RIEMANNIAN MANIFOLDS OF CONSTANT CURVATURE AND THE GROWTH FUNCTION OF SUBMANIFOLDS

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1. Let $\overline{M}$ be a Riemannian C^∞-manifold, and let M be a compact C^∞-submanifold of codimension 1, possibly with boundary. If there exists a globally defined C^∞ unit-normal field N on M, we say that M is *relatively orientable*. If $\overline{M}$ is orientable, then M is relatively orientable if and only if M is orientable. We call a submanifold of codiemnsion 1 a *hypersurface*.

Suppose M is relatively orientable, and let N be a C^∞ unit-normal field on M. For $m \in M$, let $g_m(s)$ denote the geodesic (of $\overline{M}$), parametrized by arc length s, such that $g_m(0) = m$ and $\dot{g}_m(0) = N(m)$, where $\dot{g}_m$ is the tangent vector to g_m. Let M_s be the set of points $\{g_m(s) \mid m \in M\}$. For small s, the set M_s is a C^∞-submanifold of $\overline{M}$. Denote the volume of M_s by A(s). Following H. Wu and R. A. Holzsager [3], [4], we call A(s) the *growth function* of M. Let $A^{(k)}$ denote the kth derivative of A with respect to s. Wu and Holzsager [3], [4] showed that the two-dimensional Riemannian manifolds of constant curvature equal to c are characterized by the equation $A^{(2)} + cA = 0$ for all M. Let $L = d/ds$, and let c be a constant. Let

$$L_n = (L^2 + c)(L^2 + 9c)(L^2 + 25c) \cdots (L^2 + (n-1)^2 c) \tag{1}$$

if n is even, and

$$L_n = L(L^2 + 4c)(L^2 + 16c) \cdots (L^2 + (n-1)^2 c) \tag{2}$$

if n is odd. We shall prove the following four theorems.

THEOREM 1. *Suppose $\overline{M}$ is an n-dimensional Riemannian manifold of constant curvature equal to* c. *Then the growth function* A *of each compact, relatively orientable hypersurface* M *of* $\overline{M}$ *satisfies the differential equation*

$$L_n A = 0.$$

Furthermore, this is the only differential equation of lowest order that A *satisfies for every* M.

THEOREM 2. *Suppose the growth function* A *of each compact, relatively orientable hypersurface* M *of* $\overline{M}$ *satisfies the differential equation*

$$A^{(3)} + c_2 A^{(2)} + c_1 A^{(1)} + c_0 A = 0, \tag{3}$$

where c_2, c_1, *and* c_0 *are functions of* s, *and no lower-order differential equation is satisfied by* A *for all* M; *then* $\overline{M}$ *is a three-dimensional Riemannian manifold of constant curvature, say* K, *and therefore, by Theorem* 1, $c_2 = c_0 = 0$ *and* $c_1 = 4K$.

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THEOREM 3. *Suppose the growth function* A *of each compact, relatively orientable hypersurface* M *of* $\overline{M}$ *satisfies the differential equation*

$$A^{(4)} + c_3 A^{(3)} + c_2 A^{(2)} + c_1 A^{(1)} + c_0 = 0, \tag{4}$$

where c_3, c_2, c_1, *and* c_0 *are functions of* s, *and no lower-order differential equation is satisfied by* A *for all* M; *then* $\overline{M}$ *has constant curvature on each of its connected components. Therefore, by Theorem* 1, *either* $\dim \overline{M} = 4$, *and* $c_3 = c_1 = 0$, $c_2 = 10K$, $c_0 = 9K^2$, *where* K *is the curvature of* $\overline{M}$; *or* $\dim \overline{M} = 2$, $\overline{M}$ *has precisely two connected components of curvatures* K_1 *and* K_2 $(K_1 \neq K_2)$, *and* $c_1 = c_3 = 0$, $c_2 = K_1 + K_2$, $c_0 = K_1 K_2$.

