INTEGRAL DOMAINS THAT SATISFY GAUSS'S LEMMA

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INTRODUCTION

Let D be a commutative integral domain with a unity element, and let D[x] denote the ring of polynomials with coefficients in D. For a polynomial f(x) in D[x], the *content of* f, denoted by A_f, is defined to be the ideal of D generated by the coefficients of f. The polynomial f is said to be *primitive over* D in case A_f is contained in no proper principal ideal of D, or equivalently, if no nonunit of D divides every coefficient of f. Primitive polynomials arise in the classical theory of unique factorization domains (UFD's) and in the theory of GCD-domains--those domains in which every pair of elements has a greatest common divisor--in the proof that both the class of all GCD-domains and the subclass of all UFD's are closed under polynomial extensions. Specifically, they appear in the preliminary result that if D is a GCD-domain, then the product of two primitive polynomials over D is primitive. This proposition is usually called Gauss's Lemma.

In this paper, we investigate the class of domains with the property of satisfying the conclusions in Gauss's Lemma. This property, which for obvious reasons we call the GL-property, is defined formally as follows:

Definition. A domain D has the GL-*property* if the product of two primitive polynomials over D is always a primitive polynomial.

In a related study of primitive polynomials over an arbitrary domain, H. T. Tang [6] presents a new concept closely related to primitivity, by defining a polynomial f(x) in D[x] to be *superprimitive over* D in case $A_f^{-1} = D$. Tang shows that, without any restrictions on D, every superprimitive polynomial is primitive, and furthermore, that the product of a primitive polynomial and a superprimitive polynomial is again primitive [6, Theorems C and D, p. 374]. The latter result is a generalization of Gauss's Lemma, since over a GCD-domain, a polynomial is primitive if and only if it is superprimitive [6, Theorem H, p. 375]. These results lead naturally to the study of the following property:

Definition. A domain D is said to have the PSP-*property* if every primitive polynomial over D is superprimitive.

In Section 1 of this paper, we characterize both the GL-property and the PSP-property in ideal-theoretic terms, and we derive a number of properties of domains with the GL-property, including the fact that irreducible elements are necessarily prime. Section 2 is largely devoted to the relation between the PSP-property and the GL-property. Since it follows directly from Tang's results cited earlier that the PSP-property implies the GL-property, we ask whether the reverse implication is true in general. We present a counterexample to this implication, but we also present some conditions on a domain that together with the GL-property are sufficient to imply the PSP-property. Additional results in Section 2 include an embedding theorem for a domain with the GL-property in a domain with the PSP-property,

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and an example showing that the condition that irreducible elements are prime does not imply the GL-property. The final section considers the question when the polynomial ring D[x] has the GL-property. We characterize the domains D for which D[x] has the GL-property, showing in the process that neither the class of domains with the GL-property nor the class of domains with the PSP-property is closed under polynomial extensions. Moreover, we show that for a polynomial ring, the GL-property, the PSP-property, and the property that all irreducible elements are prime are all equivalent. Also, we show that the polynomial ring in a family of variables over D has the GL-property if and only if the single-variable polynomial ring over D has the GL-property. Finally, we present some examples concerning questions raised in this section.

Throughout this paper, the word "domain" denotes a commutative integral domain with a unity element. In other respects, our notation and terminology are those of Gilmer [3]. The authors wish to thank Prof. M. Boisen for originally suggesting the study of the GL-property and for a number of other helpful suggestions in the writing of this paper.

1. IDEAL-THEORETIC PROPERTIES OF DOMAINS WITH THE GL-PROPERTY

While the terms "primitive" and "superprimitive" have been defined so far only for polynomials over a domain D, they can be applied in a rather obvious fashion to ideals of D.

1.1 *Definition.* Let I be a finitely generated ideal of D. Then I will be called *primitive* in case I is contained in no proper principal ideal of D, and I will be called *superprimitive* in case $I^{-1} = D$. (In other words, I is a primitive (respectively, superprimitive) ideal if I is the content of a primitive (respectively, superprimitive) polynomial over D.)

These definitions enable us now to characterize both the GL-property and the PSP-property without ever mentioning polynomials. These characterizations will be particularly useful in Section 3, where we consider when the polynomial ring D[x] has the GL-property.

1.2 PROPOSITION. *A domain* D *has the* GL-*property if and only if the product of any two primitive ideals of* D *is primitive. Moreover,* D *has the* PSP-*property if and only if every primitive ideal of* D *is superprimitive.*

Proof. In view of the remark in Definition 1.1, the verification of the second sentence is completely elementary. Now suppose that D has the GL-property, and that I and J are primitive ideals of D. Then $I = A_f$ and $J = A_g$ for some polynomials f and g in D[x], which are necessarily primitive. Then fg is primitive, and therefore A_{fg} is a primitive ideal; that is, A_{fg} is contained in no proper principal ideal of D. But $A_{fg} \subseteq IJ$; therefore IJ is contained in no proper principal ideal of D, and hence it is a primitive ideal.

