STABILIZATION OF SELF-EQUIVALENCES OF THE PSEUDOPROJECTIVE SPACES

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1. INTRODUCTION

For a space X with basepoint, let $\mathcal{E}(X)$ denote the group of homotopy classes of homotopy equivalences of X into itself, the group operation being composition. We refer to $\mathcal{E}(X)$ as the self-equivalence group of X. The operation of suspending one homotopy equivalence to obtain another determines a sequence of homomorphisms

$$\mathcal{E}(X) \xrightarrow{\Sigma} \mathcal{E}(\Sigma X) \xrightarrow{\Sigma} \cdots \xrightarrow{\Sigma} \mathcal{E}(\Sigma^n X) \xrightarrow{\Sigma} \cdots$$

connecting the self-equivalence groups of the iterated suspensions of X. When X is a finite CW complex, this sequence stabilizes at some stage $\mathcal{E}(\Sigma^n X)$ $(0 \leq n \leq \dim X)$, in that it consists of isomorphisms thereafter.

We describe this stabilization process in the case where X is the pseudoprojective plane of order q, denoted by P_q^1. As a starting point we take P. Olum's description [6] of the rather rich structure of $\mathcal{E}(P_q^1)$. Let Γ_q denote the quotient of the integral polynomial ring $Z[x]$ modulo the ideal generated by $1 + x + \cdots + x^{q-1}$, and let E_q denote the group whose elements are the units of Γ_q and whose multiplication $\circ$ is defined by the formula

$$\left\{\sum n_i x^i\right\} \circ \left\{\sum m_i x^i\right\} = \left\{\sum n_i x^i\right\}\left\{\sum m_i x^{is}\right\},$$

where $s = \sum n_i \pmod q$ is called the augmentation of $\left\{\sum n_i x^i\right\}$.

THEOREM 1 ([6, Theorems 3.4 and 3.5, and Remark 3.6]). *The self-equivalence group $\mathcal{E}(P_q^1)$ of the pseudoprojective plane P_q^1 is isomorphic to the group E_q. Moreover, E_q is isomorphic to the semidirect product $U_q^1 \times_\theta Z_q^*$ of the group U_q^1 (of units of Γ_q of augmentation 1) and the multiplicative group Z_q^* (of reduced residues modulo q) whose operators $\theta: Z_q^* \to \mathrm{Aut}\, U_q^1$ are given by the relation $\theta(s)\left(\left\{\sum n_i x^i\right\}\right) = \left\{\sum n_i x^{is}\right\}$.*

Since the pseudoprojective plane P_q^1 admits a two-dimensional cellular decomposition, namely, $S^1 \cup_q e^2$, the stabilization process takes at most two steps; hence the relevant suspensions are the pseudoprojective spaces $P_q^2 = S^2 \cup_q e^3$ and $P_q^3 = S^3 \cup_q e^4$. Our description of the stabilization process is summarized by the following two theorems.

THEOREM 2. *The self-equivalence group $\mathcal{E}(P_q^2)$ is isomorphic to the semidirect product $Z_q \times_\phi Z_q^*$ of the cyclic group Z_q of order q and the group Z_q^* whose operators $\phi: Z_q^* \to \mathrm{Aut}\, Z_q$ are given by the canonical isomorphism*

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$$\phi(s)(t \pmod q) = st \pmod q.$$

The self-equivalence group $\mathcal{E}(P_q^3)$ *is isomorphic to the direct product* $Z_{(2,q)} \times Z_q^*$ *of the cyclic group* $Z_{(2,q)}$ *of order* $(2, q)$ *with the group* Z_q^*.

