohomology, so that the theorem follows from Theorems 2 and 3. This is trivial when $H^n(B_G) = 0$. In the other case, let s be a generator of $H_n(S^n; Z)$. For $f_*: H_n(S^n; Z) \to H_n(B_G; Z)$, the element $f_*(s)$ is the image under the Hurewicz homomorphism of $f \in \pi_n(B_G)$, so that $m \cdot f_*(s) = 0$ for some odd m. Then, under the reduction $H_*(B_G; Z) \to H_*(B_G; Z_2)$, the classes $m \cdot f_*(s)$ and $f_*(s)$ both go to the same element, which is 0. Thus $f_*: \tilde{H}_n(S^n; Z_2) \to \tilde{H}_n(B_G; Z_2)$ is 0, and $f^*: \tilde{H}^n(B_G; Z_2) \to \tilde{H}^n(S^n; Z_2)$ is 0. ■

Now we present a version of the observation preceding Theorem 1, in which the requirement that H be normal is replaced by the requirement that $G \to H \backslash G$ be a locally trivial fiber bundle. This result has perhaps not been recognized previously.

THEOREM 5. *Let* $G \to E \xrightarrow{\pi} B$ *be a locally trivial fiber bundle. Suppose the structure group of* π *can be reduced to* $H \subset G$, *where* $G \to H \backslash G$ *is a locally trivial fiber bundle. Then there exists a fiber bundle (not necessarily principal or with structure group* H), $H \to E \xrightarrow{\bar{\pi}} B \times (H \backslash G)$, *and* π *factors as* $E \xrightarrow{\bar{\pi}} B \times (H \backslash G) \to B$.

Proof. There exists an open cover $\bigcup B_i$ of B such that up to fiber-homotopy equivalence $E = \left(\sum B_i \times G\right) / \sim$, where $\sum$ denotes topological sum and the identifications are as follows. For $(b, g) \in B_i \times G$ we denote the corresponding element of $B_i \times G \subset \sum B_i \times G$ by (b_i, g). Then, for $b \in B_i \cap B_j$, we have the equivalence

$(b_i, g) \sim (b_j, s_{ij}(b)\cdot g)$, for an appropriate $s_{ij}: B_i \cap B_j \to H$. Similarly, by employing an open cover $\bigcup V_\alpha$ of $H \setminus G$, we may construct G as $\left(\sum H \times V_\alpha\right)/\sim$, with identifications $(h, v_\alpha) \sim (h \cdot t_{\alpha\beta}(v), v_\beta)$, where $t_{\alpha\beta}: V_\alpha \cap V_\beta \to H$. If $\bar{h} \in H$, and if $g \in G$ is represented by (h, v_α), then $\bar{h}g$ is represented by $(\bar{h}h, v_\alpha)$. Therefore $E = \left(\left(\sum B_i\right) \times H \times \left(\sum V_\alpha\right)\right)/\sim$, where $(b, v) \in (B_i \times V_\alpha) \cap (B_j \times V_\beta)$ implies $(b_i, h, v_\alpha) \sim (b_j, s_{ij}(b)\cdot h \cdot t_{\alpha\beta}(v), v_\beta)$. Thus E is a bundle over $B \times (H \setminus G)$ with fiber H, trivial over each open set $B_i \times V_\alpha$.

An alternate proof, suggested by John C. Moore, dispenses with the requirement that π and $G \to H \setminus G$ be locally trivial. The classifying map of π can be factored through B_H. Let E' be the bundle over B induced by this map $B \to B_H$. Letting H act on the left on G, we see that $E = E' \times_H G$. Letting H act trivially on $H \setminus G$, we have the relations $E' \times_H (H \setminus G) \approx (E'/H) \times (H \setminus G) \approx B \times (H \setminus G)$, and we also have the fiber bundle $H \to E' \times_H G \to E' \times_H (H \setminus G)$. ■

Notice that $G \to H \setminus G$ is a fiber bundle if H is a closed subgroup of the Lie group G. Notice also that in Theorem 5 we could have dispensed with our general assumption that B is simply connected. However, we continue to assume B is simply connected, in what follows.