THEOREM 4. *If* $\dim \overline{M} = k$ *and* $L_n A = 0$ *for each compact, relatively orientable hypersurface* M *of* $\overline{M}$, *then* $k \leq n$. *If* $k = n$, *then* $\overline{M}$ *has constant curvature equal to* c. *If* $k = n - 1$, *then* $\overline{M}$ *is an Einstein manifold.*

Holzsager (see [1] and [2]) has recently obtained Theorems 1 and 4, together with related results.

2. *Proof of Theorem* 1. Suppose $\overline{M}$ is an n-dimensional Riemannian manifold of constant curvature equal to c. Let M be a compact, relatively orientable hypersurface, and let N be a C^∞ unit-normal field on M. Let Ω_s be the volume element for M_s, and set $\Omega = \Omega_0$. Since M and M_s are diffeomorphic by the mapping $m \to g_m(s)$, we may consider Ω_s as defined on M. Let B denote the second fundamental form of M, considered as a tensor of type 1 - 1; that is, for each X tangent to M, let $BX = -D_X N$, where D is the covariant differentiation in $\overline{M}$. An easy calculation will show that

$$\Omega_s = \left\{ \prod_i (1 - sb_i) \right\} \Omega \qquad \text{if } c = 0, \tag{5}$$

$$\Omega_s = \left\{ \prod_i [\cos(s/R) - Rb_i \sin(s/R)] \right\} \Omega \qquad \text{if } 1/R^2 = c > 0, \tag{6}$$

$$\Omega_s = \left\{ \prod_i [\cosh(s/R) - Rb_i \sinh(s/R)] \right\} \Omega \qquad \text{if } -1/R^2 = c < 0, \tag{7}$$

where $\{b_i \mid 1 \leq i \leq n - 1\}$ is the set of eigenvalues of B.

Since the calculation is local in nature, we may assume that $\overline{M}$ is the Euclidean space R^n, the sphere $S^n(R)$, or the hyperbolic space $H^n(R)$. We consider $S^n(R)$ as the sphere of radius R, contained in the Euclidean space R^{n+1}, and with center at the origin. We denote by E^{n+1} the Minkowski space with global coordinates $x_0, x_1, \cdots, x_n$ and pseudo-Riemannian metric determined by the quadratic form

$$q(x, y) = -x_0 y_0 + x_1 y_1 + \cdots + y_n y_n,$$

and we consider $H^n(R)$ as the submanifold of E^{n+1} defined by the equation

$$-x_0^2 + x_1^2 + \cdots + x_n^2 = -R^2 \qquad (x_0 > 0).$$

If $\overline{M} = R^n$ and O is the origin in R^n, let $\vec{x}_s(m) = \overrightarrow{Og_m}(s)$, and set $\vec{x} = \vec{x}_0$. Then

$$\vec{x}_s = \vec{x} + sN. \tag{8}$$

If $\overline{M} = S^n(R)$ and O is the origin in R^{n+1}, let $\vec{x}_s(m) = \overrightarrow{Og_m}(s)$, and set $\vec{x} = \vec{x}_0$. Then

$$\vec{x}_s = (\cos(s/R))\vec{x} + (R\sin(s/R))N. \tag{9}$$

If $\overline{M} = H^n(R)$ and O is the origin in E^{n+1}, let $\vec{x}_s(m) = \overrightarrow{Og_m}(s)$, and set $\vec{x} = \vec{x}_0$. Then

$$\vec{x}_s = (\cosh(s/R))\vec{x} + (R\sinh(s/R))N. \tag{10}$$

Using (8), (9), and (10), we can easily obtain (5), (6), and (7).

Equations (5), (6), and (7) imply Theorem 1.

