A CHARACTERIZATION OF THE COMPLEX SPHERE

Bang-yen Chen and Koichi Ogiue

1. INTRODUCTION

Let $P_{n+1}(C)$ denote the $(n+1)$-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 1, and let $z_0, z_1, \cdots, z_{n+1}$ be a homogeneous coordinate system of $P_{n+1}(C)$. Let

$$Q_n = \left\{ (z_0, z_1, \cdots, z_{n+1}) \in P_{n+1}(C) \middle| \sum z_i^2 = 0 \right\}.$$

Then, with respect to the induced Kaehler structure, Q_n is an Einstein manifold with scalar curvature $n/2$, and it is complex analytically isometric to the Hermitian symmetric space $SO(n+2)/SO(2) \times SO(n)$. We call Q_n an n-dimensional *complex sphere*. In [3], the second author proved the following.

PROPOSITION. *Let* M *be an* n-*dimensional complete Kaehler submanifold immersed in* $P_m(C)$. *If the Ricci curvature of* M *is everywhere greater than* $n/2$, *then* M *is totally geodesic.*

The purpose of this paper is to prove the following theorem, which gives a local characterization of complex spheres.

THEOREM. *Let* M *be an* n-*dimensional Kaehler submanifold immersed in* $P_m(C)$. *If the Ricci curvature of* M *is everywhere equal to* $n/2$, *then* M *is locally* Q_n *in some* $P_{n+1}(C)$ *in* $P_m(C)$.

For notation and terminology, we follow [4], unless it is otherwise stated.

2. PRELIMINARIES

We prepare a brief summary of some basic facts. Details are found in [4].

Let M be an n-dimensional Kaehler submanifold immersed in $P_m(C)$. Let g, S, and ρ denote the Kaehler metric, the Ricci tensor, and the scalar curvature of M, respectively. If we denote by σ or A_α the second fundamental form of the immersion, then

$$S(X, Y) = \frac{n+1}{2} g(X, Y) - 2 \sum g(A_\alpha^2 X, Y), \tag{1}$$

$$\rho = n(n+1) - \|\sigma\|^2 . \tag{2}$$

Moreover, σ and A_α satisfy the differential equation

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(3) $$\frac{1}{2}\Delta\|\sigma\|^2 = \|\nabla'\sigma\|^2 - 8\,\mathrm{tr}\left(\sum A_\alpha^2\right)^2 - \sum(\mathrm{tr}\, A_\lambda A_\mu)^2 + \frac{n+2}{2}\|\sigma\|^2.$$

It is also shown in [4] that

(4) $$\sum(\mathrm{tr}\, A_\lambda A_\mu)^2 \leq \frac{1}{2}\|\sigma\|^4,$$

and that equality holds if and only if rank $(\mathrm{tr}\, A_\lambda A_\mu) \leq 2$. Geometrically speaking, the equality in (4) holds if and only if the dimension of the first normal space (the complex vector space spanned by the image of the second fundamental form) is at most 1.

3. PROOF OF THE THEOREM

Since the Ricci curvature of M is everywhere equal to $n/2$, we see that $S = \frac{n}{2} g$ and hence $\rho = n^2$. Therefore, from (1), (2), (3), and (4) we obtain the inequality $\|\nabla'\sigma\|^2 \leq 0$, which implies that

$$\nabla'\sigma = 0$$

and that the equality in (4) holds. Hence the first normal space is parallel and of dimension 1, which implies that M is immersed in some $P_{n+1}(C)$ in $P_m(C)$ (compare Theorem 2.3 in [1]). Therefore, by a result of S.-S. Chern [2], M is locally Q_n.

REFERENCES

1. B.-Y. Chen and K. Ogiue, *Some extrinsic results for Kaehler submanifolds.* Tamkang J. Math. 4 (1973), 207-213.

2. S.-S. Chern, *Einstein hypersurfaces in a Kählerian manifold of constant holomorphic curvature.* J. Differential Geometry 1 (1967), 21-31.

3. K. Ogiue, *Positively curved complex submanifolds immersed in a complex projective space, II.* Hokkaido Math. J. 1 (1972), 16-20.

4. ———, *Differential geometry of Kaehler submanifolds.* Advances in Math. 13 (1974), 73-114.

Michigan State University
East Lansing, Michigan 48824