Conversely, suppose that D does not have the GL-property and that f and g are primitive polynomials in D[x] such that fg is not primitive. Then there is a positive integer m such that $(A_f)^{m+1} A_g = (A_f)^m A_{fg}$ [3, Theorem 28.1, p. 343]. Since fg is not a primitive polynomial, the right-hand side of this expression is contained in some proper principal ideal of D. But the left-hand side is a product of primitive ideals, and from this it follows inductively that some product of two primitive ideals of D is not a primitive ideal of D. ■

This result leads to another ideal-theoretic characterization of the GL-property.

1.3 PROPOSITION. *The domain* D *has the* GL-*property if and only if every ideal that is maximal with respect to the property of containing no primitive ideals is prime.*

Proof. Assume D has the GL-property, and suppose A is an ideal that is maximal with respect to the property of containing no primitive ideal. Let b and c be elements of D - A. Then A + bD and A + cD both contain primitive ideals. Therefore (A + bD)(A + cD) contains a primitive ideal, which implies that (A + bD)(A + cD) $\not\subseteq$ A. In other words, bc $\notin$ A; therefore A must be prime.

Conversely, suppose that D does not have the GL-property and that I and J are primitive ideals such that IJ is not primitive. Let $\mathscr{S}$ denote the set of all ideals that contain IJ and contain no primitive ideal. Then $\mathscr{S}$ is not empty, since IJ $\in \mathscr{S}$. Let $\mathscr{C}$ be a chain in $\mathscr{S}$, and let B denote the ideal $\bigcup_{C \in \mathscr{C}} C$. Then every finite subset of B is contained in some C $\in \mathscr{S}$; therefore, every finite subset of B generates an ideal that is not primitive. Consequently B contains no primitive ideal, and thus B is in $\mathscr{S}$. By Zorn's Lemma, $\mathscr{S}$ contains an ideal M that is maximal with respect to the property of containing no primitive ideal. But IJ $\subseteq$ M, with I $\not\subseteq$ M and J $\not\subseteq$ M. Therefore M is not prime, and the proof is complete. ■

The final results of this section involve the concept of an *irreducible* element of D, that is, an element a of D such that aD is maximal among the proper principal ideals of D, or alternately, an element a that can only be factored in D as a product of a unit of D with an associate of a in D. Recall that a sufficient condition (but not a necessary one) for a to be irreducible is that a be *prime*, that is, that aD be a prime ideal of D.

1.4 PROPOSITION. *Let* D *be a domain with the* GL-*property, and let* d *be an irreducible element of* D. *Then*

(i) d *is prime, and*

(ii) *if* I *is a primitive ideal containing* d, *then* I *is superprimitive.*

Proof. (i) Assume d is not prime. Then there exist a and b not in dD such that ab is in dD. Therefore dD ≠ (a, d), and this implies that (a, d) is primitive, since otherwise dD would be properly contained in a proper principal ideal of D. Similarly, (b, d) is primitive. But the product ideal (a, d)(b, d) is contained in dD, and therefore it is not primitive. Hence D does not have the GL-property.

(ii) Now suppose I is a primitive ideal that contains d and is not superprimitive. Pick k in I^{-1} - D. If J = (kd, d), then, since kd $\in$ D - dD, the argument used above shows that J is primitive. But then JI = (kd, d)I = kdI + dI; consequently JI is contained in dD, since k $\in I^{-1}$. Therefore JI is not primitive, and hence D does not have the GL-property. ■

2. RELATIONS BETWEEN THE GL-PROPERTY AND SOME RELATED DOMAIN PROPERTIES

In this section we consider the relations between the following conditions on a domain D:

(i) D is a GCD-domain.

(ii) D has the PSP-property.

(iii) D has the GL-property.

(iv) Every irreducible element of D is prime.

We begin by observing that each of these conditions implies its successor. That (i) implies (ii) and (ii) implies (iii) follows at once from the results of Tang mentioned in the Introduction. The final implication was proved in the preceding section. We further observe that in the presence of the ascending-chain condition on principal ideals, condition (iv) implies that D is a UFD, which in turn implies (i). Thus under this additional hypothesis, all four conditions are equivalent.

In general, however, no two of these conditions are equivalent, and we have counterexamples to each of the three reverse implications. An example showing that (ii) $\not\Rightarrow$ (i) is found in Example 3.11 of the next section.

Before we develop an example to show that (iii) $\not\Rightarrow$ (ii), we present a few positive results to suggest how closely related the GL-property and the PSP-property are. In particular, we show that in the presence of certain additional hypotheses, a domain with the GL-property necessarily has the PSP-property. Moreover, we show that every domain D with the GL-property has an overring with the PSP-property that is "close to" D in the sense that no nonunit of D is invertible in the overring.

