ON MS-FIBERINGS OF MANIFOLDS WITH FINITE SINGULAR SETS

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1. INTRODUCTION

Throughout the paper, we follow the notation and terminology of [1]. All manifolds are assumed to be closed, connected, and orientable, unless specifically stated otherwise, and they are taken in any of the categories Top, Diff, and PL. Fiberings shall be locally trivial and orientable. By a Hopf-type fibering of spheres, we mean a fibering $h: S^m \to S^q$ with fibre S^{m-q}, where $(m, q) = (3, 2), (7, 4)$, or $(15, 8)$. An MS-fibering $f: M^n \to N^p$ $(n > p)$ of manifolds is an open continuous map with the property that there exist closed, nonseparating sets A and B in M and N, respectively, satisfying the following conditions.

(i) $f(A) = B$ and $f \mid A: A \to B$ is a homeomorphism.

(ii) $f(M - A) \subset N - B$, and $f \mid (M - A): M - A \to N - B$ is a locally trivial fibration whose fibre is a manifold. The set A is referred to as the singular set of the MS-fibering f.

In [1], it is conjectured that if an MS-fibering $f: M^n \to N^p$ $(n > p)$ of manifolds has finite singular set A, then A consists of exactly two points provided

(1) f admits a spine fibering,

(2) N^p is the standard sphere S^p, and

(3) M^n is simply connected.

In this paper we prove that this conjecture is true in the following stronger form.

THEOREM A. *If* $f: M^n \to N^p$ $(n > p)$ *is an* MS-*fibering of manifolds with finite, nonempty singular set* A *and if* f *admits a spine, then* $\#(A) = 2$. *Moreover,* f *is* Top- *or* PL-*equivalent to the suspension of a Hopf-type fibering of spheres according as* f *is in* Top *or* PL (*modulo the Poincaré conjecture in dimensions* 3, 4).

Also, by means of the results in [1] one can easily prove the following assertion.

THEOREM B. *Let* $f: M^n \to N^p$ $(n > p)$ *be an* MS-*fibering of manifolds with singular set* A. *If* M^n *is* $([n/2] - 1)$-*connected and* $\#(A) = 2$, *then* f *admits a spine* (the square bracket denotes the greatest-integer function).

Combining these two theorems, we see that an MS-fibering $f: M^n \to N^p$ $(n > p)$ with singular set A is Top-equivalent to the suspension of a Hopf-type fibering if and only if M^n is $([n/2] - 1)$-connected and $\#(A) = 2$. One of the authors has discovered a large class of MS-fiberings $f: M^n \to S^p$ in Diff, with A-finite, nonempty and M^n $([n/2] - 1)$-connected [2]. It follows from our results that the suspension of a Hopf-type fibering is the only one among them admitting a spine.

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2. A SKETCH OF THE PROOF OF THEOREM A

It is easy to show that if a finite set of points A is removed from a given manifold M and the resulting set has a manifold as deformation retract, then $\#(A) \leq 2$. It will be shown that the case $\#(A) = 1$ cannot occur when A is the singular set of an MS-fibering. This is not to say, however, that the case $\#(A) = 1$ cannot occur more generally. For example, if FP^n denotes either real, complex, quaternionic, or Cayley projective space, then $FP^{n-1} \subseteq FP^n - *$ is a deformation retract of $FP^n - *$. Indeed, this property characterizes projective spaces to a large extent, and it is the topic of a separate paper of the authors.

Assuming that $\#(A) = 2$, it remains to show that the MS-fibering is actually Top- or PL-equivalent to the suspension of a Hopf-type fibering of spheres. As usual, $(n, p) = (4, 3)$, $(8, 5)$, and $(16, 9)$ are the only possible pairs of dimensions, as the local results in [3] show. Thus we have only to prove that $M^n = S^n$ and $N^p = S^p$, assuming $\#(A) = 2$. This is done in Section 4.

3. COLLAPSING POINT COMPLEMENTS INTO SUBMANIFOLDS

We need the following proposition for our proof of Theorem A.

PROPOSITION 3.1. *Let* M^p *be a manifold (not necessarily orientable) in* Top, Diff, *or* PL, *and let* $x \in M$. *If* $M - x$ *has the homotopy type of a Poincaré complex* Y *of formal dimension* s, *then*

$$H^*(Y, Z_2) = Z_2[\alpha]/(\alpha^{\ell+1}) \quad \text{and} \quad H^*(M, Z_2) = Z_2[\alpha]/(\alpha^{\ell+2}),$$

where $\deg \alpha = p - s$.

Proof. Throughout the proof, we use Z_2-coefficients. We first prove statements (1) to (4) mentioned below and use them for proving the proposition.

(1) $H^j(M) \simeq H^j(Y)$ for $0 \leq j \leq p - 1$.