THEOREM 3. *The first suspension homomorphism* $\Sigma\colon \mathcal{E}(P_q^1) \to \mathcal{E}(P_q^2)$ *can be identified with the product homomorphism*

$$U_q^1 \times_\theta Z_q^* \to Z_q \times_\phi Z_q^*$$

of the trivial homomorphism $0\colon U_q^1 \to Z_q$ *and the identity homomorphism* $1\colon Z_q^* \to Z_q^*$, *while the second suspension homomorphism* $\Sigma\colon \mathcal{E}(P_q^2) \to \mathcal{E}(P_q^3)$ *can be identified with the product homomorphism*

$$Z_q \times_\phi Z_q^* \to Z_{(2,q)} \times Z_q^*$$

of the unique epimorphism $Z_q \to Z_{(2,q)}$ *and the identity homomorphism* $1\colon Z_q^* \to Z_q^*$.

In each of the identifications of Theorems 1 and 2, the factor $Z_q^* \approx \operatorname{Aut} Z_q$ of a self-equivalence reflects the automorphism it induces on the first nonvanishing homotopy group. Thus each automorphism can be realized by some self-equivalence, and the complementary factors U_q^1, Z_q, and $Z_{(2,q)}$ measure the possible variations, the dwindling of which constitutes the stabilization process.

The process under consideration is intimately connected with the more basic stabilization process

$$\pi_2(P_q^1) \xrightarrow{\Sigma} \pi_3(P_q^2) \xrightarrow{\Sigma} \pi_4(P_q^3)$$

involving the suspension homomorphisms connecting the second nontrivial homotopy groups of the pseudoprojective spaces. Section 2 contains an analysis of the latter stabilization process. Section 3 presents a description of the second suspension homomorphism $\Sigma\colon \mathcal{E}(P_q^2) \to \mathcal{E}(P_q^3)$. Section 4 begins with a formulation of the isomorphism $\mathcal{E}(P_q^1) \approx E_q$ of Theorem 1 in terms compatible with the description in Section 3 of $\mathcal{E}(P_q^2)$, and continues with the calculation of the first suspension homomorphism $\Sigma\colon \mathcal{E}(P_q^1) \to \mathcal{E}(P_q^2)$. Section 5 presents a self-contained proof of P. Olum's isomorphism $\mathcal{E}(P_q^1) \approx E_q$ quoted in Theorem 1.

2. STABILIZATION OF THE HOMOTOPY GROUPS

Since the pseudoprojective plane P_q^1 is the two-dimensional space $S^1 \cup_q e^2$ obtained from the 1-sphere S^1 by attaching a 2-cell by a map of degree q, its fundamental group $\pi_1 = \pi_1(P_q^1)$ is cyclic of order q and is generated by some element a. If $C\colon (B^2, S^1) \to (P_q^1, S^1)$ is the characteristic map for the 2-cell, then the images $\rho^i[C]$ of the homotopy class $[C]$ in $\pi_2(P_q^1, S^1)$ under the action of ρ^i in $\pi_1(S^1)$ satisfy the relation $\rho^i[C] = \rho^j[C]$ if and only if $i \equiv j \pmod q$, and they determine a basis

$$[C], \rho[C], \cdots, \rho^{q-1}[C]$$

for $\pi_2(P_q^1, S^1)$. Thus we can identify $\pi_2(P_q^1, S^1)$ with the integral group ring $Z[\pi_1]$ of the fundamental group π_1, by mapping $\rho^i[C]$ to a^i. The exact homotopy sequence for the pair (P_q^1, S^1) shows that $\pi_2(P_q^1)$ can be identified with the ideal in $Z[\pi_1]$ generated by $\alpha = 1 - a$, so that as an abelian group, $\pi_2(P_q^1)$ is free of rank $q - 1$, and as a π_1-module, $\pi_2(P_q^1)$ has a single generator α, subject solely to the relation $(1 + a + \cdots + a^{q-1})\alpha = 0$.