2.1 PROPOSITION. *Let* D *be a domain with the* GL-*property in which every finitely generated proper ideal is in the radical of some principal proper ideal. Then* D *has the* PSP-*property.*

Proof. We shall prove this result by showing that every primitive ideal is equal to D itself, hence trivially superprimitive. Let I be a proper ideal of D that is finitely generated. Then by hypothesis, $I \subseteq \sqrt{(d)}$ for some nonunit d of D; thus $I^k \subseteq (d)$ for some natural number k. Hence I is not primitive. ∎

2.2 COROLLARY. *If* D *is a domain with the* GL-*property and the* QR-*property* [3, p. 334], *then* D *has the* PSP-*property as well.*

2.3 COROLLARY. *If* D *is a one-dimensional domain with the* GL-*property and with only a finite number of maximal ideals, then* D *has the* PSP-*property.*

Proofs. Corollary 2.2 follows at once from the proposition and the result of Pendleton [5, Theorem 5, p. 500] that in a domain with the QR-property, every finitely generated ideal shares its radical with a principal ideal. A domain satisfying the hypotheses of Corollary 2.3 has the property that every maximal ideal is the radical of a principal ideal. This follows immediately from the well-known result that the union of a finite set of prime ideals cannot contain a prime ideal unrelated to those in the set. ∎

2.4 THEOREM. *If* D *is a domain with the* GL-*property, then there exists an overring* E *of* D *that contains the inverse of no nonunit of* D *and has the* PSP-*property.*

Proof. Let K denote the quotient field of D, and let S denote the set of all primitive ideals of D.

Since S is a multiplicatively closed set of ideals, the set

$$\{x \in K \mid xI \subseteq D \text{ for some } I \in S\}$$

is an overring of D, which we shall denote by E. We begin by showing that no nonunit of D is invertible in E. Assume $d \in D$ and $d^{-1} \in E$. Then $d^{-1}I \subseteq D$ for some primitive ideal I. Hence, $I \subseteq dD$; therefore d must be a unit of D.

Next let J be a primitive ideal of E, with generators $j_1, j_2, \cdots, j_n$. We wish to show J is superprimitive. For each index k, let I_k denote a primitive ideal of D such that $j_k I_k \subseteq D$. We know that the product $A = I_1 \cdots I_n$ is a primitive ideal of D and that $j_k A \subseteq D$ for every index k. Now let B denote $j_1 A + \cdots + j_n A$, a finitely generated ideal of D. We claim that B is primitive. If not, then $B \subseteq dD$ for some nonunit d of D. Therefore, $j_k A \subseteq dD$ for all k, so that $d^{-1} j_k A \subseteq D$ for all k. Hence $d^{-1} j_k$ is in E for all k, which means j_k is in dE for all k. But by the argument in the preceding paragraph, d is a nonunit of E; thus we conclude that $J \subseteq dE \neq E$. This contradicts the way we chose J, and we conclude that B is primitive ideal of D.

Now let x be an element of J^{-1}. Then $xJ \subseteq E$; in fact, xJ is a finitely generated ideal of E, with generators $xj_1, \cdots, xj_n$. By the reasoning used earlier to construct the ideal A, there is a primitive ideal C of D such that $xj_k C \subseteq D$ for all k. Clearly, $xj_k AC \subseteq D$ for all k, which implies that $xj_1AC + \cdots + xj_nAC \subseteq D$. This means that $x(j_1 A + \cdots + j_n A)C \subseteq D$, so that $xBC \subseteq D$. But we know BC is a primitive ideal of D, and hence we conclude that x is in E. This implies that $J^{-1} = E$, which is what we wanted to show. ∎

Next we construct the counterexample to show that (iii) $\not\Rightarrow$ (ii).

2.5 *Example.* Let Q^+ denote the set of positive rational numbers, and let Q_0^+ denote $Q^+ \cup \{0\}$. Let F denote the field of two elements, and let J be the ring $F[\{X^\alpha \mid \alpha \in Q_0^+\}, \{Y^\alpha \mid \alpha \in Q_0^+\}, \{Z^\alpha \mid \alpha \in Q_0^+\}]$ of all "polynomials" over F in which the exponents for the indeterminates range over Q_0^+ instead of merely the nonnegative integers. Finally, let D denote the subring

$$F[\{X^\alpha \mid \alpha \in Q_0^+\}, \{Y^\alpha \mid \alpha \in Q_0^+\}, \{X^\alpha Z^\beta \mid \alpha \in Q^+, \beta \in Q^+\}, \{Y^\alpha Z^\beta \mid \alpha \in Q^+, \beta \in Q^+\}]$$

of J. We shall show that D has the GL-property (Proposition 2.8) but not the PSP-property (Proposition 2.9). We need the following two lemmas.

2.6 LEMMA. J *is a* GCD-*domain.*

Proof. J may be considered as the semigroup ring over F of the semigroup $S = Q_0^+ \times Q_0^+ \times Q_0^+$. Since S is clearly a GCD-semigroup, J is a GCD-domain [4, Theorem 6.4, p. 76]. ∎

2.7 LEMMA. *Assume* f *and* g *are in* J *and* fg *is in* D. *Then* f *is in* D *or* g *is in* D. *If, in addition,* f *is in* D *and* f *has nonzero constant term, then* g *is in* D.