(2) $H^j(Y) \simeq H^{j+k(p-s)}(Y)$ whenever $j \geq 0$, $k \geq 0$, and $j + k(p \cdot s) < p$.

(3) $s = \ell(p \cdot s)$ for some integer $\ell \geq 0$. Moreover,

$$H^{k(p-s)}(Y) = Z_2 \text{ for } 0 \leq k \leq \ell,$$

$$H^j(Y) = 0 \quad \text{otherwise.}$$

(4) $H^{k(p-s)}(M) = Z_2$ for $0 \leq k \leq \ell + 1$,

$$H^j(M) = 0 \quad \text{otherwise.}$$

Actually, (1) is an immediate consequence of the isomorphisms

$$H^j(M) \simeq H_{p-j}(M) \simeq H_{p-j}(M, x) \simeq H^j(M - x) \simeq H^j(Y);$$

the first isomorphism is due to Poincaré duality, and the third to Lefschetz duality. The second is valid whenever $j \leq p - 1$.

From the isomorphisms $H^s(M - x) \simeq H^s(Y) \simeq Z_2$ and $H^i(M - x) = 0$ for $i \geq p - 1$ it follows that $s \leq p - 1$. For each $j \geq 0$, we have the isomorphisms

$$H^j(Y) \simeq H^{s-j}(Y) \simeq H^{s-j}(M) \simeq H^{j+(p-s)}(M).$$

The first and the last of these are consequences of Poincaré duality and the universal-coefficient theorem. The second isomorphism is due to (1). Also, from (1) we see that

$$H^{j+(p-s)}(M) \simeq H^{j+(p-s)}(Y)$$

whenever $j + (p - s) < p$. Thus $H^j(Y) \simeq H^{j+(p-s)}(Y)$ if $j + (p - s) < p$. Induction on k yields (2).

Corresponding to each integer j satisfying the condition $0 \leq j \leq s$, there exists an integer $k \geq 0$ such that $p > j + k(p - s) \geq s$. From (2) we get the isomorphism $H^j(Y) \simeq H^{j+k(p-s)}(Y)$. In case $j + k(p - s) > s$, this implies that $H^j(Y) = 0$. Since $H^0(Y) \neq 0$, it follows that there exists an integer $\ell \geq 0$ such that $\ell(p - s) = s$. This immediately gives (3).

(4) is an immediate consequence of (3) and (1).

Let α_k generate $H^{k(p-s)}(M)$. We shall prove by induction on k that $\alpha_k = \alpha_1^k$ for $0 \leq k \leq \ell + 1$. This is clear for $k = 0$. Let $\bar{\alpha}_j$ generate $H^{j(p-s)}(Y)$ for $0 \leq j \leq \ell$. Assume $1 \leq k \leq \ell + 1$ and that we have proved $\alpha_{k-1} = \alpha_1^{k-1}$, in other words, $\alpha_1^{k-1} \neq 0$. Then $\bar{\alpha}_1^{k-1} \neq 0$ and hence $\bar{\alpha}_1^{k-1} = \bar{\alpha}_{k-1}$. By Poincaré duality for Y^s, we see that $\bar{\alpha}_1^{k-1} \cup \bar{\alpha}_{\ell-k+1} = \bar{\alpha}_\ell$. It follows that $\bar{\alpha}_1^{k-1} \cup \bar{\alpha}_{\ell-k+1} = \bar{\alpha}_\ell$. This in turn implies that $\alpha_1^{k-1} \cup \alpha_{\ell-k+1} = \alpha_\ell$. However, Poincaré duality for M yields the relation $\alpha_1 \cup \alpha_\ell = \alpha_{\ell+1}$. Hence $\alpha_1^k \cup \alpha_{\ell-k+1} = \alpha_{\ell+1}$. This implies $\alpha_1^k \neq 0$ and hence $\alpha_1^k = \alpha_k$.

Proposition 3.1 now follows immediately.

COROLLARY 3.2. *With the assumptions of Proposition* 3.1, $\frac{s}{p} = \frac{\ell}{\ell + 1}$.

COROLLARY 3.3. *Suppose* M *and* Y *occurring in Proposition* 3.1 *are orientable manifolds and* $s \geq 1$. *Then* $p - s > 1$.

Proof. This will follow if we show that a closed manifold V satisfying the condition $H^*(V, Z_2) \simeq Z_2[\alpha]/(\alpha^{2k+1})$ with $\deg \alpha = 1$ and $k \geq 1$ is necessarily nonorientable. If w_1 denotes the first Stiefel-Whitney class of V, then w_1 is also the first Wu class of V, and we have the equations

$$w_1 \cup \alpha^{2k-1} = Sq^1 \alpha^{2k-1} = (Sq^1 \alpha) \cup \alpha^{2k-2} + \alpha \cup Sq^1 \alpha^{2k-2}$$

$$= \alpha^2 \cup \alpha^{2k-2} = \alpha^{2k} \neq 0.$$

Hence $w_1 \neq 0$. This means V is nonorientable.

