HOMOTOPY GROUPS OF $\mathscr{E}(S^n, S^{n+r})$

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E. C. Zeeman conjectured that $\pi_k(\mathscr{E}(S^n, S^{n+r})) = 0$ for $k + 3 \leq r$ [8, Chapter 8]. L. S. Husch [4] and E. Lusk [7] gave affirmative answers, using a general-position lemma for maps and a taming lemma, respectively. The purpose of this note is to present a new, elementary proof of this result.

The idea of our proof is to build a series of fibrations, so that we can reduce the computations to rather trivial cases in the framework of [6]. The precise statement and the proof of our result will be found in Section 2.

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1. DEFINITIONS AND PREPARATIONS

We work in the piecewise linear (PL) category, and the symbol PL will often be omitted.

We use standard notation such as

$$R^n, \quad \partial X \text{ (boundary)}, \quad \Delta_k \text{ (standard } k\text{-simplex)},$$

$$I^n = [0, 1]^n, \quad D^n = [-1, 1]^n, \quad S^n = \partial D^{n+1},$$

$$\text{Int } X \text{ (interior)}, \quad \text{Cl}(X) \text{ (closure)}.$$

Sometimes, Δ_k, D^k, and I^k are identified. For $r > 0$, $rD^n = [-r, r]^n$.

Let Q denote a PL-manifold of dimension q and M a submanifold of Q of dimension m.

Definition 1. By $\mathscr{E}(M, Q)$ we denote the semisimplicial (s. s.) complex whose typical k-simplex is a k-isotopy

$$f: \Delta_k \times M \to \Delta_k \times Q$$

such that

(E) $\qquad$ f is a restriction of a k-homeotopy F of Q

(that is, $F: \Delta_k \times Q \to \Delta_k \times Q$ is a surjective isotopy).

Remarks. 1. If $q - m \geq 3$ and $f \mid 0 \times M$ extends to a homeomorphism of Q, the condition (E) is automatically satisfied, in view of the unknotting theorem of Zeeman [8] (see Hudson [2]).

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2. M and Q can be manifolds with boundary. The condition (E) implies that if $N \subset M$ lies in ∂Q, then $f(x \times N) \subset x \times \partial Q$, and if $N \subset x \times \operatorname{Int}(Q)$, then $f(x \times N) \subset \operatorname{Int} Q$ for each $x \in \Delta_k$.

Definition 2. Let M and Q be as in Definition 1, and suppose Y is a submanifold of M.

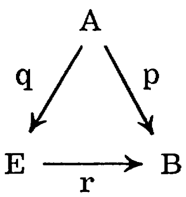
(1) $\mathscr{E}_Y(M, Q)$ is a subcomplex of $\mathscr{E}(M, Q)$ defined by an extra condition that a k-simplex $f: \Delta_k \times M \to \Delta_k \times Q$ of $\mathscr{E}(M, Q)$ belongs to $\mathscr{E}_Y(M, Q)$ if $f \mid \Delta_k \times Y$ is the standard inclusion. When Y is a union of spaces $\bigcup_{i=1}^q Y_i$, we often write $Y_1, Y_2, \cdots, Y_q$ instead of $\bigcup_{i=1}^q Y_i$.

(2) The complex $\mathscr{E}_{\partial I^p \times M \cup I^p \times Y}(I^p \times M, I^p \times Q)$ will be abbreviated as $\mathscr{E}^p_Y(M, Q)$.

(3) $G\,\mathscr{E}_0(D^n, D^{n+r})$ denotes the quotient complex obtained from $\mathscr{E}_0(D^n, D^{n+r})$ by the following equivalence relation: two k-simplices f and g are equivalent if $f \mid \Delta_k \times N(0) = g \mid \Delta_k \times N(0)$ for some neighborhood $N(0)$ of 0 in D^n.

To prove our theorem, we need several lemmas in the framework of [6].

LEMMA 1. *Let the* s.s. *maps* $r: E \to B$ *and* $q: A \to E$ *be surjective, and let* $p: A \to B$ *be a Kan fibration. If the following diagram is commutative and* E *is a Kan complex, then* r *is a Kan fibration with an appropriate fibre.*



The proof follows directly from the definition in D. M. Kan's paper [5].