3. To prove Theorems 2 and 3, we must calculate the first four derivatives of A with respect to s. Let I_k be the integrand for $A^{(k)}$; that is, let

$$A^{(k)} = \int_{M_s} I_k \Omega_s,$$

where Ω_s is the volume element for M_s. Let $S = \dot{g}_m(s)$; then S is a unit-normal field on M_s. Let B denote the second fundamental form of M_s, considered as a tensor of type 1-1; that is, for each X tangent to M_s, let $BX = -D_X S$, where D is the covariant differentiation in $\overline{M}$. In [4], it is shown that

$$I_1 = -(\mathrm{tr}\, B) \tag{11}$$

and

$$I_2 = (\mathrm{tr}\, B)^2 - (\mathrm{tr}\, B^2) - \langle \mathscr{R}(S), S \rangle, \tag{12}$$

where tr stands for trace, $\mathscr{R}$ is the Ricci tensor of $\overline{M}$ (as in [4]), and $\langle\ ,\ \rangle$ is the inner product. Equation (11) implies that

$$I_{k+1} = -(\mathrm{tr}\, B)I_k + SI_k. \tag{13}$$

It is a tedious but straightforward task to show that

$$\begin{aligned} I_3 = {} & -(\mathrm{tr}\, B)^3 + 3(\mathrm{tr}\, B)(\mathrm{tr}\, B^2) - 2(\mathrm{tr}\, B^3) \\ & + 3\langle \mathscr{R}(S), S\rangle (\mathrm{tr}\, B) - 2\sum_{i,k} \langle R(S, E_i)S, E_k\rangle B_{ki} - \langle (D_S\mathscr{R})S, S\rangle \end{aligned} \tag{14}$$

and

$$\begin{aligned} I_4 = {} & (\mathrm{tr}\, B)^4 - 6(\mathrm{tr}\, B)^2(\mathrm{tr}\, B^2) + 8(\mathrm{tr}\, B)(\mathrm{tr}\, B^3) + 3(\mathrm{tr}\, B^2)^2 - 6(\mathrm{tr}\, B^4) \\ & - 6\langle \mathscr{R}(S), S\rangle (\mathrm{tr}\, B)^2 + 6\langle \mathscr{R}(S), S\rangle (\mathrm{tr}\, B^2) + 3\langle \mathscr{R}(S), S\rangle^2 \\ & - \langle (D_S^2\mathscr{R})S, S\rangle + 4\langle (D_S\mathscr{R})S, S\rangle (\mathrm{tr}\, B) - 2\sum_{i,k}\langle R(S, E_i)S, E_k\rangle^2 \\ & + 8\sum_{i,k}\langle R(S, E_i)S, E_k\rangle (\mathrm{tr}\, B) B_{ki} - 8\sum_{i,k,j}\langle R(S, E_i)S, E_k\rangle B_{kj}B_{ji} \\ & - 2\sum_{i,k}\langle (D_S R)(S, E_i)S, E_k\rangle B_{ki}, \end{aligned} \tag{15}$$

where $\{E_i \mid 1 \le i \le n-1\}$ is an orthonormal basis of the tangent space of M_s at each point of M_s, the symbols B_{ki} denote the components of B with respect to this basis, and R is the curvature tensor of $\overline{M}$ (as in [4]).

LEMMA 1. *Let* B_{ij} *denote an* $(n-1)$-*by*-$(n-1)$ *symmetric matrix, let* m *be a point of* $\overline{M}$, *and let* $E_1, \cdots, E_{n-1}, S$ *constitute an orthonormal basis of the tangent space of* $\overline{M}$ *at* m. *Then there exists a compact, relatively orientable hypersurface* M *of* $\overline{M}$, *containing* m, *with normal* S *at* m, *and such that the second fundamental form of* M *at* m, *expressed with respect to* $E_1, \cdots, E_{n-1}$, *has the components* B_{ij}.

Proof of Lemma 1. Let $x_1, \cdots, x_n$ be normal coordinates around m with $(0, \cdots, 0)$ corresponding to m, and with

$$\left(\frac{\partial}{\partial x_i}\right)_m = E_i \quad \text{for } 1 \le i \le n-1 \quad \text{and} \quad \left(\frac{\partial}{\partial x_n}\right)_m = S\,.$$

Define a hypersurface M by the equation $x_n = \frac{1}{2}\sum_{i,j} B_{ij}x_ix_j$, and restrict the values of x_i suitably. Then M has the desired properties.