The cell structure for P_q^1 that consists of a 0-cell e^0, a 1-cell e^1, and a 2-cell e^2 determines a cell structure for its reduced product $J(P_q^1)$ (see [3]), in which the 2-skeleton consists of P_q^1 together with a new 2-cell $e^1 \square e^1$ whose attaching map is inessential. Hence $J(P_q^1)^2 \cong P_q^1 \vee S^2$, and as a π_1-module, the second homotopy group of $P_q^1 \vee S^2$ has two generators, α coming from P_q^1 and β coming from S^2. Passage to the 3-skeleton of the reduced product $J(P_q^1)$ introduces three 3-cells $e^1 \square e^1 \square e^1$, $e^2 \square e^1$, and $e^1 \square e^2$ whose attaching maps represent the elements $a\beta - \beta$, $\alpha + \left(\sum a^i\right)\beta$, and $\alpha - \left(\sum a^i\right)\beta$, respectively, in $\pi_2(P_q^1 \vee S^2)$.

Thus, α and β are π_1-generators of $\pi_2(J(P_q^1))$, and they are subject solely to the relations

$$a\beta = \beta, \quad \alpha = -\left(\sum a^i\right)\beta, \quad \alpha = \left(\sum a^i\right)\beta, \quad \left(\sum a^i\right)\alpha = 0 .$$

It follows from the first three relations that the action of π_1 on $\pi_2(J(P_q^1))$ is trivial, that $\alpha = q\beta$, and that $2q\beta = 0$; therefore the fourth relation gives $q^2\beta = 0$. Thus, β serves as a single Z-generator of $\pi_2(J(P_q^1))$ of order $(2q, q^2)$.

Since the suspension homomorphism $\Sigma\colon \pi_n(X) \to \pi_{n+1}(\Sigma X)$ can be identified with the homomorphism $j_\#\colon \pi_n(X) \to \pi_n(J(X))$ induced by the inclusion $j\colon X \to J(X)$ of a connected finite CW complex into its reduced product, the previous calculations imply the following:

LEMMA 1. *The group* $\pi_3(P_q^2)$ *is cyclic of order* $(2q, q^2)$. *Moreover, the suspension homomorphism* $\Sigma\colon \pi_2(P_q^1) \to \pi_3(P_q^2)$ *sends the* π_1*-generator of* $\pi_2(P_q^1)$ *to* q *times a generator of* $\pi_3(P_q^2)$.

It follows directly from the exact homotopy sequence of the pair (P_q^2, S^2) that one generator of $\pi_3(P_q^2)$ is the Hopf map $S^3 \to S^2$ composed with the inclusion map $S^2 \to P_q^2$. We call this the Hopf map generator, and we denote it by $h\colon S^3 \to P_q^2$.

We now apply the same device to P_q^2 with its cell structure consisting of a 0-cell e^0, a 2-cell e^2, and a 3-cell e^3. The 3-skeleton of its reduced product $J(P_q^2)$ is merely P_q^2, and the 4-skeleton contains a single 4-cell $e^2 \square e^2$ that attaches to $S^2 \subset P_q^2$ via the Whitehead product $[j, j] \in \pi_3(P_q^2)$ of the inclusion map $j\colon S^2 \to P_q^2$ with itself. This product $[j, j]$ is known to be exactly twice the Hopf map generator of $\pi_3(P_q^2)$ [1, Chapter VI, Theorem 2.15]. Thus we have the following result:

LEMMA 2. *The group* $\pi_4(P_q^3)$ *is cyclic of order* $(2, q)$, *and the suspension homomorphism* $\Sigma\colon \pi_3(P_q^2) \to \pi_4(P_q^3)$ *is the unique epimorphism from the cyclic group of order* $(2q, q^2)$ *to the cyclic group of order* $(2, q)$.