Now suppose A is a finite set of points in a manifold M^n such that M - A has a manifold Y^s as a deformation retract. Then clearly $s \leq n - 1$, and the integral cohomology sequence

$$\begin{array}{ccccccc} \cdots \to & H^0(M) & \to & H^0(A) & \to & H^1(M, A) & \to \cdots \\ & \wr\| & & & & \wr\| & \\ & Z & & & & H_{n-1}(M - A) & \end{array}$$

implies rank $H^0(A) = \#(A) \leq 2$, because

$$H_{n-1}(M - A) \simeq H_{n-1}(Y^s) = \begin{cases} 0 & \text{if } s < n - 1, \\ Z & \text{if } s = n - 1. \end{cases}$$

Suppose that an MS-fibering $f: M^n \to N^p$ as in the hypothesis of Theorem A is given. Then we must have the spine diagram

$$\begin{array}{ccc} M^n - A & \xrightarrow{r_M} & X^r \\ \downarrow f & & \downarrow \pi \\ N^p - f(A) & \xrightarrow{r_N} & Y^s \end{array}$$

with deformation retractions r_M and r_N and spine fibering $\pi: X^r \to Y^s$. It follows from the argument above that $\#(A) \leq 2$. It remains to show that $\#(A) = 1$ is an impossibility.

LEMMA 3.4. *If* $\#(A) = 1$, *then the spine fibering* $\pi: X^r \to Y^s$ *of* $f: M^n \to N^p$ $(n > p)$ *has codimension one* (*namely,* $r = n - 1$ *and* $s = p - 1$).

Proof. Local results in [3] reduce the problem to three cases, namely $(n, p) = (4, 3)$, $(8, 5)$, or $(16, 9)$, and in each case the fibre is a homotopy $(n - 1)$-sphere.

Case (4, 3). We have only to prove that $s = 2$. If $s = 0$, then $r = 1$ and $r/n = 1/4$, and this is not of the form $\ell/(\ell + 1)$ for any integer ℓ. Likewise, $s = 1$ implies $s/p = 1/3$, which is also not of the form $\ell/(\ell + 1)$. Corollary 3.2 gives the desired result.

Case (8, 5). In this case we have to show that $s = 4$. For $s = 1, 2, 3$ it is clear that $s/p = s/5 \neq \ell/(\ell + 1)$ for any integer ℓ. If $s = 0$, we see that $r/n = 3/8 \neq \ell/(\ell + 1)$. Again, Corollary 3.2 gives the desired conclusion.

Case (16, 9). The proof is similar to the proofs in the other two cases, and we omit it.

Now, X^{n-1} and M^n are both orientable. From Corollary 3.3 it follows that X^{n-1} cannot be a deformation retract of $M^n - A$, when $\#(A) = 1$. This completes the proof of the first part of Theorem A.

4. COMPLETION OF PROOF OF THEOREM A

PROPOSITION 4.1. *Let* s *be any integer* $(s \geq 2)$. *Let* N^{s+1} *be a manifold,* $A \subset N$ *a set consisting of two points. If* $N - A$ *has the homotopy type of an* s-*manifold* Y^s, *then* N *and* Y *are homotopy spheres of dimension* $s + 1$ *and* s, *respectively.*

Proof. We prove that the top-dimensional homology class of Y is spherical. Then it will follow from Lemma 4.2 below that Y is a homotopy s-sphere.

Let D_1 and D_2 be two disjoint, locally flatly imbedded discs around the two points of A in N^{n+1}. Let B_1 and B_2 denote the interiors of D_1 and D_2, respectively. Then $V^{s+1} = N - B_1 \cup B_2$ is a connected manifold with $\partial V = S^s_1 - S^s_2$ (two disjoint copies of S^s). Further, V has the homotopy type of Y^s. By excision,

$H_i(V, \partial V) \simeq H_i(N, D_1 \cup D_2)$. Also, $H_i(N, D_1 \cup D_2) \simeq H_i(N)$ for $i \geq 2$. Hence $H_i(V, \partial V) \simeq H_i(N)$ for $i \geq 2$.