Proof of Theorem 2. If equation (3) is satisfied for all compact, relatively orientable hypersurfaces M of $\overline{M}$, then

$$I_3 + c_2 I_2 + c_1 I_1 + c_0 = 0 \tag{16}$$

for every compact, relatively orientable hypersurface M of $\overline{M}$. By Lemma 1, the sum of the terms on the left-hand side of (16) of a certain degree in the entries of B must vanish. Let $m \in \overline{M}$, and let $\{E_1, \cdots, E_{n-1}, S\}$ be an orthonormal basis of the tangent space of $\overline{M}$ at m.

Step 1. The equation of degree 3 is

$$-(\text{tr } B)^3 + 3(\text{tr } B)(\text{tr } B^2) - 2(\text{tr } B^3) = 0\,. \tag{17}$$

Suppose $\dim \overline{M} \ge 4$. Let $B_{11} = B_{22} = B_{33} = 1$ and $B_{ij} = 0$ if

$$(i, j) \neq (1, 1), (2, 2), (3, 3)\,.$$

Choose a hypersurface as in Lemma 1. Then, at m, the left-hand side of (17) is -6. Thus, $\dim \overline{M} \le 3$.

Step 2. Suppose $\dim \overline{M} = 2$. Let $B_{11} = 1$, and choose a hypersurface as in Lemma 1. Then, at m, the equation of degree 1 implies that

$$K - c_1 = 0\,,$$

where K is the Gaussian curvature of $\overline{M}$ at m and c_1 is evaluated at $s = 0$. Since m is arbitrary, we conclude that $\overline{M}$ has constant Gaussian curvature.

Step 3. Suppose $\dim \overline{M} = 3$. Let $B_{11} = B_{22} = 1$ and $B_{12} = 0$. Choose a hypersurface as in Lemma 1. Then, at m, the equation of degree 1 implies that

$$4\left\langle \mathscr{R}(S), S \right\rangle - 2c_1 = 0\,,$$

where c_1 is evaluated at $s = 0$. Since m is arbitrary and S is an arbitrary unit vector at m, we conclude that $\overline{M}$ is an Einstein manifold. Since $\dim \overline{M} = 3$, we conclude that $\overline{M}$ has constant curvature.

Proof of Theorem 3. If equation (4) is satisfied for all compact, relatively orientable hypersurfaces M of $\overline{M}$, then

$$I_4 + c_3 I_3 + c_2 I_2 + c_1 I_1 + c_0 = 0 \tag{18}$$

for every compact, relatively orientable hypersurface M of $\overline{M}$. By Lemma 1, the sum of the terms on the left-hand side of (18) of a certain degree in the entries of B must vanish. Let $m \in \overline{M}$, and let $\{E_1, \cdots, E_{n-1}, S\}$ be an orthonormal basis of the tangent space of $\overline{M}$ at m.

Step 1. The equation of degree 4 is

$$0 = (\operatorname{tr} B)^4 - 6(\operatorname{tr} B)^2(\operatorname{tr} B^2) + 8(\operatorname{tr} B)(\operatorname{tr} B^3) + 3(\operatorname{tr} B^2)^2 - 6(\operatorname{tr} B^4) . \tag{19}$$

Suppose $\dim \overline{M} \geq 5$. Let $B_{11} = B_{22} = B_{33} = B_{44} = 1$, and let $B_{ij} = 0$ if $(i, j) \neq (1, 1), (2, 2), (3, 3), (4, 4)$. Choose a hypersurface as in Lemma 1. Then, at m, the left-hand side of (19) is 24. Thus $\dim \overline{M} \leq 4$.

