PIERCING DISKS WITH TAME ARCS

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We give here a short proof of the following theorem of R. H. Bing.

THEOREM 0 (Bing [5]; see [3] for an earlier related theorem). *Suppose that* S *is a 2-sphere in* E^3, *that* A *is a rectilinear segment in* E^3, *and that* $\varepsilon > 0$. *Then there exists an* ε*-ambient isotopy* $H: E^3 \times I \to E^3 \times I$ *of* E^3, *fixed outside* $N(A \cap S, \varepsilon)$, *such that* $H_1(A) \cap S$ *is finite.* (*The map* $H_1: E^3 \to E^3$ *is the homeomorphism defined by the restriction* $H \mid E^3 \times \{1\}$.)

Bing's original proof of the Side Approximation Theorem (S.A.T.) [4] (see [7] for a new proof) was considerably complicated by the lack of a proof for Theorem 0; Bing first proved Theorem 0 by using both the Side Approximation Theorem and a number of its deep consequences. Our proof of Theorem 0 depends only on the existence of abundantly many *tame* arcs in S [2]; the existence of such tame arcs has recently been established independently of the S.A.T. (see [6, Sections 2 and 3]).

Definition. A compact metric space K is said to be a *regular compactum* if for each $\varepsilon > 0$ there exist a finite subset K_ε of K and a separation

$$K - K_\varepsilon = K_1 \cup \cdots \cup K_r \qquad \text{(separated)}$$

of $K - K_\varepsilon$ such that each of the sets $K_1, \cdots, K_r$ has diameter less than ε.

LEMMA. *If* K *is a regular compactum in* E^2 *and* A *is an arc in* E^2, *then for each* $\varepsilon > 0$ *there exists an* ε*-isotopy* $H: E^2 \times I \to E^2 \times I$ *of* E^2, *fixed outside* $N(A \cap K, \varepsilon)$, *such that* $H_1(A) \cap K$ *is finite.*

Proof. Since K is at most 1-dimensional, we may assume that $A \cap K$ is a 0-dimensional subset of Int A. Then $K \cap A$ is covered by the interiors of finitely many disjoint ε-subarcs $A_1, \cdots, A_k$ of Int A. There exist disjoint ε-disks $D_1, \cdots, D_k$ in E^2 such that for each i the intersection $D_1 \cap A = A_i$ is a spanning arc of D_i. It clearly suffices to show that Bd A_i bounds a spanning arc B_i of D_i such that $B_i \cap K$ is finite.

Since K is a regular compactum, it follows from [8, (3.2) Theorem, p. 35] that there are finitely many disks $E_1, \cdots, E_n$ in Int D_i whose interiors cover $K \cap A_i$ and such that, for each j, the set (Bd E_j) $\cap$ K is finite. Clearly, there is an arc B_i bounded by Bd A_i in the set

$$\left[A_i - \bigcup_{j=1}^{n} \operatorname{Int} E_j \right] \cup \left[\bigcup_{j=1}^{n} \operatorname{Bd} E_j \right].$$

It is equally clear that this B_i must satisfy the requirements of the preceding paragraph. This completes the proof of the lemma.

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Proof of Theorem 0. It is an immediate consequence of [2] (see [6; Results 2C.7(2), 2C.7(2).1, 3.1 and 3.2]) that there exists a sequence $T_1, T_2, \cdots$ of curvilinear triangulations of S such that, for $n = 1, 2, \cdots$,

(i) each 1-simplex of each T_n has a locally tame interior;

(ii) the mesh of T_n is less than $1/n$;

(iii) T_{n+1} refines T_n.

It is an inductive consequence of [1] (see [6; Result 3.1]) that we may assume that each 1-simplex of each T_n has locally polyhedral interior. Let D be a triangular disk in E^3 that contains A in its interior. A slight ambient isotopy of E^3 will move the vertices of D into general position so that the adjusted D misses the countably many endpoints of the 1-simplexes of the triangulations T_n and is in general position with respect to the locally polyhedral interiors of their 1-simplexes. As a consequence, D has finite intersection with the 1-skeleton of each T_n. Thus, for the adjusted D, the intersection $D \cap S$ is a regular compactum. By the lemma, there is a small isotopy of D such that the final image of A has finite intersection with $D \cap S$. Since D is tame, this isotopy may be extended to a small isotopy of E^3. This completes the proof.

Addendum to Theorem 0. We may assume that at each point of $H_1(A) \cap S$ the arc $H_1(A)$ pierces S.

Proof (R. H. Bing). We show how to remove any intersection $p \in H_1(A) \cap S$ at which A does not pierce S. It suffices to consider the case where $H_1(A) \cap S$ is the single point $p \in \text{Int}\,[H_1(A)]$, where $H_1(A)$ is a rectilinear segment, and where $H_1(A) - \{p\} \subset \text{Int}\, S$. In this case, there exists an arc $\alpha \subset \text{Int}\, S$ such that $J = \alpha \cup H_1(A)$ is a simple closed curve and there is a solid right circular cylinder C containing p in its interior, intersecting J precisely in a subarc β of $H_1(A)$ that is also the axis of C, and having diameter less than ε. If there were any component R of $(\text{Bd}\, C) \cap S$ that separates the endpoints of β in Bd C, then J would link some simple closed curve J' in S that is very close to R. This is impossible, since J lies in the set $(S \cup \text{Int}\, S) - R$, which is a set having trivial first homology. Thus no component of $(\text{Bd}\, C) \cap S$ separates the endpoints of β in Bd C, and therefore some arc γ in $(\text{Bd}\, C) - S$ joins the endpoints of β. A small ambient isotopy fixes $A - \beta$ and moves β to γ. This removes the intersection p of $H_1(A)$ with S, and it completes the proof of the Addendum.

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