From the diagram (with Z_p-coefficients)

$$
\begin{array}{ccccccc}
 & & & & H^0(N) \simeq Z_p & & \\
 & & & & \downarrow & & \\
 & & & & H^0(A) \simeq Z_p \oplus Z_p & & \\
 & H_s(Y) & & & & & \\
 \wr\!/ & & \backslash\!\wr & & & & \\
 & & & & \downarrow & & \\
H_s(V) & \simeq & H_s(N - A) & \simeq & H^1(N, A) & \simeq & Z_p \\
\downarrow & & \downarrow & & \downarrow & & \\
H_s(V, \partial V) & \simeq & H_s(N) & \simeq & H^1(N) & & \\
 & & & & \downarrow & & \\
 & & & & H^1(A) = 0 & &
\end{array}
$$

with exact vertical sequence we see that $H^1(N; Z_p) = 0$. Since $H_s(N; Z_p) = 0$ for every prime p. This in turn implies that $H_s(N; Z) = 0$. Hence $H_s(V, \partial V; Z) = 0$. From the exact sequence (integer coefficients)

$$
\begin{array}{l}
0 \to H_{s+1}(V, \partial V) \to H_s(\partial V) \to H_s(V) \to H_s(V, \partial V) \to \\
\hphantom{0 \to H_{s+1}(V, \partial V) \to H_s(\partial V) \to H_s(V) \to H_s(V,} \,\, \| \\
\hphantom{0 \to H_{s+1}(V, \partial V) \to H_s(\partial V) \to H_s(V) \to H_s(V,} \,\, 0
\end{array}
$$

we see that the top class γ of $(V, \partial V)$ gets mapped to $\alpha_1 - \alpha_2$, where α_j is the top class of S_j^s $(j = 1, 2)$. Hence α_1 gets mapped to a generator of $H_s(V) \simeq H_s(Y) \simeq Z$. It follows that Y is a homotopy sphere.

From the condition $s + 1 \geq 3$ we see that $\pi_1(N - A) \simeq \pi_1(N)$. But $\pi_1(N - A) \simeq \pi_1(Y) = 0$. Thus $\pi_1(N) = 0$. Also, for $2 \leq j \leq s$, we have the isomorphisms $H^j(N) \simeq H^j(N, A) \simeq H_{s+1-j}(N - A) = 0$. It follows that N is also a homotopy sphere.

LEMMA 4.2. *Let Y^s be a manifold of dimension $s \geq 1$ with spherical top-dimensional homology class. Then Y is a homotopy s-sphere.*

Proof. We only need to prove the lemma for $s \geq 2$. Let $f: S^s \to Y^s$ represent the top class. Let $\tilde{Y}^s$ be the simply connected covering space of Y. Since $\pi_1(S^s) = 0$, there exists a lift $\tilde{f}: S^s \to \tilde{Y}$ of f. It follows that $H_s(\tilde{Y}) \neq 0$. Hence $\tilde{Y}$ is compact, and therefore it is a finite covering of Y. If $p: \tilde{Y} \to Y$ denotes the covering projection, then $\deg p = 0(\pi_1(Y))$ (the order of the fundamental group of Y). Since $1 = \deg f = \deg p \cdot \deg \tilde{f}$, we see that $0(\pi_1(Y)) = 1$. Hence Y itself is simply connected.

It is well known that if Y^s is simply connected with a map $S^s \xrightarrow{f} Y^s$ of degree 1, then f is a homotopy equivalence. Thus Y^s is a homotopy sphere.

This completes the proof of Theorem A. As for Theorem B, we observe that if M^n is $([n/2] - 1)$-connected, one can easily show that N^p is a homotopy sphere. Then Theorem 3.1 of [2] gives the desired result.

5. REMARKS ON NONORIENTABLE MS-FIBERINGS

To our knowledge, MS-fiberings with nonorientable manifolds have so far not been studied. When we take the MS-fibering $S^1 \to S^4 \to S^3$ obtained as the suspension of the Hopf fibering $S^1 \to S^3 \to S^2$, we may be tempted to believe that antipodal identification will give us an MS-fibering $S^1 \to RP^4 \to PR^3$ with one singular point and having a spine. But this is not true; for if it is an MS-fibering and if there is a spine, the spine sequence $S^1 \to X^3 \to Y^2$ must satisfy the conditions $X^3 \sim PR^3$ and $Y^2 \sim RP^2$. Then, by a well-known result of Serre, $S^1 \to X^3 \to Y^2$ will be an orientable fibering. From the Gysin sequence

$$\to H_2(Y^2) \to H_3(X^3) \to H_3(Y^2) \to H_1(Y^2)$$

we get a contradiction, because $H_3(X^3) \simeq Z$ and $H_2(Y^2) = H_3(Y^2) = 0$.

Similar arguments show that the MS-fibering $S^3 \to S^8 \to S^5$ obtained as the suspension of the Hopf fibering $S^3 \to S^7 \to S^4$ will not yield an MS-fibering with a spine under the Z_2-action on S^8 and S^5. We are unable to make a similar statement in the case of the MS-fibering $S^7 \to S^{16} \to S^9$, since S^7 is not a group and we do not know whether Serre's result can be applied in this case.

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