Step 2. Suppose $\dim \overline{M} = 2$. Let $B_{11} = 1$. Then, at m, the equation of degree 1 implies that

$$2 D_S K + c_3 K - c_1 = 0 , \tag{20}$$

where c_1 and c_3 are evaluated at $s = 0$ and K is the Gaussian curvature of $\overline{M}$. Since m is arbitrary and S is an arbitrary unit vector at m, it is not difficult to see that equation (20) implies that K is constant on each connected component of $\overline{M}$.

Step 3. Suppose $\dim \overline{M} = 3$. Let $B_{11} = -B_{22} = 1$ and $B_{12} = 0$. Choose a hypersurface as in Lemma 1. Then, at m, the equation of degree 2 implies that

$$12 \left\langle \mathscr{R}(S), S \right\rangle - 8K(S \wedge E_1) - 8K(S \wedge E_2) - 2c_2 = 0 , \tag{21}$$

where c_2 is evaluated at $s = 0$ and $K(X \wedge Y)$ is the sectional curvature in $\overline{M}$ of the plane spanned by X and Y. We may write equation (21) as

$$4 \left\langle \mathscr{R}(S), S \right\rangle - 2c_2 = 0 .$$

Since m is arbitrary and S is an arbitrary unit vector at m, we conclude that $\overline{M}$ is an Einstein manifold. Since $\dim \overline{M} = 3$, we conclude that $\overline{M}$ has constant curvature.

Step 4. Suppose $\dim \overline{M} = 4$. Let $B_{11} = -B_{22} = 1$ and $B_{ij} = 0$ if $(i, j) \neq (1, 1), (2, 2)$. Choose a hypersurface as in Lemma 1. Then, at m, the equation of degree 2 implies that (21) holds. Similarly, at m, we can obtain the relation

$$12 \left\langle \mathscr{R}(S), S \right\rangle - 8K(S \wedge E_2) - 8K(S \wedge E_3) - 2c_2 = 0 . \tag{22}$$

Comparing (21) and (22), we obtain the equation

$$K(S \wedge E_1) = K(S \wedge E_3) .$$

Since $\{E_1, E_2, E_3, S\}$ is an arbitrary orthonormal frame at m and m is arbitrary, we conclude that $\overline{M}$ has constant curvature.

4. Let $\dim \overline{M} = k$, and let M be a compact, relatively orientable hypersurface. Let N be a unit-normal field on M. Let Ω_s be the volume element of M_s, and set $\Omega = \Omega_0$. Since M and M_s are diffeomorphic by the mapping $m \to g_m(s)$, we may consider Ω_s as defined on M. Let f(s, m) be defined by the equation

$$\Omega_s = f(s, m)\,\Omega .$$

If $L_n A = 0$ for all compact, relatively orientable hypersurfaces, then $L_n f = 0$.

Let $p \in \overline{M}$, and let $M(r)$ be the geodesic sphere of radius r with center p; that is, let

$$M(r) = \{q \mid q = \exp_p rX,\ \langle X, X\rangle = 1\} .$$

For small, positive r, $M(r)$ is a smooth, compact hypersurface. For a fixed, small $\varepsilon > 0$, write $M = M(\varepsilon)$. Let N be the inward-pointing unit normal on M; that is, if $m = \exp_p \varepsilon X$, $\langle X, X\rangle = 1$, and $\alpha(t) = \exp_p tX$, let $N(m) = -\dot{\alpha}(\varepsilon)$, where $\dot{\alpha}(t)$ is the tangent vector to $\alpha(t)$. It is not difficult to see that the radial geodesics $\exp_p tX$ intersect $M(r)$ orthogonally. Thus, $M_s = M(\varepsilon - s)$. Let $\Omega(r)$ be the volume element of $M(r)$, and consider $\Omega(r)$ as defined on $M(\varepsilon)$. Then $\Omega_s = \Omega(\varepsilon - s)$. Thus