In the next lemma, we use special notation for certain subspaces of S^m as follows: We denote the base point (north pole) of any sphere by a. The disk D^m is identified with the northern hemisphere of S^m, and the origin $0 \in D^m$ is identified with the north pole a. By ∂ we mean the boundary of the disk D^m, and $\bigcup_\delta \mathscr{E}_{\delta D^{n-p}}(S^{n-p}, S^{n+r-p})$ denotes an s.s. subcomplex of $\mathscr{E}^p(S^{n-p}, S^{n+r-p})$ defined as follows: a k-simplex

$$f: \Delta_k \times I^p \times S^{n-p} \to \Delta_k \times I^p \times S^{n+r-p}$$

of $\mathscr{E}^p(S^{n-p}, S^{n+r-p})$ belongs to $\bigcup_\delta \mathscr{E}^p_{\delta D^{n-p}}(S^{n-p}, S^{n+r-p})$ if and only if $f \mid \Delta_k \times I^p \times \delta D^{n-p}$ is the standard inclusion map for some δ.

LEMMA 2. *The following are Kan fibrations:*

(a) $\mathscr{E}^p_a(S^{n-p}, S^{n+r-p}) \hookrightarrow \mathscr{E}^p(S^{n-p}, S^{n+r-p}) \xrightarrow{r} \mathscr{E}^p(a, S^{n+r-p})$,

(b) $\bigcup_\delta \mathscr{E}^p_{\delta D^{n-p}}(S^{n-p}, S^{n+r-p}) \hookrightarrow \mathscr{E}^p_a(S^{n-p}, S^{n+r-p}) \xrightarrow{\gamma \circ r} G\,\mathscr{E}^p_0(D^{n-p}, S^{n+r-p})$,

(c) $\bigcup_\delta \mathscr{E}^p_{\delta D^{n-p},\partial}(D^{n-p}, D^{n+r-p}) \hookrightarrow \mathscr{E}^p_{\partial,0}(D^{n-p}, D^{n+r-p})$

$$\xrightarrow{\gamma} G\,\mathscr{E}^p_0(D^{n-p}, D^{n+r-p}).$$

Here r *is the obvious restriction, and the symbol* γ *means that we take germs along* $I^p \times 0$ *in* $I^p \times S^{n+r-p}$ *or* $I^p \times D^{n+r-p}$.

Proof. First we prove (a). Let $A = \mathscr{H}^p(S^{n+r-p})$ be an s.s. group whose k-simplex is a k-isotopy

$$f\colon \Delta_k \times I^p \times S^{n+r-p} \to \Delta_k \times I^p \times S^{n+r-p}$$

such that $f \mid \Delta_k \times \partial I^p \times S^{n+r-p}$ is the identity map. To use Lemma 1, take E to be $\mathscr{E}^p(S^{n-p}, S^{n+r-p})$ and B to be $\mathscr{E}^p(a, S^{n+r-p})$. Obviously, p and q are restriction maps. By the definition of $\mathscr{E}^p(M, Q)$, the maps p and q are s.s. principal fibre bundles (see [1] or Kuiper and Lashof [6], for example). Here we use the n-isotopy-extension theorem of Hudson [2] to guarantee that the extension keeps $\Delta_k \times \partial I^p \times S^{n+r-p}$ fixed. The condition (E) guarantees the local triviality. The diagram in Lemma 1 commutes, under the present interpretation. Hence r is a Kan fibration. We can prove (b) and (c) similarly by taking A to be $\mathscr{H}^p_a(S^{n+r-p})$ and $\mathscr{H}^p_{\partial,0}(D^{n+r-p})$, respectively. ■

By considering D^{n+r-p} as the northern hemisphere of S^{n+r-p}, as mentioned before, we have an inclusion map

$$i\colon G\,\mathscr{E}^p_0(D^{n-p}, D^{n+r-p}) \to G\,\mathscr{E}^p_0(D^{n-p}, S^{n+r-p}),$$

because it is easy to see that if a k-simplex $f \in G\,\mathscr{E}^p_0(D^{n-p}, D^{n+r-p})$ is represented by the restriction of a k-homeotopy of $I^p \times D^{n+r-p}$, then i(f) is represented by a k-homeotopy of $I^p \times S^{n+r-p}$. Also, by identifying $I^p \times \left(D^m - \operatorname{Int}\left(\frac{1}{2} D^m\right)\right)$ with $I^{p+1} \times S^{m-1}$, we obtain an inclusion map

$$j\colon \mathscr{E}^{p+1}(S^{n-p-1}, S^{n+r-p-1}) \to \bigcup_\delta \mathscr{E}^p_{\delta D^{n-p},\partial}(D^{n-p}, D^{n+r-p}).$$

LEMMA 3. *The inclusion maps* i *and* j *are homotopy equivalences.*

Proof. We can construct an inverse map $\bar{i}$ for i, because each element of $G\,\mathscr{E}^p_0(D^{n-p}, S^{n+r-p})$ is represented by an element of $G\,\mathscr{E}^p_0(D^{n-p}, D^{n+r-p})$. We need to show that these spaces satisfy condition (E). We can do this by an argument similar to that of Lemmas 0.1 and 0.2 of [6], using the isotopy extension theorem of Hudson [2] and the uniqueness of the regular neighborhood [3]; we omit the details.