THEOREM 6. *Theorem 1 holds if the condition that H is normal is replaced by the condition that $G \to H \setminus G$ is a fiber bundle.*

Proof. As in the proof of Theorem 1, we see that the result holds for $b = 1$. The fiber-bundle map $G \to H \setminus G$ has the factorization

$$G \to E = E' \times_H G \to E \times_H (H \setminus G) = B \times (H \setminus G) \to H \setminus G;$$

therefore the proof for $\bar{g} = 1$ may also be repeated. ■

COROLLARY 2. *Corollary 1 also holds with the condition that H is normal replaced by the condition that $G \to H \setminus G$ is a fiber bundle.*

2. EXAMPLES AND APPLICATIONS

In this section, we present some applications of the spectral sequence operations and of the results of Section 1. Descriptions of many of the fibrations that are employed, and of the cohomology and homotopy groups of the Lie groups and homogeneous spaces involved in these fibrations, can be found in [6, p. 217] and [5, pp. 132-133]. Other references are supplied with some of the examples.

We shall now consider certain examples of a principal bundle E over a sphere, with a Lie group as fiber, using our techniques to calculate $H^*(E)$ as an A-algebra.

Example 1. (See [4, Section 17].) Let $G = G_2$ (the exceptional Lie group). Then

$$H^*(G) = \Delta(x_3, x_5, x_6),$$

$$H^*(B_G) = Z_2[y_4, y_6, y_7],$$

$$\pi_4(B_G) = Z.$$

If $f: S^4 \to B_G$ is of odd degree in $\pi_4(B_G)$, then $f^*: H^4(B_G) \to H^4(S^4)$ is an epimorphism. The fiber-square spectral sequence for the diagram $\begin{array}{ccc} E & \to & E_G \\ \pi\downarrow & & \downarrow \\ S^4 & \to & B_G \end{array}$ has

$$FSE_2 = FSE_\infty = \mathrm{Tor}_{H^*(B_G)}(H^*(S^4), Z_2)$$

$$= \mathrm{Tor}_{Z_2[y_4^2, y_6, y_7]}(Z_2, Z_2) = E[z_7, z_5, z_6].$$

The next-to-last equality in this sequence involves a standard technique for calculating Tor (see [10, Theorem 2.4]). In the Serre spectral sequence for π, $E_2 = E[s_4] \otimes \Delta(x_3, x_5, x_6)$. Thus E_∞ is generated by $s_4 \otimes x_3$, $1 \otimes x_5$, and $1 \otimes x_6$. We shall denote the unique classes in these cosets by $s_4 * x_3$, $1 * x_5$, and $1 * x_6$. As a ring, $H^*(E) = E[s_4 * x_3, 1 * x_5, 1 * x_6]$. Since $S^1(1 \otimes x_5) = 1 \otimes x_6$ and $1 * x_6$ is the only 6-dimensional class in $1 \otimes x_6$, we conclude that $Sq^1(1 * x_5) = 1 * x_6$, and that $Sq^1(1 * x_6) = Sq^1 Sq^1(1 * x_5) = 0$. We cannot immediately determine whether $Sq^2(1 * x_5) = s_4 * x_3$. However, $Sq^i(s_4 * x_3) = 0$ $(i \geq 1)$, since this must be true for dimensional reasons when $i = 1, 2$, or 3, and when $i > 3$, $Sq^i(s_4 * x_3)$ must have filtration greater than or equal to 5.

The exact homotopy sequence of the fibration $S^3 \to G_2 \xrightarrow{\rho} SO(7)/SO(5)$ can be employed to show that $\pi_3(S^3) \to \pi_3(G)$ is an epimorphism. Thus π can be reduced to an S^3-bundle, and there is a fibration $S^3 \to E \xrightarrow{\bar{\pi}} S^4 \times (SO(7)/SO(5))$. Since $\mathrm{Im}\, \rho^* = \Delta(x_5, x_6)$, the image of $\bar{\pi}^*$ must be $E[1 * x_5, 1 * x_6]$. Since this ring is an A-subalgebra of $H^*(E)$, we conclude that $Sq^2(1 * x_5) = 0$.