Proof. Let I denote the ideal of J generated by the set

$$\{X^\alpha \mid \alpha \in Q^+\} \cup \{Y^\alpha \mid \alpha \in Q^+\}.$$

Then I is also an ideal of D. Furthermore, every element f of J can be written as $f_1 + f_2$, where $f_1 \in I$ and $f_2 \in F[\{Z^\alpha \mid \alpha \in Q_0^+\}]$. Write f and g as $f_1 + f_2$ and $g_1 + g_2$, as above. Then $fg = f_1g_1 + f_2g_1 + f_1g_2 + f_2g_2$, and since $fg \in D$ and $f_1g_1 + f_2g_1 + f_1g_2 \in I \subseteq D$, it follows that $f_2g_2 \in D$. Thus the only possibility for

f_2g_2 is that it is a constant polynomial in $F[\{Z^\alpha \mid \alpha \in Q_0^+\}]$. If it is the zero polynomial, then either $f_2 = 0$ or $g_2 = 0$; thus either f or g is in D. In case f_2g_2 is nonzero, then both f_2 and g_2 must be nonzero constant polynomials, and the sums $f_1 + f_2$ and $g_1 + g_2$ must both be in D. Finally, if f is in D with nonzero constant term, then $f_2 \neq 0$. Consequently either $f_2g_2 = 0$, in which case $g_2 = 0$, so that $g = g_1 \in D$; or $f_2g_2 \neq 0$, in which case the preceding sentence applies. In either case, g is in D, as required. ■

2.8 PROPOSITION. D *has the* GL-*property.*

Proof. Assume $A = (a_1, \cdots, a_m)$ and $B = (b_1, \cdots, b_n)$ are ideals of D such that the product AB is not primitive. We shall show that A and B are not both primitive. Let c be a nonunit of D such that $AB \subseteq cD$. Then c is also a nonunit of J, and

$$(AJ)(BJ) = (AB)J \subseteq (cD)J = cJ \neq J.$$

Since J is a GCD-domain, it follows that not both AJ and BJ are primitive ideals in J. If we let λ and μ denote the gcd's in J of $\{a_1, \cdots, a_m\}$ and $\{b_1, \cdots, b_n\}$, respectively, then not both λ and μ are units of J. Still taking gcd's in J, we see that $a_i\mu = \gcd(a_ib_1, \cdots, a_ib_n)$ for each i. But $a_ib_j \in AB \subseteq cJ$ for each j, and hence $\gcd(a_ib_1, \cdots, a_ib_n)$ is in cJ for each i. In other words, $a_i\mu \in cJ$ for each i, and consequently $\gcd(a_1\mu, \cdots, a_m\mu)$ is in cJ. But

$$\gcd(a_1\mu, \cdots, a_m\mu) = \mu \gcd(a_1, \cdots, a_m) = \lambda\mu.$$

Therefore $\lambda\mu \in cJ$. Since J is a GCD-domain, $c = \alpha\beta$, where $\alpha, \beta \in J$ and $\lambda \in \alpha J$, $\mu \in \beta J$ [2, Theorem 2.4, p. 256]. By the preceding lemma, either α or β is in D, since their product c is in D. Moreover, if α is a unit of D, then α is a constant polynomial, so that $\beta = c\alpha^{-1}$ is also in D, and therefore β is a nonunit, since c is. Hence either α or β, say α, is a nonunit of D. Then $AJ \subseteq \lambda J \subseteq \alpha J$; consequently, for each i there exists $\gamma_i \in J$ such that $a_i = \alpha\gamma_i$. We consider two cases on α. If α has nonzero constant term, then by the preceding lemma, each γ_i is in D, because the product $\alpha\gamma_i$ is in D. Then $a_i \in \alpha D$ for all i, so that $A \subseteq \alpha D$ and hence A is not primitive. The other case is where α has constant term zero. In this case, we can find a nonunit δ in D such that $\delta^2 = \alpha$, simply by summing the terms obtained on taking each term of α and dividing the exponents by 2 (since F is the field of two elements). Moreover, δ is in the ideal I of D defined in the proof of Lemma 2.7; thus $\gamma_i\delta \in \gamma_i I \subseteq JI = I \subseteq D$. Since $a_i = (\gamma_i\delta)\delta$ for all i, it follows that $A \subseteq \delta D$. Again we conclude that A is not primitive, and the proof is complete. ■

2.9 PROPOSITION. D *does not have the* PSP-*property.*

Proof. Consider the ideal $A = XD + YD$. Since X and Y are contained in no proper principal ideal of J, and since nonunits of D are nonunits of J, we see that X and Y are contained in no proper principal ideal of D. Hence A is a primitive ideal of D. But $Z \in A^{-1} - D$, since $ZA \subseteq D$ and Z is in the quotient field of D. Therefore A is not superprimitive, and the proof is complete. ■

The final result of this section is the construction of a domain to illustrate that (iv) $\not\Rightarrow$ (iii).

2.10 *Example.* Let F be the field of two elements, and let

$$D_0 = F[\{X_1^\alpha, X_2^\alpha, Y_1^\alpha, Y_2^\alpha, Z^\alpha \mid \alpha \in Q_0^+\}].$$

Let D be $D_0[\{X_i^{\alpha} Y_j^{\beta} Z^{-\delta} \mid \alpha, \beta, \delta \in Q^+;\ i, j = 1, 2\}]$. Then D does not have the GL-property; but irreducible elements of D are prime (since every nonunit of D is reducible).