The proof that j is a homotopy equivalence is essentially the same as that of Lemma 2.4 of [6], and we omit it also.

LEMMA 4. *The following spaces are contractible.*

(a) $\bigcup_\delta \mathscr{E}^p_{\delta D^{n-p}}(S^{n-p}, S^{n+r-p})$,

(b) $\mathscr{E}^p_{\partial,0}(D^{n-p}, D^{n+r-p})$.

Proof. By an argument analogous to that of the preceding lemma, it is easy to see that the space (a) is homotopy-equivalent to $\mathscr{E}^p_\partial(D^{n-p}, D^{n+r-p})$, which is again homotopy-equivalent to (b). By the Alexander trick (see [6], for example), (b) is contractible. ■

COROLLARY 1. (i) *The mapping* $\gamma \circ r$ *in Lemma* 2 (b) *is a homotopy equivalence.*

$$\text{(ii)}\ \pi_k(G\,\mathscr{E}^p_0(D^{n-p}, D^{n+r-p})) \cong \pi_{k-1}\left(\bigcup_\delta\ \mathscr{E}^p_{\delta D^{n-p},\partial}(D^{n-p}, D^{n+r-p})\right).$$

The corollary follows immediately from Lemma 1.4 and from the homotopy exact sequences in parts (b) and (c) of Lemma 1.2.

LEMMA 5. *If* $m > 0$ *and* $n + r - 2p > m$ *or* $m = 0$ *and* $n + r - p \geq 3$, *then* $\pi_m(\mathscr{E}^p(a, S^{n+r-p})) \cong 0$.

Proof. Let $f\colon \Delta_m \times I^p \times a \to \Delta_m \times I^p \times S^{n+r-p}$ represent an element of $\pi_m(\mathscr{E}^p(a, S^{n+r-p}))$. The composition of this with the projection

$$\pi\colon \Delta_m \times I^p \times S^{n+r-p} \to S^{n+r-p}$$

defines an element of $\pi_{m+p}(S^{n+r-p})$. If $m + p < n + r - p$, we can find a point $b \in S^{n+r-p}$ such that $b \notin \pi \circ f(\Delta_m \times I^p \times a)$. Now we may consider f to be an element of $\mathscr{E}^p(0, D^{n+r-p})$, by deleting a small open-disk neighborhood of b from S^{n+r-p} and making a suitable identification. If $m > 0$, the condition (E) for this is guaranteed by the theorem of Hudson. By the Alexander trick again, $\mathscr{E}^p(0, D^{n+r-p})$ is contractible. If $m = 0$, the condition (E) is guaranteed by the unknotting theorem of Zeeman ([8] or [9]). ■

2. STATEMENT AND PROOF OF THE THEOREM

In this section we prove the following theorem and some related results:

THEOREM. *If* $k + 3 \leq r$, *then* $\pi_k(\mathscr{E}(S^n, S^{n+r})) \cong 0$.

Proof. Suppose $k \geq n$. Then Lemma 5 asserts that $\pi_{k-n}(\mathscr{E}^n(a, S^r)) \cong 0$ if $k + 3 \leq r$. Applying the argument in the proof of Lemma 5, we also see that $\pi_{k-n}(\mathscr{E}^n_a(S^0, S^r)) \cong 0$. By the homotopy exact sequence in the fibration Lemma 2 (a), this implies that $\pi_{k-n}(\mathscr{E}^n(S^0, S^r)) \cong 0$. Using the homotopy equivalences j and i of Lemma 3, as well as $\gamma \circ r$ (Corollary 1), we deduce that

$$\pi_{k-n+1}(\mathscr{E}^{n-1}_a(S^1, S^{n+r-n+1})) \cong 0\,.$$

Actually, the same argument implies that $\pi_m(\mathscr{E}^{n-1}_a(S^1, S^{n+r-n+1})) \cong 0$ for all $m \leq k - n + 1$. Hence

$$\pi_{k-n+1}(\mathscr{E}^{n-1}(S^1, S^{n+r-n+1})) \cong \pi_{k-n+1}(\mathscr{E}^{n-1}(a, S^{n+r-n+1}))\,.$$