$$\Omega(r) = \tilde{f}(r, m)\,\Omega ,$$

where $\tilde{f}(r, m) = f(\varepsilon - r, m)$. Set $r = \varepsilon - s$, and write $\tilde{L} = \frac{d}{dr}$. Let $\tilde{L}_n$ be defined by the right-hand side of (1) or (2), according as n is even or odd, with L replaced by $\tilde{L}$. Since $\frac{d}{dr} = -\frac{d}{ds}$, the equation $L_n f = 0$ implies the equation $\tilde{L}_n \tilde{f} = 0$. Note that $\tilde{L}_n \tilde{f} = 0$ implies that

$$\tilde{f}(r, m) = \sum_{i=1}^{n} a_i(m) \sin^{n-i}(r/R) \cos^{i-1}(r/R) \quad \text{if } 0 < c = 1/R^2, \tag{23}$$

$$\tilde{f}(r, m) = \sum_{i=1}^{n} a_i(m) \sinh^{n-i}(r/R) \cosh^{i-1}(r/R) \quad \text{if } 0 > c = -1/R^2, \tag{24}$$

$$\tilde{f}(r, m) = \sum_{i=0}^{n-1} a_i(m)\, r^i \quad \text{if } c = 0 . \tag{25}$$

PROPOSITION 1. *Let* $\{E_1, \cdots, E_k\}$ *be an orthonormal frame at* p, *and let* $m = \exp_p \varepsilon E_k$. *Then*

$$\tilde{f}(r, m) = a(m) \left\{ r^{k-1} - \frac{(K_{1k} + K_{2k} + \cdots + K_{k-1,k})}{6}\, r^{k+1} + o(r^{k+2}) \right\}, \tag{26}$$

where K_{ik} *is the sectional curvature of* $\overline{M}$ *at* p *of the plane spanned by* E_i *and* E_k.

We postpone the proof of Proposition 1 to Section 6.

We shall now prove the first statement in Theorem 4. Suppose $1/R^2 = c > 0$. Expand the right-hand side of (23) in a Taylor series about $r = 0$, and compare the result with (26). We conclude that $k \leq n$. A similar argument holds if $c < 0$ or $c = 0$.

PROPOSITION 2. *Suppose* $k = n$. *Let* $E_1, \cdots, E_n$, *and* m *be as in Proposition* 1. *Then, with the assumptions of Theorem* 4,

$$K_{1n} + K_{2n} + \cdots + K_{n-1,n} = (n - 1)c . \tag{27}$$

Proof. Suppose $0 > c = 1/R^2$. Expand the right-hand side of (23) in a Taylor series about $r = 0$, and compare the result with (26). We conclude that $a_i = 0$ for $i > 1$. Thus

$$\tilde{f}(r, m) = a_1(m)\left\{\left(\frac{r}{R}\right)^{n-1} - \frac{(n-1)}{6}\left(\frac{r}{R}\right)^{n-1} + o(r^{n+2})\right\}. \tag{28}$$

Comparing (28) with (26), we conclude that

$$K_{1n} + K_{2n} + \cdots + K_{n-1,n} = (n-1)/R^2 .$$

A similar argument holds if $c < 0$ or $c = 0$.

Since $\{E_1, \cdots, E_n\}$ is an arbitrary orthonormal frame at p and p is arbitrary, we conclude that $\overline{M}$ is an Einstein manifold.

In a similar way we can prove the following result.

PROPOSITION 3. *Suppose* $k = n - 1$. *Let* $E_1, \cdots, E_{n-1}$, *and* m *be as in Proposition* 1. *Then, with the assumptions of Theorem* 4,

$$K_{1,n-1} + K_{2,n-1} + \cdots + K_{n-2,n-1} = (n+1)c .$$

Since $\{E_1, \cdots, E_{n-1}\}$ is an arbitrary orthonormal frame at p and p is arbitrary, we conclude that $\overline{M}$ is an Einstein manifold.