Proof. It will be convenient to think of D as the semigroup ring over F of the semigroup S of 5-tuples of rational numbers in which the first four coordinates are always nonnegative and either the fifth is nonnegative or there is a positive entry in one of the first two entries and a positive entry in one of the third and fourth entries. We claim that the ideal $I = (X_1, X_2)$ is primitive in D. Let d be a divisor of X_1. Then d must be a monomial [4, Lemma 4.1, p. 70], and by the way D was defined d must be of the form X_1^{α}. Similarly, a divisor of X_2 must be of the form X_2^{α}. Hence X_1 and X_2 have no nonunit common divisor; consequently (X_1, X_2) is primitive. The same argument shows that $J = (Y_1, Y_2)$ is primitive. Now $IJ = (X_1Y_1, X_2Y_1, X_1Y_2, X_2Y_2)$ is contained in the proper principal ideal ZD, and hence is not primitive. Therefore D does not have the GL-property.

The example will be complete when we have shown that D has no irreducible nonunits. For this, we need only observe that every monomial has a square root, and consequently, since F is the field of two elements, every element of D has a square root. ■

3. THE GL-PROPERTY IN POLYNOMIAL RINGS

In this section we consider conditions under which the polynomial ring D[x] has the GL-property. A natural starting place for this inquiry is the question whether the GL-property in D implies the GL-property in D[x], and conversely. Our first proposition shows that the converse implication holds; in fact, the GL-property in D[x] implies the stronger condition that D has the PSP-property. This result provides a negative answer to the original question whether the GL-property in D implies the GL-property in D[x]. If D is the domain in Example 2.5, which has the GL-property but not the PSP-property, then by the proposition, D[x] cannot have the GL-property. Later results will show that even when D has the PSP-property, D[x] does not necessarily have the GL-property. In other words, neither the class of domains with the GL-property nor the class of domains with the PSP-property is closed under the adjunction of a polynomial indeterminate.

3.1 PROPOSITION. *If* D *is an integral domain such that* D[x] *has the* GL-*property, then* D *has the* PSP-*property.*

Proof. Suppose D does not have the PSP-property, and let I be a finitely generated ideal of D that is primitive but not superprimitive, and that has a minimal number of generators among ideals with this property. Let $\{a_0, a_1, \cdots, a_n\}$ be a minimal set of generators of I, and note that $n \neq 0$, since principal primitive ideals are always superprimitive. Let p(x) denote the primitive polynomial $a_0 + a_1 x + \cdots + a_n x^n$ in D[x]. We shall show that p(x) is irreducible. Suppose that $p(x) = g(x)h(x)$, where g(x) and h(x) are polynomials in D[x] of degree strictly smaller than n. Since the product of two superprimitive polynomials is superprimitive [6, Theorem F, p. 374], and the product of a nonprimitive polynomial with an arbitrary polynomial is clearly nonprimitive, it follows that either g(x) or h(x), say g(x), is a primitive polynomial that is not superprimitive. Thus A_g is a primitive ideal that is not superprimitive and has fewer than n generators, contradicting our choice of I. Hence p(x) is irreducible, as we wanted to show. Now, by Proposition 1.4, the polynomial p(x) must be prime. However, Tang [6, Theorem A, p. 372] shows that A_p, which is simply I, must be superprimitive, and this contradicts our assumptions about I. Hence D has the PSP-property. ■

Observe that the essence of the preceding proof is the fact that when D[x] has the GL-property, all irreducible elements of D[x] are prime. This implication is reversed in the following result, which shows that for a ring of polynomials, the GL-property is equivalent to having only prime irreducibles. Note that this equivalence does not hold in general for domains other than polynomial rings, as is shown in Example 2.10.

3.2 THEOREM. *Let* D *be an integral domain. Then* D[x] *has the* GL-*property if and only if all irreducible elements of* D[x] *are prime.*

Proof. We need only show that the second condition implies the first. We begin by proving that every nonconstant primitive polynomial f(x) in D[x] can be expressed as a product of irreducible polynomials. We proceed by induction on deg(f). If deg(f) = 1, then each factorization of f includes a constant polynomial that is necessarily primitive, and hence a unit. Thus f is irreducible. In the general case, f is either irreducible or has a factorization f = gh, where neither g nor h is a unit. But g and h are both primitive, and consequently, neither can have degree zero. Hence each has degree less than deg(f), and by hypothesis each is a product of irreducibles; hence f itself is a product of irreducibles.

Now assume that all irreducible elements of D[x] are prime. Consider two finitely generated primitive ideals I and J of D[x], and suppose IJ is not primitive. Then IJ is contained in some proper principal ideal (f(x)) of D[x]. We consider two cases for f(x).

Case I. The polynomial f(x) is primitive. By the preceding, f(x) is a product of irreducibles, hence a product of primes. Let p(x) be one such prime. Then $IJ \subseteq (f(x)) \subseteq (p(x))$, and (p(x)) is a proper prime ideal of D[x]. This implies that either $I \subseteq (p(x))$ or $J \subseteq (p(x))$, which contradicts the assumption that I and J are primitive ideals.