3. THE SUSPENSION HOMOMORPHISM $\Sigma: \mathcal{E}(P_q^2) \to \mathcal{E}(P_q^3)$

We actually describe the suspension homomorphism $\Sigma: \mathcal{E}(M) \to \mathcal{E}(\Sigma M)$ for a Moore space $M = M(G, n)$ (see [5]) whose single nonvanishing homology group G occurs in dimension $n \geq 2$. Given the abelian group G as a quotient F/R of a free abelian group F modulo a subgroup R, we can construct the Moore space M up to homotopy type from a wedge N of n-spheres by attaching $(n+1)$-cells so that the boundary homomorphism $\partial: \pi_{n+1}(M, N) \to \pi_n(N)$ coincides with the inclusion homomorphism $i: R \to F$. Equivalently, M has the homotopy type of the mapping cone of a map $g: L \to N$ between wedges of n-spheres for which the induced homomorphism $g_\#: \pi_n(L) \to \pi_n(N)$ may be identified with the inclusion homomorphism $i: R \to F$. From this it follows that the exact sequence determined by the Puppe sequence of the map g takes the form indicated in the diagram

$$
(1)\quad
\begin{array}{ccccccccc}
\mathrm{Hom}(F, \pi_{n+1}(X)) & \xrightarrow{i^*} & \mathrm{Hom}(R, \pi_{n+1}(X)) & & & & \mathrm{Hom}(F, \pi_n(X)) & \xrightarrow{i^*} & \mathrm{Hom}(R, \pi_n(X)) \\
\| & & \| & & & & \| & & \| \\
[\Sigma N, X] & \xrightarrow{\Sigma g^\#} & [\Sigma L, X] & \xrightarrow{k^\#} & [M, X] & \xrightarrow{j^\#} & [N, X] & \xrightarrow{g^\#} & [L, X]
\end{array}
$$

Since $n \geq 2$, the entire Puppe sequence determined by the map g can be desuspended, and hence the exact sequence (1) involves additive groups and homomorphisms. By localizing the exactness of (1) at $[M, X]$, we obtain the well-known Universal Coefficient Sequence for homotopy groups with coefficients in G ([2, p. 30]):

$$
(2)\quad
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{coker}\ \Sigma g^\# & \xrightarrow{\omega} & [M, X] & \xrightarrow{\#} & \ker g^\# & \longrightarrow & 0 \\
 & & \| & & & & \| & & \\
 & & \mathrm{Ext}(G, \pi_{n+1}(X)) & & & & \mathrm{Hom}(G, \pi_n(X)) & &
\end{array}
$$

Here the homomorphism ω is induced by the quotient map $k: M \to M/N = \Sigma L$, and it sends the coset $\{\alpha\} \in \mathrm{coker}\ \Sigma g^\#$ associated with $\alpha \in [\Sigma L, X]$ to $\alpha \circ k \in [M, X]$, while the homomorphism $\#$ is induced by the inclusion map $j: N \to M$ and sends $f \in [M, X]$ to $f_\#: G = \pi_n(M) \to \pi_n(X)$.

For the special case in which $X = M$, the sequence (2) is more than just an exact sequence of additive groups and homomorphisms. When $n \geq 3$ $(n = 2)$, the multiplications induced by composition make the last two entries rings with unit (near-rings with unit, possibly lacking commutativity of addition and right distributivity of multiplication over addition), and they make the intervening homomorphism a (near-) ring homomorphism. It is precisely the units of this multiplication in $[M, M]$ in which we are interested, and therefore we note the following proposition.

LEMMA 3. *For each integer* $n \geq 2$, *the modified sequence*

$$
0 \longrightarrow \mathrm{Ext}(G, \pi_{n+1}(M(G, n))) \xrightarrow{\bar{\omega} = \omega + 1} [M(G, n), M(G, n)] \xrightarrow{\#} \mathrm{Hom}(G, G) \longrightarrow 1
$$

is a short exact sequence of multiplicative semigroups with unit, provided the first entry retains its original additive operation.