5. Suppose $\dim \overline{M} = n$. Let $p \in M$, and let $\{E_1, \cdots, E_n\}$ be an orthonormal frame at p. Let $\gamma(u) = \exp_p uE_1$ $(-\delta < u < \delta)$. Let $\{E_1(u), \cdots, E_n(u)\}$ be a parallel, orthonormal frame field along γ with $E_1(u) = \dot{\gamma}(u)$, where $\dot{\gamma}(u)$ is the tangent vector to $\gamma(u)$. Let M(r) be the geodesic cylinder of radius r about γ; that is, let

$$M(r) = \{q \mid q = \exp_{\gamma(u)} rX,\ \langle X, X\rangle = 1,\ \langle X, E_1\rangle = 0\} .$$

For small positive r and δ, M(r) is a smooth, compact hypersurface with boundary. For a fixed small $\varepsilon > 0$, let $M = M(\varepsilon)$. Let N be the inward-pointing unit normal on M, so that if $m = \exp_{\gamma(u)} \varepsilon X$, $\langle X, X\rangle = 1$, $\langle X, E_1\rangle = 0$, and $\alpha(t) = \exp_{\gamma(u)} tX$, then $N(m) = -\dot{\alpha}(\varepsilon)$. It is not difficult to see that the radial geodesics $\alpha(t) = \exp_{\gamma(u)} tX$ $(\langle X, E_1\rangle = 0)$ intersect M(r) orthogonally. Thus, $M_s = M(\varepsilon - s)$. Let $\tilde{f}(r, m)$ and $\tilde{L}_n$ be defined as in Section 4.

PROPOSITION 4. *Let* $E_1, \cdots, E_n$ *be as above, and let* $m = \exp_p \varepsilon E_n$. *Then*

$$\tilde{f}(r, m) = a(m)\left\{r^{n-2} - \frac{(K_{2n} + \cdots + K_{n-1,n} + 3K_{1n})}{6} r^n + o(r^{n+1})\right\}, \tag{29}$$

where K_{in} *is the sectional curvature of* $\overline{M}$ *at* p *of the plane spanned by* E_i *and* E_n.

We postpone the proof of Proposition 4 to Section 6.

PROPOSITION 5. *Let* $E_1, \cdots, E_n$, *and* m *be as in Proposition* 4. *Then, with the assumptions of Theorem* 4,

$$K_{2n} + \cdots + K_{n-1,n} + 3K_{1n} = (n+1)c . \tag{30}$$

Proof. Suppose $0 < c = 1/R^2$. The assumptions of Theorem 4 imply that $\tilde{L}_n \tilde{f} = 0$. Thus equation (23) holds. Expanding the right-hand side of (23) in a Taylor series about $r = 0$, and comparing the result with (29), we conclude that $a_i = 0$ if $i \neq 2$. Thus

$$\tilde{f}(r, m) = a_2(m)\left\{\left(\frac{r}{R}\right)^{n-2} - \frac{n+1}{6}\left(\frac{r}{R}\right)^{n} + o(r^{n+1})\right\} . \tag{31}$$

Comparing (29) and (31), we conclude that

$$K_{2n} + \cdots + K_{n-1,n} + 3K_{1n} = (n+1)/R^2 .$$

A similar argument holds if $c < 0$ or $c = 0$.

To prove the second statement in Theorem 4, note that equations (27) and (30) imply the equation $K_{1n} = c$. Since this is true for each orthonormal frame at p and p is arbitrary, we conclude that $\overline{M}$ has constant curvature equal to c.

6. We shall now prove Propositions 1 and 4. Let $M(r)$ be the geodesic sphere of Section 4. Let $E_1, \cdots, E_k$, and m be as in Proposition 1. Let $\alpha(t)$ be the geodesic $\alpha(t) = \exp_p tE_k$, and let $T = \dot{\alpha}(t)$. Consider the Jacobi fields $V_1, \cdots, V_{k-1}$ defined along $\alpha(t)$ by the initial conditions $V_i(0) = 0$ and $(D_T V_i)(0) = E_i$, where D is the covariant differentiation in $\overline{M}$. The Jacobi field V_i is induced by the geodesic variation

