GREEN'S MEASURE AND HARMONIC MEASURE

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Although our main result remains valid in more general settings, the present discussion will be carried out merely for the case of bounded plane regions. Let us thus take Ω as such a region, fix z as any point of Ω, and denote by $G_z(\zeta)$ the Green's function at ζ ($\in \Omega$) with pole at z. Relative to this pole, the Green's lines are the maximal orthogonal trajectories, issuing from z, of the level lines of $G_z(\zeta)$. Green's lines were introduced into analysis by G. C. Evans [4], and they have been explored in greater depth by M. Brelot and G. Choquet [3] and others. An independent treatment of the plane case was given by M. Arsove and G. Johnson [2].

Let us recall briefly some of the fundamental notions concerning Green's lines and Green's measure. All Green's lines have well-defined initial angles, and to each initial angle ϕ there corresponds a unique Green's line Γ^ϕ. Given a subset A of $\partial\Omega$, the Green's measure of A is defined as the normalized Lebesgue measure of the set of angles ϕ for which the Green's lines Γ^ϕ terminate in points of A (provided, of course, that this set of angles is Lebesgue-measurable). Green's measure will be denoted by $g_z(A)$, the normalizing condition being $g_z(\partial\Omega) = 1$. In the sense of Green's measure, almost all Green's lines terminate in points of $\partial\Omega$.

Brelot and Choquet proved that the inner and outer Green's measures of each boundary set are bracketed between the inner and outer harmonic measures of the set (see [3, p. 255]). Thus, harmonic measurability implies Green's measurability and equality of the two measures. (See also [2, pp. 23-24] for an alternative derivation of this property.) Our present aim is to complete the picture by establishing the converse. We give an elementary argument to show that Green's measurability implies harmonic measurability, so that the two measures actually coincide.

THEOREM. *A set* $A \subset \partial\Omega$ *is measurable with respect to Green's measure at* z *if and only if it is measurable with respect to harmonic measure at* z, *and the two measures are then equal.*

In proving that Green's measurability implies harmonic measurability, we note that, contrary to what might perhaps be expected, there is no need to appeal to the theory of analytic sets. Our proof will be carried out along simpler lines, and the key lies in the use of Lusin's theorem in a manner similar to that employed in a different context by P. R. Ahern and D. Sarason [1].

We start with a g_z-measurable subset A of $\partial\Omega$, and we denote by α the set of angles ϕ on $[0, 2\pi)$ for which the Green's lines Γ^ϕ terminate in points of A. It is convenient to introduce the Green's mapping T at this point, although no appeal will be made to the fact that T is conformal in the plane case. As initially defined (see [2, p. 8]), T is given by the relation $T(\rho e^{i\phi}) = \zeta$, where ζ is the unique point of Ω lying on the Green's line Γ^ϕ and on the level line $[G_z = \log(1/\rho)]$. We shall regard T as extended to the set of points $e^{i\phi}$ for which Γ^ϕ terminates in a point ζ of $\partial\Omega$ by the defining property $T(e^{i\phi}) = \zeta$. To simplify the notation, let us now write

Received July 12, 1971.

This research was supported by the National Science Foundation under Grant GP 22821.

Michigan Math. J. 19 (1972).

$f(\phi) = T(e^{i\phi})$, wherever $T(e^{i\phi})$ is defined on $[0, 2\pi)$. The important fact about f is that, as a limit of continuous functions, f is measurable relative to Lebesgue measure on $[0, 2\pi)$, so that $f \mid \alpha$ is a measurable function. We can assume without loss of generality that ρ tends to 1 as ζ tends to $\partial\Omega$ along Γ^{ϕ} $(\phi \in \alpha)$, since the points of A where this fails form a set of harmonic measure zero (hence, Green's measure zero).

Invoking Lusin's theorem, we obtain an increasing sequence $\{\mathscr{K}_n\}$ of compact subsets of α such that $f \mid \mathscr{K}_n$ is continuous for each n and

$$|\alpha| = \left|\bigcup_{n=1}^{\infty} \mathscr{K}_n\right|, \tag{1}$$

where the bars denote Lebesgue measure. By continuity of f, the sets

$$K_n = f(\mathscr{K}_n) \qquad (n = 1, 2, \cdots)$$

are compact, and it follows that

$$P = \bigcup_{n=1}^{\infty} K_n$$

is a Borel subset of A. Hence, P is harmonic measurable, and

$$m_z(P) = g_z(P) = g_z(A), \tag{2}$$

in view of (1). The same argument applied to the complement of A leads at once to the existence of a Borel set Q containing A, with $m_z(Q) = g_z(A)$. In conjunction with (2), this shows that the set $A - P$ $(\subset Q - P)$ has harmonic measure zero. It follows that A must be harmonic measurable, and the proof is complete.

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