Example 2. Again let $G = G_2$, and let $f: S^4 \to B_G$ be of even degree. Then f^* is necessarily 0 on reduced cohomology. Therefore the spectral sequence of π collapses. Using the reducibility of π to an S^3 bundle, we conclude that $H^*(E)$ has an A-subalgebra that is isomorphic to $E[s_4] \otimes \Delta(x_5, x_6)$ and is generated by $s_4 * 1 \in s_4 \otimes 1$ and $1 * x_5 \in 1 \otimes x_5$, together with $1 * x_6 \in 1 \otimes x_6$. There are also unique classes $1 * x_3 \in 1 \otimes x_3$ and $s_4 * x_3 \in s_4 \otimes x_3$. The products involving these last two elements and elements of the subalgebra can be computed by employing dimensional considerations and the ring structure of $E(\pi)$. For instance, $(1 * x_3) \cdot (1 * x_5) \in 1 \otimes x_3 x_5$, and thus this product is uniquely determined, there being only one 8-dimensional class in $H^*(E)$. We conclude that $H^*(E) \approx H^*(S^4 \times G_2)$ as a ring. Some Steenrod squares can be determined. As above,

$$Sq^1(1 * x_5) = 1 * x_6,$$

and $Sq^i(1 * x_5) = 0 = Sq^i(1 * x_6)$ for $i > 1$. Since $S^3(s_4 \otimes x_3) = s_4 \otimes x_6$, which contains only one class, it follows that $Sq^3(s_4 * x_3) = s_4 * x_6$. We cannot resolve whether $Sq^1(1 * x_3) = 0$ or $s_4 * 1$.

Example 3. (See [4, Section 22].) Again let $G = G_2$. Then $\pi_9(B_G) = Z_2$ and $H^9(B_G) = 0$. Consider the nontrivial bundle $G_2 \to E \to S^9$. Since $H^*(G_2)$ is generated over A by its classes of dimension less than or equal to 7, we conclude (using Corollary 1) that $H^*(E) \approx H^*(S^9 \times G_2)$ as an A-algebra.

Example 4. Let $G = F_4$. Then

$H^*(G) = \Delta(x_3, x_5, x_6, x_{15}, x_{23})$,

$H^*(B_G) = Z_2[y_4, y_6, y_7, y_{16}, y_{24}]$,

$\pi_9(B_G) = Z_2$,

$H^9(B_G) = 0$.

The subalgebra over A of $H^*(G)$ generated by the elements of dimension less than or equal to 7 is $\Delta(x_3, x_5, x_6)$. Thus for the nontrivial bundle $F_4 \to E \to S^9$, the algebra $H^*(E)$ has a subalgebra over A which is isomorphic to $H^*(S^9 \times G_2)$.

Example 5. Let G = Spin(7). Then

$H^*(G) = \Delta(x_3, x_5, x_6, x_7)$,

$H^*(B_G) = Z_2[w_4, w_6, w_7, b_8]$,

$\pi_9(B_G) = Z_2 \oplus Z_2$,

$H^9(B_G) = 0$.

The subalgebra of $H^*(G)$ generated by the elements of dimension less than or equal to 7 is the whole ring, so that for each bundle $G \to E \to S^9$ we have the isomorphism $H^*(E) \approx H^*(S^9 \times G)$ as A-algebras.

Example 6. Again let G = Spin(7). Then $\pi_4(B_G) = Z$. If f: $S^4 \to B_G$ is of odd order, then f^* is an epimorphism in cohomology. In the fiber-square spectral sequence for the diagram

$$\begin{array}{ccc} E & \to & E_G \\ \pi\downarrow & & \downarrow \\ S^4 & \xrightarrow{f} & B_G \end{array}$$

we have the relation

$$FSE_2 = \mathrm{Tor}_{Z_2[w_4^2, w_6, w_7, b_8]}(Z_2, Z_2) = E[u_7, u_5, u_6, v_7].$$

Since the generators of this last algebra are in filtration -1, we conclude that $FSE_2 = FSE_\infty$, and that in the Serre spectral sequence for $G \to E \xrightarrow{\pi} S^4$, the stage E_∞ is generated by $s_4 \otimes x_3$ and $1 \otimes x_5$ and $1 \otimes x_6$, together with $1 \otimes x_7$.