Proof. Since the function $\#$ sends the identity mapping to the identity homomorphism, it follows from the exactness of the original additive sequence (2) that an element in the multiplicative kernel of # differs from the identity mapping $1\colon M(G, n) \to M(G, n)$ by an element in the image of ω, and hence is itself in the image of $\overline{\omega} = \omega + 1$. It remains to prove that the injective function $\overline{\omega} = \omega + 1$ is a homomorphism from the additive structure of Ext to the multiplicative structure of $[M(G, n), M(G, n)]$.

Assume that $n \geq 3$. We see that for $\alpha, \beta \in [M/N, M]$,

$$(\omega(\{\alpha\}) + 1) \circ (\omega(\{\beta\}) + 1) = \omega(\{\alpha\}) \circ \omega(\{\beta\}) + \omega(\{\alpha\}) + \omega(\{\beta\}) + 1 .$$

Hence we need merely show that $\omega(\{\alpha\}) \circ \omega(\{\beta\}) = 0$. In fact,

$$\omega(\{\alpha\}) \circ \omega(\{\beta\}) = \alpha \circ k \circ \beta \circ k ,$$

and it is sufficient to show that $k \circ \beta$ is null-homotopic. Since

$$j_{\#}\colon \pi_{n+1}(N) \to \pi_{n+1}(M)$$

is surjective and M/N is a wedge of $(n + 1)$-spheres, the map

$$j_{\#}\colon [M/N, N] \to [M/N, M]$$

is also surjective. Therefore $\beta\colon M/N \to M$ admits a factorization $\beta = j \circ \beta'$. Since $k \circ j$ is null-homotopic, $k \circ j \circ \beta' = k \circ \beta$ is null-homotopic. In the case $n = 2$, the right-hand distributive law is not available, and we require a slightly more elaborate argument involving the comultiplication for the pair (M, N).

Since a map $M(G, n) \to M(G, n)$ $(n \geq 2)$ is a homotopy equivalence if and only if it induces an automorphism of $G = \pi_n(M(G, n))$, we have the following immediate conclusion from Lemma 3.

THEOREM 4. *The group* $\mathcal{E}(M(G, n))$ *of self-equivalences of a Moore space* $M(G, n)$ *associated with an abelian group* G *and an integer* $n \geq 2$ *is a group extension*

$$0 \to \operatorname{Ext}(G, \pi_{n+1}(M(G, n))) \xrightarrow{\overline{\omega}} \mathcal{E}(M(G, n)) \xrightarrow{\#} \operatorname{Aut} G \to 1$$

of Ext *by the automorphism group* Aut G *of* G.

Because $M(G, n)$ is $(n - 1)$-connected and admits an $(n + 1)$-dimensional cell structure, the suspension homomorphism $\Sigma\colon \mathcal{E}(M(G, n)) \to \mathcal{E}(M(G, n + 1))$ is an isomorphism if $n \geq 3$, and is an epimorphism if $n = 2$. One easily notes from the construction of the extensions in Theorem 4 that the suspension process induces a morphism $(\operatorname{Ext}(1, \Sigma), \Sigma, 1)$ from the nth extension to the $(n + 1)$st extension. Thus, the following situation for the case $n = 2$ obtains.

COROLLARY 1. *The suspension homomorphism* $\Sigma\colon \mathcal{E}(M(G, 2)) \to \mathcal{E}(M(G, 3))$ *is an epimorphism whose kernel is exactly that of the homomorphism*

$$\operatorname{Ext}(1, \Sigma)\colon \operatorname{Ext}(G, \pi_3(M(G, 2))) \to \operatorname{Ext}(G, \pi_4(M(G, 3))) .$$

In the case where G is the cyclic group Z_q, we have available the calculations of the previous section. We find that

$$\operatorname{Ext}(Z_q, \pi_3(P_q^2)) \approx Z_q \quad \text{and} \quad \operatorname{Ext}(Z_q, \pi_4(P_q^3)) \approx Z_{(2,q)} .$$

Moreover, under these identifications $\mathrm{Ext}(1, \Sigma)$ corresponds to the unique epimorphism $Z_q \to Z_{(2,q)}$. From Corollary 1, we deduce that the kernel of the suspension homomorphism $\Sigma: \mathcal{E}(P_q^2) \to \mathcal{E}(P_q^3)$ is cyclic of order $q/(2, q)$. A generator of this kernel is best described in terms of Puppe action, which we now consider.