Case II. The polynomial f(x) is not primitive. There is a nonunit d of D such that $IJ \subseteq (f(x)) \subseteq dD[x]$. Let $\{g_0, g_1, \cdots, g_m\}$ be a generating set for I. We construct a polynomial g^* in I of the form $g_0 x^{r_0} + g_1 x^{r_1} + \cdots + g_m x^{r_m}$ for some choice of $r_0, r_1, \cdots, r_m$ satisfying the inequality $r_{i+1} - r_i > \deg(g_i)$ for each i. In other words, the set of coefficients of g^* is the union of the sets of coefficients of $g_0, g_1, \cdots, g_n$. Therefore g^* is a primitive polynomial, because otherwise there would be a nonunit c of D such that $g_i \in cD[x]$ for each i, and hence $I \subseteq cD[x]$. Similarly, we construct a primitive polynomial h^* in J. As we mentioned earlier, the proof of Proposition 3.1 shows that when irreducible elements of D[x] are prime, then D has the GL-property. Hence $g^* h^*$ is a primitive polynomial. But $g^* h^* \in IJ \subseteq dD[x] \neq D[x]$, which is a contradiction.

Since both Case I and Case II lead to contradictions, we conclude that IJ is a primitive ideal and hence that D[x] has the GL-property. ■

Our next result shows that the GL-property in a polynomial extension of D is independent of the number of indeterminates being adjoined. It also gives conditions on D that are necessary and sufficient for D[x] to have the GL-property. We already have a necessary condition, the PSP-property, but it is not sufficient, as we show in a later example. However, by adding integral closure and a third condition on D, we obtain a characterization of the domains D for which D[x] has the GL-property. The third property involves the concept of the v-operation on fractional ideals of D. For convenience, we list the properties of this operation that we need here. (1) If B is a fractional ideal of D, then B_v is the intersection of all principal fractional ideals of D that contain B, or equivalently, $B_v = (B^{-1})^{-1}$. (2) For any two fractional ideals B and C, $(BC)_v = (B_v C_v)_v$. (3) If (k) is a principal fractional ideal, $(k)_v = (k)$. (4) For a polynomial f in D[x], $(A_f)_v = D$ if and only if f is superprimitive. A detailed discussion of this v-operation can be found in Section 34 of Gilmer's book [3].

3.3 THEOREM. *The following statements are equivalent for an integral domain* D.

(i) D[x] *has the* GL-*property*.

(ii) $D[\{x_\alpha\}_{\alpha \in A}]$ *has the* GL-*property for every nonempty set* $\{x_\alpha\}_{\alpha \in A}$ *of indeterminates*.

(iii) D *satisfies the three conditions*

(α) D *has the* PSP-*property*,

(β) D *is integrally closed, and*

(γ) *if* B *and* C *are finitely generated fractional ideals of* D *such that* $(BC)_v = D$, *then* B_v *is a principal fractional ideal*.

For the proof of this theorem, we need the following result.

3.4 LEMMA. *Let* D *be a domain with the property that irreducible elements in* D[x] *are prime. If* Q *is a prime ideal of* D[x] *such that* $Q \cap D = (0)$, *and if* Q *contains a primitive polynomial over* D, *then* Q *is a principal ideal*.

Proof. Assume Q is a prime ideal in D[x] containing a primitive polynomial and satisfying the condition $Q \cap D = (0)$. By the argument in the proof of Theorem 3.2, for an arbitrary domain D, every primitive polynomial in D[x] is a product of irreducible polynomials. By hypothesis, irreducible elements of D[x] are prime; therefore Q contains a prime element p. But Q is a minimal prime ideal [3, Corollary 30.4, p. 356]; thus Q must equal (p), as we wanted to show.

Proof of Theorem 3.3. We shall prove the sequence (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i), of which the last implication is trivial. Throughout the proof, K will denote the quotient field of D.

(i) $\Rightarrow$ (iii). Assume D[x] has the GL-property. We already know D has the PSP-property. To show D is integrally closed, assume t is an element of K that is integral over D. Let f(x) be the monic polynomial of minimal degree in D[x] with the property that f(t) = 0. If f(x) were reducible in D[x], we could write f(x) = g(x) h(x), where g(x) and h(x) are both monic of smaller degree; in this case, either g(t) or h(t) would have to be zero, contradicting the minimality of deg (f). Thus f(x) is irreducible in D[x], hence prime; therefore, f(x) is irreducible over K [6, Theorem A, p. 372]. But f(t) = 0; therefore in K[x] the polynomial f(x) must equal (x - t) g(x), for some g(x) in K[x]. Consequently, g(x) must be a constant polynomial; in fact, it must be the constant polynomial 1, since f is monic. Thus, x - t equals f(x) and hence x - t is in D[x]. Thus $t \in D$, and therefore D is integrally closed.