The exact homotopy sequence of $G_2 \to \mathrm{Spin}(7) \to S^7$ shows that π can be reduced to a G_2-bundle. The spectral sequence of $G_2 \to E \xrightarrow{\bar{\pi}} S^4 \times S^7$ can be used to show that there is a class $1 * y_7$ in $1 \otimes y_7$ such that $E[i * y_7]$ is an A-subalgebra of $H^*(E)$. Thus $Sq^i(1 * y_7) = 0$ for $i > 0$.

Similarly, from the homotopy sequence of $\mathrm{Spin}(5) \to \mathrm{Spin}(7) \to SO(7)/SO(5)$ we conclude that the unique classes $1 * x_5 \in 1 \otimes x_5$ and $1 * x_6 \in 1 \otimes x_6$ generate an A-subalgebra of $H^*(E)$. Thus $Sq^1(1 * x_5) = 1 * x_6$, and it is also true that $Sq^i(1 * x_5) = 0$ for $i > 1$, and $Sq^i(1 * x_6) = 0$ for $i > 0$. Let $s_4 * x_3$ denote the unique element in $s_4 \otimes x_3$. Then, reasoning as in Example 1, we conclude that $Sq^i(s_4 * x_3)$ must be 0 for $i > 0$. Thus $H^*(E) \approx E[s_4 * x_3, 1 * x_7, 1 * x_5, 1 * x_6]$ as an algebra, and the only nontrivial Steenrod square is $Sq^1(1 * x_5) = 1 * x_6$.

If f is of even order, the spectral sequence of π collapses. In this case we are left with some unanswered questions. For instance, for the unique classes $1 * x_3 \in 1 \otimes x_3$ and $s_4 * 1 \in s_4 \otimes 1$, does $Sq^1(1 * x_3) = s_4 * 1$?

Example 7. Let G = Spin(9). Then

$$H^*(G) = \Delta(x_3, x_5, x_6, x_7, y_{15}),$$

$$H^*(B_G) = Z_2[w_4, w_6, w_7, w_8, b_{16}],$$

$$\pi_9(B_G) = Z_2 \oplus Z_2,$$

$$H^9(B_G) = 0.$$

For each nontrivial bundle $G \to E \xrightarrow{\pi} S^9$, the spectral sequence collapses. The A-subalgebra of $H^*(G)$ generated by the elements of dimension less than or equal to 7 is $\Delta(x_3, x_5, x_6, x_7)$; thus $H^*(E)$ has an A-subalgebra generated by a class $s_9 * 1 \in s_9 \otimes 1$, a class $1 * x_3 \in 1 \otimes x_3$, a class $1 * x_5 \in 1 \otimes x_5$, a class $1 * x_6 \in 1 \otimes x_6$, and a class $1 * x_7 \in 1 \otimes x_7$. Looking at the exact homotopy sequence of $\mathrm{Spin}(7) \to \mathrm{Spin}(9) \xrightarrow{\rho} S^{15}$, we see that π can be reduced to a Spin(7)-bundle. It is easily seen that $\rho^*(s_{15}) = y_{15}$. Suppose then that $\mathrm{Spin}(7) \to E \xrightarrow{\bar{\pi}} S^9 \times S^{15}$ is a fiber bundle, and let $1 * y_{15}$ denote $\bar{\pi}^*(s_{15})$. Then, as an A-algebra,

$$H^*(E) \approx H^*(S^9) \otimes \Delta(1 * x_3, 1 * x_5, 1 * x_6, 1 * x_7, 1 * y_{15}) \approx H^*(S^9 \times G).$$

Example 8. (See [3, Section 9].) Let G = SU(3). Then

$$H^*(G) = \Delta(x_3, x_5),$$

$$H^*(B_G) = Z_2[y_4, y_6],$$

$$\pi_7(B_G) = Z_2 \oplus Z_3.$$

Since the A-subalgebra of $H^*(G)$ generated by all elements of dimension less than or equal to 5 is the whole ring, we conclude that for each bundle $G \to E \to S^7$, the A-algebra isomorphism $H^*(E) \approx H^*(S^7 \times G)$ holds. The same is true when S^7 is replaced by a sphere of higher odd dimension. Similar results hold for other SU(n), since $H^*(B_{SU(n)})$ has only even-dimensional cohomology. For instance, one can exploit the fact that the first non-0 homotopy group of odd dimension of $B_{SU(4)}$ is $\pi_9 = Z_8 \oplus Z_3$.

