REMARKS ON THE INVARIANT-SUBSPACE PROBLEM

Carl Pearcy, J. R. Ringrose, and Norberto Salinas

1. The *invariant-subspace problem* for operators on Hilbert space is the question whether every bounded, linear operator on a separable, infinite-dimensional, complex Hilbert space $\mathscr{H}$ maps some (closed) subspace different from (0) and $\mathscr{H}$ into itself. In this note we prove three theorems, all of which are concerned with equivalent reformulations of this problem. The main tools employed are the Lomonosov technique and the theory of subdiagonalization of compact operators.

In what follows, $\mathscr{L}(\mathscr{H})$ will denote the algebra of all bounded, linear operators on $\mathscr{H}$. Moreover, all subalgebras of $\mathscr{L}(\mathscr{H})$ under consideration will be assumed to contain the identity operator $1_{\mathscr{H}}$. The lattice of invariant subspaces of a subalgebra $\mathscr{A}$ of $\mathscr{L}(\mathscr{H})$ will be denoted by Lat$(\mathscr{A})$, and the algebra $\mathscr{A}$ will be called *transitive* if Lat$(\mathscr{A}) = \{(0), \mathscr{H}\}$. The lattice of invariant subspaces of a single operator T will be denoted by Lat(T).

THEOREM 1.1. *Let* $\mathscr{A}$ *be a subalgebra of* $\mathscr{L}(\mathscr{H})$. *Then the following statements are equivalent:*

(1) $\mathscr{A}$ *is transitive.*

(2) *For every nonzero, quasinilpotent, compact operator* K *on* $\mathscr{H}$, *there exist an operator* $A = A_K$ *in* $\mathscr{A}$ *and a nonzero vector* $x = x_K$ *in* $\mathscr{H}$ *such that* $AKx = x$.

(3) *For every nonzero, nilpotent operator* N *on* $\mathscr{H}$ *of rank one, there exist an operator* $A = A_N$ *in* $\mathscr{A}$ *and a nonzero vector* $x = x_N$ *in* $\mathscr{H}$ *such that* $ANx = x$.

Proof. That (1) implies (2) follows from the basic Lomonosov theorem (see [1] and [2]): *if* $\mathscr{A}$ *is a transitive subalgebra of* $\mathscr{L}(\mathscr{H})$ *and* K *is a nonzero compact operator on* $\mathscr{H}$, *then there exist an operator* $A = A_K$ *in* $\mathscr{A}$ *and a nonzero vector* $x = x_K$ *such that* $AKx = x$. That (2) implies (3) is obvious; therefore we complete the proof by showing that (3) implies (1). Arguing contrapositively, we suppose that $\mathscr{M}$ is a subspace different from (0) and $\mathscr{H}$ that is invariant under $\mathscr{A}$. Let x_0 and y_0 be any two nonzero vectors in $\mathscr{H}$ such that y_0 belongs to $\mathscr{M}$ and x_0 is orthogonal to $\mathscr{M}$, and let N be the (unique) nilpotent operator of rank one that maps x_0 to y_0. Then the range of N is contained in $\mathscr{M}$, and it follows that for each A in $\mathscr{A}$ the range of AN is contained in $\mathscr{M}$. Thus $NAN = 0$ and $(AN)^2 = 0$ for every A in $\mathscr{A}$, from which it follows that there cannot exist an operator A_N in $\mathscr{A}$ and a nonzero x_N in $\mathscr{H}$ satisfying the equation $A_N N x_N = x_N$. This contradicts (3), and thus completes the proof.

We remark that one may argue independently of the Lomonosov theorem to show that conditions (1) and (3) above are equivalent. The reader may supply the details of this argument himself.

COROLLARY 1.2. *Let* A *be an operator in* $\mathscr{L}(\mathscr{H})$. *Then* A *has a nontrivial invariant subspace if and only if there exists a nonzero compact operator* K *such that* p(A)K *is quasinilpotent for every polynomial* p.

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2. A difficulty in trying to apply Theorem 1.1 to solve the invariant-subspace problem is that the equivalence of conditions (1) and (3) is near the surface; thus (3) is not likely to be useful. Furthermore, the verification of (2) is complicated by the necessity of determining the validity of a certain statement for every quasinilpotent compact operator, and it is not always obvious whether an operator is compact. It seems worthwhile to find a condition intermediate to (2) and (3) that is equivalent to (1), and in this section we establish a framework in which there is such a condition.

A family Γ of subspaces of $\mathscr{H}$ that is linearly ordered by inclusion is a *chain* of subspaces. If Γ is a chain of subspaces and $\mathscr{M} \in \Gamma$, we denote by $\mathscr{M}_-$ the closure of the union $\bigcup\{\mathscr{N} \in \Gamma: \mathscr{N} \subsetneq \mathscr{M}\}$. A chain Γ will be called a *simple chain* if it satisfies the following conditions:

(i) $(0) \in \Gamma$ and $\mathscr{H} \in \Gamma$,

(ii) if Γ_0 is any subfamily of Γ, then the subspaces $\bigcap\{\mathscr{M}: \mathscr{M} \in \Gamma_0\}$ and $\mathrm{cl}\left[\bigcup\{\mathscr{M}: \mathscr{M} \in \Gamma_0\}\right]$ are in Γ,

(iii) for each $\mathscr{M}$ in Γ, the subspace $\mathscr{M} \ominus \mathscr{M}_-$ is at most one-dimensional.