Finally, we show that condition (γ) holds in D, as follows: Let B and C be finitely generated fractional ideals of D such that $(BC)_v = D$. Pick polynomials f(x) and g(x) in K[x] such that $A_f = B$ and $A_g = C$, and let $f(x) = a^{-1} h(x)$, with $a \in D$ and $h(x) \in D[x]$. Now we know that D is integrally closed, hence $(A_{gf})_v = (A_g A_f)_v$ [3, Proposition 34.8, p. 424]. In other words, $(A_{gf})_v = D$, which means that g(x) f(x) is a primitive, in fact, a superprimitive polynomial in D[x]. Moreover, $A_g A_f \subseteq D$, so that $A_g \subseteq A_f^{-1}$. In particular, $g(x) \in A_f^{-1} D[x]$. Therefore,

$$g(x)f(x) \in A_f^{-1} f(x)D[x] = A_h^{-1} h(x)D[x],$$

and by [6, Theorem B, p. 373], the right-hand member is

$$h(x)K[x] \cap D[x] = a^{-1} h(x)K[x] \cap D[x] = f(x)K[x] \cap D[x].$$

Consider a minimal prime ideal Q of $D[x]$ that contains $h(x)$ and satisfies the condition $Q \cap D = (0)$. Viewing $K[x]$ as the quotient overring of $D[x]$ with respect to $D - \{0\}$, we see that Q extends to a minimal prime ideal Q^e of $K[x]$ that contains $f(x)$ and contracts to Q. Therefore Q contains the ideal $(f(x))K[x] \cap D[x]$, which contains the primitive polynomial $g(x)f(x)$. By Lemma 3.4, Q is a principal prime ideal, and consequently its generator $p(x)$ is superprimitive [6, Theorem A, p. 372]. Now, since the prime ideals of $D[x]$ satisfying the assumptions for Q correspond to the minimal prime ideals of $K[x]$ containing $f(x)$, there are a finite number of them, say $Q_1, \cdots, Q_n$, each generated by a superprimitive polynomial $p_i(x)$ in $D[x]$. Moreover, $p_i(x)$ generates Q_i^e in $K[x]$; therefore in $K[x]$ the polynomial $f(x)$ must factor as $u \cdot p_1(x)^{m_1} \cdots p_n(x)^{m_n}$, where u is a unit of $K[x]$, that is, a constant polynomial. Now, by repeated application of Theorem 34.8 of Gilmer [3], it follows that

$$(A_f)_v = (A_{up_1(x)^{m_1}\cdots p_n(x)^{m_n}})_v = ((A_u)(A_{p_1(x)^{m_1}}) \cdots (A_{p_n(x)^{m_n}}))_v$$

$$= ((A_u)_v (A_{p_1(x)})_v^{m_1} \cdots (A_{p_n(x)})_v^{m_n})_v .$$

But $(A_{p_i(x)})_v = D$ for each i, since $p_i(x)$ is superprimitive. Moreover, $(A_u)_v$ is merely the ideal (u), and therefore $(A_f)_v$ is the principal fractional ideal (u). This is what we wanted to show, and it completes the proof that (i) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (ii). Assume D satisfies conditions (α), (β), and (γ) of (iii). Let D' denote $D[\{x_\alpha\}_{\alpha\in A}]$ for some nonempty set of indeterminates $\{x_\alpha\}_{\alpha\in A}$, and let E denote $K[\{x_\alpha\}_{\alpha\in A}]$. To show that D' has the GL-property, it will suffice by Theorem 3.2 to show that irreducible elements of D' are prime. Let f denote an irreducible polynomial in D'. Then f is necessarily primitive over D, hence superprimitive, by assumption (α). Since E is a unique factorization domain, it has a principal prime ideal P containing f. If q is a polynomial in E that generates P, then $f = qr$ for some polynomial r in E. Since D is integrally closed, by assumption (β), we see that $(A_f)_v = (A_q A_r)_v$ [3, Proposition 34.8, p. 424]. Since f is superprimitive, $D = (A_q A_r)_v$, and by assumption (γ), $(A_q)_v$ is a principal ideal, say equal to kD.

Now, letting q_1 denote the polynomial $k^{-1}q$ in E, we see that

$$(A_{q_1})_v = (k^{-1}A_q)_v = k^{-1}(A_q)_v = k^{-1}(kD) = D.$$

In other words, q_1 is in D', and it is superprimitive over D. Moreover, by our choice of q_1, we know it generates the prime ideal P in E. Thus, $P \cap D' = q_1 E \cap D' = q_1 D'$ [3, Corollary 34.9, p. 424], and hence q_1 generates a prime ideal of D' that contains f. Since f is irreducible, $fD' = q_1 D'$, and hence f is a prime element of D', as we wanted to show. ∎

As we noted in the introduction, Gauss's Lemma is a step in the proof that polynomial rings over GCD-domains are again GCD-domains [3, Theorem 34.10,

p. 424]. In particular, if D is a Bezout domain, then D[x] is a GCD-domain [3, Exercise 12, p. 79], and hence it has the GL-property. On the other hand, if D is a Prufer domain and B is a finitely generated ideal of D, then B is invertible, so that $B = (B^{-1})^{-1} = B_v$ [3, Theorem 22.1, p. 276]. Consequently, if D is a Prüfer domain satisfying condition (γ) of Theorem 3.3, then D is a Bezout domain, since every finitely generated ideal is a principal ideal. In short, we have proved the following corollary, characterizing the Prüfer domains D for which D[x] has the GL-property.

3.5 COROLLARY. *Let* D *be a Prüfer domain. Then* D[x] *has the* GL-*property if and only if* D *is a Bezout domain.*

Another corollary to Theorem 3.3 is the following result, namely, that for D[x] the GL-property and the PSP-property are equivalent. This result and Theorem 3.2 together show that the last three of the four nonequivalent conditions considered in Section 2 are equivalent, for a ring of polynomials. The first condition from Section 2, that of being a GCD-domain, is still not equivalent to the other three in this case, as we shall show in Example 3.11.