Example 9. Let G = Sp(2). Then

$$H^*(G) = \Delta(x_3, x_7),$$

$$H^*(B_G) = Z_2[y_4, y_8],$$

$$\pi_5(B_G) = \pi_6(B_G) = Z_2.$$

Consider a bundle $G \to E \xrightarrow{\pi} S^5$. The A-subalgebra of $H^*(G)$ generated by elements of dimension less than or equal to 3 is $E[x_3]$. Thus $H^*(E)$ has an A-subalgebra generated by the unique classes $s_5 * 1 \in s_5 \otimes 1$ and $s_5 * x_3 \in s_5 \otimes x_3$. The exact homotopy sequence of $\mathrm{Sp}(1) \to \mathrm{Sp}(2) \xrightarrow{\rho} S^7$ shows that π can be reduced to a

$Sp(1) = S^3$-bundle. Since $\rho^*(s_7) = x_7$, we see that for the unique class $1 * x_7 \in 1 \otimes x_7$, the algebra $E[s_5 * 1, 1 * x_7]$ is also an A-subalgebra of $H^*(E)$. Thus $H^*(E) \approx H^*(S^5 \times Sp(2))$ as an A-algebra. The same reasoning with S^5 replaced by S^6 shows that for each bundle $G \to E \to S^6$, the A-algebra isomorphism $H^*(E) \approx H^*(S^6 \times Sp(2))$ holds.

The following examples employ Theorem 5 but do not require the spectral-sequence operations in any essential way. We shall occasionally use the fact that the Steenrod operations commute with the homomorphisms induced by the projection and fiber inclusion in a fibration.

We show, in the case of certain classical fibrations consisting of the projection of a Lie group on a homogeneous space, $G \to K \xrightarrow{\pi} K/G$, that the structure group cannot be reduced to a particular Lie subgroup H of G. Under the assumption that this reduction is possible, there is, by Theorem 5, a fibration $H \to K \xrightarrow{\bar{\pi}} (K/G) \times (H \setminus G)$. The existence of this fibration will lead to a contradiction.

Example 10. (See [3, Section 9].) Let $G = SU(3)$, let $K = G_2$, and let $H = SU(2) = S^3$. Then $K/G = S^6$ and $H \setminus G = S^5$. If the group G of the bundle π were reducible to H, we would have the spectral sequence

$$E_2(\bar{\pi}) = H^*(S^6 \times S^5) \otimes H^*(H) = E[s_6, s_5] \otimes E[s_3],$$

which converges to $H^*(K) = \Delta(x_3, x_5, x_6)$. By comparing ranks in each dimension, we see that this spectral sequence collapses. We must have the relations $\bar{\pi}^*(s_5) = x_5$ and $\bar{\pi}^*(s_6) = x_6$. But $Sq^1(x_5) = x_6$, while $Sq^1(s_5) = 0$.

Example 11. Let $G = Spin(6)$, let $K = Spin(7)$, and let $H = Spin(5)$. Then $K/G = S^6$ and $H \setminus G = S^5$. If the group G of π were reducible to H, we would have the relations

$$E_2(\bar{\pi}) = H^*(S^6 \times S^5) \otimes H^*(H) = E[s_6, s_5] \otimes \Delta(x_3, x_7),$$

and this spectral sequence would converge to $H^*(K) = \Delta(y_3, y_5, y_6, y_7)$. Thus, it would be true that $\bar{\pi}^*(s_5) = y_5$ and $\bar{\pi}^*(s_6) = y_6$. However, these statements are contradictory, since $Sq^1(y_5) = y_6$, while $Sq^1(s_5) = 0$.