We repeat this argument, starting from $\pi_{k-n+1}(\mathscr{E}^{n-1}(a, S^{n+r-n+1})) \cong 0$, because $k + 3 \leq r$ implies $k - n + 1 + 3 \leq n + r - 2(n - 1)$. Now we obtain the relation

$$\pi_{k-n+2}(\mathscr{E}^{n-2}(S^2, S^{n+r-n+2})) \cong \pi_{k-n+2}(\mathscr{E}^{n-2}(a, S^{n+r-n+2}))\,.$$

In the end, we have the relation $\pi_k(\mathscr{E}(S^n, S^{n+r})) \cong \pi_k(\mathscr{E}(a, S^{n+r})) \cong 0$.

If we suppose $k \leq n$, we can still apply the same argument, because Lemma 6 that follows guarantees that we can use an argument similar to that above.

By virtue of the condition $k + 3 \leq r$, the rest of the argument is again the same as above. ■

LEMMA 6. *If* $k + 3 \leq r$, *then* $\pi_0(\mathscr{E}^k(S^{n-k}, S^{n+r-k})) \cong 0$.

Proof. This is proved by induction on $n - k$. When $n - k = 0$, this is proved in the first half of the proof of Theorem. Suppose it is true for all values less than $n - k > 0$.

From Lemma 2(a), we have the exact sequence

$$\to \pi_0(\mathscr{E}^k_a(S^{n-k}, S^{n+r-k})) \to \pi_0(\mathscr{E}^k(S^{n-k}, S^{n+r-k})) \to \pi_0(\mathscr{E}^k(a, S^{n+r-k})),$$

and the last term is zero, by Lemma 5, because $k + 3 \leq r$. We want to show that $\pi_0(\mathscr{E}^k_a(S^{n-k}, S^{n+r-k})) \cong 0$. Because of Lemma 4 and Lemma 3 applied to Lemma 2, it is sufficient to show that $\pi_0(G\,\mathscr{E}^k_0(D^{n-k}, D^{n+r-k})) \cong 0$.

Let $f\colon I^k \times D^{n-k} \to I^k \times D^{n+r-k}$ represent an element of

$$\pi_0(G\,\mathscr{E}_0(D^{n-k}, D^{n+r-k})).$$

By definition, f is proper and extends to a homeomorphism F of $I^k \times D^{n+r-k}$. Using the uniqueness theorem for relative regular neighborhoods [3], we may further assume that

$$F(I^k \times \partial D^{n+r-k}) = I^k \times \partial D^{n+r-k}.$$

This guarantees that $f \mid I^k \times \partial D^{n-k}$ represents an element of

$$\pi_0(\mathscr{E}^k(S^{n-k-1}, S^{n+r-k-1})).$$

By the induction hypothesis and the assumption $k + 3 \leq r$, this is trivial. Using the theorem of Hudson, we can assume that $f \mid I^k \times \partial D^{n-k}$ was the identity function, to start with. Now we have the equation $f \mid \partial(I \times D^{n-k}) =$ identity. Using the Alexander trick (see [6, Lemma 1.5]), we see that f is isotopic to the identity and that it fixes the boundary $\partial(I^k \times D^{n-k})$ and $I^k \times 0$. This means that $\pi_0(G\,\mathscr{E}^k_0(D^{n-k}, D^{n+r-k}))$ is trivial. ■

Using our main result (Theorem) and the Alexander trick, and looking at appropriate Kan fibrations, we can obtain the following results. The proof is left to the reader.

COROLLARY 2. *If* $k + 3 \leq r$, *then*

(1) $\pi_k(\mathscr{E}(D^{n+1}, D^{n+r+1})) \cong 0$,

(2) $\pi_k\left(\mathscr{E}\left(\frac{1}{2}D^n, D^{n+r}\right)\right) \cong 0$,

(3) $\pi_k(\mathscr{E}(0 * SD^n, D^{n+r+1})) \cong 0$, *where* SD^n *is the southern hemisphere of* $S^n \subset \partial D^{n+r+1}$,

(4) $\pi_k(\mathscr{E}(S^{n-1}, D^{n+r})) \cong 0$, *where* S^{n-1} *is identified with* $\partial\left(\frac{1}{2}D^n\right) \subset \operatorname{Int} D^{n+r}$,

(5) $\pi_k(\mathscr{E}(S^{n-1}, R^{n+r})) \cong 0$.

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