Finally, an operator T will be said to have *trivial reducing kernel* if kernel $T \cap$ kernel $T^* = (0)$.

Our program announced above begins with the following result.

THEOREM 2.1. *Let* $\mathscr{A}$ *be a subalgebra of* $\mathscr{L}(\mathscr{H})$, *and suppose that* $\mathrm{Lat}(\mathscr{A})$ *contains a simple chain. Then*

(2') *there exists a nonzero, quasinilpotent Hilbert-Schmidt operator* K *on* $\mathscr{H}$ *with trivial reducing kernel such that for every* A *in* $\mathscr{A}$ *the operator* AK *is quasinilpotent.*

The proof of Theorem 2.1 depends upon the following facts (see, for example, [3, Theorems 4.3.10, 4.4.6, 4.4.10]), which we state for completeness.

PROPOSITION 2.2. *Let* K *be a compact operator on* $\mathscr{H}$, *and let* Γ *be a simple chain in* Lat(K). *Then* K *is quasinilpotent if and only if* $K\mathscr{M} \subset \mathscr{M}_-$ *for every* $\mathscr{M}$ *in* Γ.

PROPOSITION 2.3. *Let* Γ *be a simple chain of subspaces of* $\mathscr{H}$, *and let* J *be a self-adjoint Hilbert-Schmidt operator on* $\mathscr{H}$ *such that, for each* $\mathscr{M}$ *in* Γ, *the compression of* J *to* $\mathscr{M} \ominus \mathscr{M}_-$ *is the zero operator. Then there exists a quasinilpotent Hilbert-Schmidt operator* K *on* $\mathscr{H}$ *such that* $\Gamma \subset \mathrm{Lat}(K)$ *and such that* J *is the imaginary part of* K $\left(\textit{that is, } J = \frac{1}{2i}(K - K^*)\right)$.

Proof of Theorem 2.1. Let Γ be a simple chain in $\mathrm{Lat}(\mathscr{A})$. Since $\mathscr{H}$ is separable, there exist at most countably many subspaces $\mathscr{M}$ in Γ such that $\mathscr{M} \ominus \mathscr{M}_-$ is one-dimensional. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for $\mathscr{H}$ with the property that if $\mathscr{M} \in \Gamma$ and $\mathscr{M} \ominus \mathscr{M}_-$ is one-dimensional, then $\mathscr{M} \ominus \mathscr{M}_-$ contains one of the vectors e_j. Let S be the shift operator on $\mathscr{H}$ defined by the equations $Se_n = \frac{1}{n}e_{n+1}$ $(1 \leq n < \infty)$, and set $J = S + S^*$. Then J is a Hilbert-Schmidt operator, and an easy calculation shows that J has trivial kernel. Furthermore, it is clear from the construction that for every $\mathscr{M}$ in Γ, either $\mathscr{M} = \mathscr{M}_-$, or $\mathscr{M} \ominus \mathscr{M}_-$ is one-dimensional and the compression of J to $\mathscr{M} \ominus \mathscr{M}_-$ is zero. (In other words, the diagonal entries of the matrix for J with respect to the basis $\{e_n\}$

are all zero.) According to Proposition 2.3, there exists a quasinilpotent Hilbert-Schmidt operator K on $\mathscr{H}$ such that $\Gamma \subset \text{Lat}(K)$ and such that $J = \frac{1}{2i}(K - K^*)$. Since J has trivial kernel, it follows that K has trivial reducing kernel. Moreover, by Proposition 2.2, $K\mathscr{M} \subset \mathscr{M}_-$ for each $\mathscr{M}$ in Γ, and since $\Gamma \subset \text{Lat}(\mathscr{A})$, $AK\mathscr{M} \subset \mathscr{M}_-$ for each A in $\mathscr{A}$. Applying Proposition 2.2 once again, we see that AK is quasinilpotent for each A in $\mathscr{A}$; this completes the proof.

We can now employ condition (2') to state an equivalent formulation of the invariant-subspace problem for the class of quasinilpotent operators, for the class of operators with compact imaginary part, and for some other interesting classes of operators. Instead of treating each of these classes separately, however, we shall formulate one theorem that applies to all the classes simultaneously, and that is the purpose of the following discussion.