Example 12. (See [3, Section 10] and [4, Theorems 8.7 and 8.8].) A situation quite similar to that above is obtained if we specialize the following to the case where n is odd. For any $n > 1$ and $k < n$, let $G = SO(n - 1)$, let $K = SO(n)$, and let $H = SO(n - k - 1)$. Then $K/G = S^{n-1}$. Here of course we are considering the well-understood problem, whether the $(n - 1)$-sphere has k independent vector fields. Suppose k is the largest power of 2 dividing n. If the group G of π were reducible to H, we would have the fibration

$$SO(n - k - 1) \to SO(n) \xrightarrow{\bar{\pi}} S^{n-1} \times (SO(n - 1)/SO(n - k - 1)).$$

There are classes $x \in H^{n-k-1}(H \setminus G)$ and $y \in H^{n-k-1}(K)$ such that $\bar{\pi}^*(x) = y$. But, as calculated in Theorems 6.1 and 7.2 of Chapter 4 of [11], the class $Sq^k(y)$ is the generator of $H^{n-1}(K)$, while for dimensional reasons $Sq^k(x) = 0$.

Example 13. (See [2] and [12, p. 151].) Let $G = F_4$, let $K = E_6$, and let $H = Spin(9)$. Then

$$H \setminus G = \Pi = \text{the Cayley projective plane}.$$

If the group G of π were reducible to H, we would have the relations

$$E_2(\bar{\pi}) = H^*((K/G) \times \Pi) \otimes H^*(H)$$

$$= (\Delta(z_9, z_{17}) \otimes Z_2[r_8]/r_8^3) \otimes \Delta(w_3, w_5, w_6, w_7, y_{15}),$$

and this spectral sequence would converge to

$$H^*(K) = \Delta(x_3, x_5, x_6, x_9, x_{15}, x_{17}, x_{23}).$$

Since $H^7(K) = 0$, the class $1 \otimes w_7$ cannot persist to a non-0 class in $E_\infty(\bar{\pi})$. However, since $E_2^{p\ q}(\bar{\pi}) = 0$ for $2 \leq p \leq 7$, the class $1 \otimes w_7$ persists to E_∞. Yet a class in $E_2^{0\ 7}(\bar{\pi})$ cannot be in the image of any d_r, giving a contradiction.

Example 14. (See [2] and [12, p. 151].) Let $G = E_6$, let $K = E_7$, and let $H = F_4$. If the group G of π were reducible to H we would have the relations

$$E_2(\bar{\pi}) = H^*((K/G) \times (H \setminus G)) \otimes H^*(H)$$

$$= (\Delta(z_{10}, z_{18}, z_{27}) \otimes \Delta(w_9, w_{17})) \otimes \Delta(x_3, x_5, x_6, x_{15}, x_{23}),$$

and this spectral sequence would converge to

$$H^*(K) = (Z_2[r_3, r_5, r_9]/(r_3^4, r_5^4, r_9^4)) \otimes \Delta(r_{15}, r_{17}, r_{23}, r_{27}).$$

The class $w_{17} \otimes 1$ persists to E_∞. The only possibility for it to be in the image of some d_r occurs if it is equal to $d_7(z_{10} \otimes z_6)$; however,

$$d_7(z_{10} \otimes x_6) = (z_{10} \otimes 1) \cdot d_7(1 \otimes x_6) = 0.$$

Thus $\bar{\pi}^*(w_{17})$ must be the unique 17-dimensional class r_{17}. But $Sq^1(r_{17}) = r_9^2$, while $Sq^1(w_{17}) = 0$.

Example 15. (See [4, Section 20].) Let $G = \mathrm{Spin}(9)$, let $K = F_4$, and let $H = \mathrm{Spin}(7)$. Then $K/G = \Pi$ and $H \setminus G = S^{15}$. If the group G of π were reducible to H, we would have the relations

$$E_2(\bar{\pi}) = H^*(\Pi \times S^{15}) \otimes H^*(H) = ((Z_2[e_8]/e_8^3) \otimes E[s_{15}]) \otimes \Delta(x_3, x_5, x_6, y_7),$$

and this spectral sequence would converge to $H^*(F_4) = \Delta(w_3, w_5, w_6, w_{15}, w_{23})$. The class $s_{15} \otimes 1$ must persist to E_∞. The only possibility for it to be an image occurs when it equals $d_7(e_8 \otimes x_6)$, which is however

$$(1 \otimes x_6) \cdot d_7(e_8 \otimes 1) + (e_8 \otimes 1) \cdot d_7(1 \otimes x_6) = 0.$$

Thus $\bar{\pi}^*(s_{15}) = w_{15}$. This gives a contradiction, since $Sq^8(w_{15}) = w_{23}$ while $Sq^8(s_{15}) = 0$.

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