3.6 COROLLARY. *Let* D *be an integral domain. Then* D[x] *has the* GL-*property if and only if* D[x] *has the* PSP-*property.*

Proof. If D[x] has the GL-property, then by the fact that (i) $\Rightarrow$ (ii) in the preceding theorem, we know D[x, y] has the GL-property. Now, considering D[x, y] as D[x][y], we see from (i) $\Rightarrow$ (iii) of the same theorem that D[x] has the PSP-property. ■

The rest of this paper is devoted to the construction of examples. Since the examples we obtain are all special cases of a general construction, the D + M construction [1], we shall develop some common notation and preliminary results before stating the examples explicitly.

3.7 *Notation.* Let K be a field, and let V be a rank-one valuation ring of the form K + M, where M is the maximal ideal of V. We shall consider subrings of V of the form F + M, where F is a proper subfield of K. Since our examples are concerned with the PSP-property, we characterize those rings F + M with this property in the following lemma.

3.8 LEMMA. *Let* D *denote* F + M, *as in Paragraph* 3.7. *Then* D *has the* PSP-*property if and only if* V *is a nondiscrete valuation ring.*

Proof. Assume V is a nondiscrete valuation ring. Then its value group is a subgroup of the real numbers having no smallest positive element. Let A be a finitely generated ideal contained in M. Then there is an element x of V with smaller positive value than any element of A. Hence $A \subseteq xM \subseteq xD$, and since x has positive value, $xD \neq D$. Therefore A is not primitive, and consequently no proper ideal of D is primitive. Hence D has the PSP-property trivially.

Now assume V is a discrete valuation ring with the integers as its value group. Then, if m is an element of M of value 1, it is irreducible in D, but not prime, since it doesn't generate M as an ideal of D. By Proposition 1.4, D does not have the GL-property, hence not the PSP-property, either. ■

We also need the following result.

3.9 LEMMA. *If* D *is as in Lemma* 3.8, *then* D *satisfies condition* (γ) *of Theorem* 3.3.

Proof. If we consider V as a fractional ideal of D, then $V^{-1} = M$. Hence $M_v = (M^{-1})^{-1} = ((V^{-1})^{-1})^{-1} = V^{-1} = M$. Consequently, for every ideal I of D contained in M, $I_v \subseteq M_v \subseteq M$. Hence, if A and B are finitely generated fractional ideals of D such that $(AB)_v = D$, then $AB \subseteq D$ and $AB \not\subseteq M$. In other words, AB = D, so that A is an invertible ideal of D. Since D is quasilocal, A must be a principal fractional ideal, as we wanted to show [3, Proposition 7.4, p. 72]. ■

3.10 *Remark.* The argument used above proves a slightly more general result, which may be of assistance in generating examples of domains satisfying the conditions in Theorem 3.3. The result we have in mind reads as follows: Let D be a semi-quasi-local domain with the property that the only superprimitive ideal of D is D itself. Then D satisfies condition (γ) of Theorem 3.3.

The first example we construct shows that for a polynomial domain D[x], the PSP-property does not imply that D[x] is a GCD-domain. This answers the question raised in the remarks preceding Corollary 3.6.

3.11 *Example.* Let D be as in Lemma 3.8, where F is algebraically closed in K, and V is nondiscrete. Then D[x] has the PSP-property but is not a GCD-domain.

Proof. Since F is algebraically closed in K, D is integrally closed, [1, Theorem 2.1(b), p. 80]. By Lemmas 3.8 and 3.9, D satisfies the other two conditions in Theorem 3.3 (iii); therefore, by Corollary 3.6, D[x] has the PSP-property. But D is not a GCD-domain [1, Theorem 3.13, p. 85]; therefore neither is D[x], since some pair of constant polynomials will not have a greatest common divisor. ■

The remaining examples arise from the consideration of the relations between the three conditions on D in Theorem 3.3 (iii). Specifically, we want to know whether any one of them is superfluous in the sense of being implied by the other two. That the PSP-property is not implied by the other two is shown by an example in which F is algebraically closed in K and V is a discrete rank-one valuation ring. On the other hand, if F is not algebraically closed in K, and V is nondiscrete, then the resulting example shows that integral closure is not a consequence of the other two conditions. Both of these results follow immediately from the lemmas. Note also that in the second example, D has the PSP-property but D[x] does not; from this we conclude that the class of domains with the PSP-property is not closed under polynomial extensions.

The question remains whether condition (γ) of Theorem 3.3 (iii) is implied by the other two conditions. We do not know the answer, but we conjecture that it is negative. Even in some very special cases, the answer does not seem to be known. For example, it seems to be an open question whether a Prüfer domain with the PSP-property satisfies condition (γ). Recalling the observation preceding Corollary 3.5 that a Prüfer domain satisfying (γ) is a Bezout domain, we can recast the question whether a Prüfer domain with the PSP-property is a Bezout domain. Even in the very special case of a Prüfer domain D in which every maximal ideal is principal (so that D has the PSP-property trivially), it seems to be an open question whether D is a Bezout domain.

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