We consider classes $\mathscr{C}(\mathscr{H})$ of operators on $\mathscr{H}$ with the property that if U is any unitary operator in $\mathscr{L}(\mathscr{H})$, then the transformation $T \to UTU^*$ leaves $\mathscr{C}(\mathscr{H})$ fixed. If $\mathscr{C}(\mathscr{H})$ is such a class, and if $\mathscr{K}$ is any other Hilbert space of dimension $\aleph_0$, we denote by $\mathscr{C}(\mathscr{K})$ the class of all operators T in $\mathscr{L}(\mathscr{K})$ such that T is unitarily equivalent to some operator in $\mathscr{C}(\mathscr{H})$. If $\mathscr{I}(\mathscr{H})$ is such a unitarily invariant class, then $\mathscr{I}(\mathscr{H})$ will be called an *inheriting class* of operators provided the following conditions are satisfied.

(a) If T belongs to $\mathscr{I}(\mathscr{H})$ and $\mathscr{M}$ is an infinite-dimensional subspace in Lat(T), then $T \mid \mathscr{M} \in \mathscr{I}(\mathscr{M})$.

(b) If $T \in \mathscr{I}(\mathscr{H})$ and $\mathscr{M}^\perp$ is an infinite-dimensional subspace such that $\mathscr{M} \in \text{Lat}(T)$, then $T^* \mid \mathscr{M}^\perp \in [\mathscr{I}(\mathscr{M}^\perp)]^*$.

The following proposition, whose proof is routine and is therefore omitted, sets forth some interesting inheriting classes of operators on $\mathscr{H}$.

PROPOSITION 2.4. *Each of the following subsets of* $\mathscr{L}(\mathscr{H})$ *is an inheriting class of operators:*

I. $\mathscr{L}(\mathscr{H})$,

II. {T: T *is quasinilpotent*},

III. {T: T *has compact imaginary part*},

IV. {T: *the approximate point spectrum* $\pi(T)$ *of* T *is contained in a fixed closed set that does not separate the plane*},

V. {T: *the left Calkin spectrum* $\sigma_{\ell e}(T)$ *of* T *is contained in a fixed closed set that does not separate the plane* },

VI. {T: $\pi(T)$ *has empty interior and doesn't separate the plane* }, *or, equivalently,*

{T: $\sigma_{\ell e}(T)$ *has empty interior and doesn't separate the plane* }.

The following theorem can be applied to each of these examples of inheriting classes; thus it provides an equivalent formulation of the invariant-subspace problem for each of the classes.

THEOREM 2.5. *Let* $\mathscr{I}(\mathscr{H})$ *be any inheriting class of operators. Then every operator in* $\mathscr{I}(\mathscr{H})$ *has a nontrivial invariant subspace if and only if the following condition is satisfied:*

(2") *For every* T *in* $\mathscr{I}(\mathscr{H})$, *there exists a Hilbert-Schmidt operator* K_T *with trivial reducing kernel such that for every polynomial* p, *the operator* $p(T)K_T$ *is quasinilpotent.*

Proof. Suppose first that (2") is valid. Then it follows immediately from Corollary 1.2 that every operator in $\mathscr{I}(\mathscr{H})$ has a nontrivial invariant subspace. Conversely, suppose that this last condition is satisfied, and consider an arbitrary operator T in $\mathscr{I}(\mathscr{H})$. If $\mathscr{M}$ is a nontrivial invariant subspace for T and dim $\mathscr{M} > 1$, then $T \mid \mathscr{M}$ itself has a nontrivial invariant subspace, by virtue of the hypothesis and the definition of an inheriting class. Moreover, if $\mathscr{M}^{\perp}$ is not one-dimensional, then $T^* \mid \mathscr{M}^{\perp}$ also has a nontrivial invariant subspace. From these facts it readily follows that Lat(T) contains a simple chain. Since Lat(T) coincides with the lattice of invariant subspaces of the algebra consisting of all polynomials p(T), the fact that (2") is satisfied follows immediately from Theorem 2.1.

We choose one corollary from many to exemplify the contrapositive of Theorem 2.5.

COROLLARY 2.6. *Suppose that* T *is a quasinilpotent operator on* $\mathscr{H}$, *and suppose that for every quasinilpotent Hilbert-Schmidt operator* K *on* $\mathscr{H}$ *without reducing kernel there exists a polynomial* p_K *such that* $p_K(T)K$ *is not quasinilpotent Then some quasinilpotent operator on* $\mathscr{H}$ *has no nontrivial invariant subspace.*

REFERENCES

1. V. I. Lomonosov, *On invariant subspaces of families of operators, commuting with a completely continuous operator.* (Russian) Funkcional. Anal. i Priložen 7, no. 3 (1973), 55-56.
2. C. Pearcy and A. L. Shields, *A survey of the Lomonosov technique in the theory of invariant subspaces.* Amer. Math. Soc. Mathematical Surveys, Number 13 (to appear).
3. J. R. Ringrose, *Compact non-self-adjoint operators.* Van Nostrand, London, 1971.

C. Pearcy and N. Salinas:
University of Michigan, Ann Arbor, Michigan 48104

J. R. Ringrose:
University of Newcastle upon Tyne, Newcastle upon Tyne, England