$$\phi(t, u) = \exp_p t(E_k \cos u + E_i \sin u) .$$

From this, it is not difficult to see that the vector fields $V_1, \cdots, V_{k-1}$ span the tangent space of $M(r)$ at $\alpha(r)$, for small $r > 0$. Furthermore, the mapping $M(\varepsilon) \to M(r)$ given by the rule $\exp_p \varepsilon X \to \exp_p rX$ maps $V_i(\varepsilon)$ to $V_i(r)$. Thus

$$\tilde{f}(r, m) = \frac{(\operatorname{Det} | \langle V_i(r), V_j(r) \rangle |)^{1/2}}{(\operatorname{Det} | \langle V_i(\varepsilon), V_j(\varepsilon) \rangle |)^{1/2}} . \tag{32}$$

LEMMA 2. *Let* $V_1, \cdots, V_{k-1}$ *be as above. Then*

$$\langle V_i, V_i \rangle = r^2 - \left(\frac{K_{ik}}{3}\right) r^4 + o(r^5) \tag{33}$$

and

$$\langle V_i, V_j \rangle = o(r^4) \quad \textit{if } i \neq j . \tag{34}$$

We can easily prove Lemma 2 by using the Jacobi equation $D_T^2 V_i = R(V_i, T)T$.

Lemma 2, equation (32), and the equation

$$(1+x)^{1/2} = 1 + x/2 + o(x^2) \tag{35}$$

immediately imply Proposition 1.

Alternatively, one may use the Jacobi equation to show that

$$V_i = E_i r + \sum_j \langle R(E_i(0), E_k(0))E_k(0), E_j(0) \rangle \, E_j \frac{r^3}{6} + o(r^4) , \tag{36}$$

where $E_1, \cdots, E_k$ have been extended to parallel vector fields along α. One may then evaluate $(\operatorname{Det} | \langle V_i, V_j \rangle |)^{1/2}$ by using the equation

$$V_1 \wedge \cdots \wedge V_{k-1} = (\mathrm{Det}\,|\langle V_i, V_j\rangle|)^{1/2} E_1 \wedge \cdots \wedge E_{k-1}. \tag{37}$$

Let $M(r)$ be the geodesic cylinder of radius r about γ, as in Section 5. Let $E_1, \cdots, E_n$, and m be as in Proposition 4. Let $\alpha(t)$ be the geodesic $\alpha(t) = \exp_p tE_n$, and let $T = \dot\alpha(t)$. Consider the Jacobi fields $V_1, \cdots, V_{n-1}$ defined along $\alpha(t)$ by the initial conditions $V_1(0) = E_1$, $(D_T V_1)(0) = 0$, $V_i(0) = 0$, and $(D_T V_i)(0) = E_i$ $(i \geq 2)$. The Jacobi field V_1 is induced by the geodesic variation

$$\phi(t, u) = \exp_{\gamma(u)} tE_n.$$

(Recall that E_n is parallel along γ.) As in the case of the geodesic sphere, we again obtain equation (32).

LEMMA 3. *Let* $V_1, \cdots, V_{n-1}$ *be as above. Then, for* $i, j > 1$ *and* $i \neq j$, *equations* (33) *and* (34) *are satisfied; also,*

$$\langle V_1, V_1\rangle = 1 - K_{1n} r^2 + o(r^3)$$

and

$$\langle V_1, V_j\rangle = o(r^2) \quad \textit{if } j > 1.$$

We can easily prove Lemma 3 by using the Jacobi equation.

Lemma 3 and equations (32) and (35) imply Proposition 4.

Alternatively, one may use the Jacobi equation to show that

$$V_1 = E_1 + \sum_j \langle R(E_1(0), E_n(0))E_n(0), E_j(0)\rangle E_j \frac{r^2}{2} + o(r^3),$$

where $E_1, \cdots, E_n$ have been extended to parallel vector fields along α. If $i > 1$, then V_i satisfies (36). One then uses (37) with $k = n$.

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