If M is the mapping cone associated with a map $g: L \to N$, there is a co-operation $c: M \to \Sigma L \vee M$. This map, which simply collapses the mid-belt of the cone in M, determines an action of $\alpha \in [\Sigma L, M]$ on $f \in [M, M]$, namely,

$$f^\alpha = \nabla \circ \alpha \vee f \circ c: M \to \Sigma L \vee M \to M \vee M \to M.$$

If $g: L \to N$ can be desuspended, then the mapping cone M inherits a suspension comultiplication $\mu: M \to M \vee M$ for which $c \simeq k \vee 1 \circ \mu: M \to \Sigma L \vee M$, where $k: M \to M/N = \Sigma L$ is the indicated quotient map. In this case, $f^\alpha \simeq \alpha \circ k + f$.

It follows that for $n \geq 2$, the homomorphism

$$\overline{\omega}: \mathrm{Ext}(Z_q, \pi_{n+1}(P_q^n)) \to \mathcal{E}(P_q^n)$$

is given by the correspondence $\{\alpha\} \to 1^\alpha$ for $\alpha: S^{n+1} \to P_q^n$. Since one generator of the cyclic group $\mathrm{Ext}(Z_q, \pi_3(P_q^2))$ is the coset of the Hopf map $h: S^3 \to P_q^2$, a generator of the kernel of the suspension homomorphism $\Sigma: \mathcal{E}(P_q^2) \to \mathcal{E}(P_q^3)$ is given by 1^h when q is odd, and by $1^{2h} = 1^h \circ 1^h$ when q is even.

A similar description of the suspension homomorphism

$$\Sigma: \mathcal{E}(M(G, 2)) \to \mathcal{E}(M(G, 3))$$

for a noncyclic group G requires knowledge of the crucial homomorphism $\Sigma: \pi_3(M(G, 2)) \to \pi_4(M(G, 3))$. Undoubtedly, this information could be obtained for finitely generated abelian groups G by means of I. M. James's reduced-product construction, which we used in the previous section. We are content to mention that $\pi_4(M(G, 3)) \approx Z_2 \otimes G$ ([1, Chapter VIII, Theorem 2.4]), so that we can draw the following conclusion from Theorem 4.

COROLLARY 2. *For a finitely generated abelian group* G, *each automorphism of* G *admits exactly* 2^{n^2} *distinct realizations by a self-equivalence* $M(G, 3) \to M(G, 3)$, *where* n *is the length of the* 2-*primary component of* G. *In particular,* $\#: \mathcal{E}(M(G, 3)) \approx \mathrm{Aut}\, G$ *if and only if* G *has no* 2-*torsion.*

4. THE SUSPENSION HOMOMORPHISM $\Sigma: \mathcal{E}(P_q^1) \to \mathcal{E}(P_q^2)$

To take advantage of Olum's isomorphism $\mathcal{E}(P_q^1) \approx E_q$, we first formulate it in terms compatible with our previous description of $\mathcal{E}(P_q^2)$. For this we need the following data.

In Section 2 we saw that the π_1-module $\pi_2(P_q^1)$ can be identified with the ideal in $Z[\pi_1]$ generated by $1 - a$. Moreover, if the ideal of $Z[\pi_1]$ generated by $1 + a + \cdots + a^{q-1}$ is denoted by I, and if the quotient ring $Z[\pi_1]/I$ is denoted by Γ_q, then $\pi_2(P_q^1)$ can be further identified with the ideal in Γ_q generated by the coset $\{1 - a\}$. This ideal consists of the cosets $\left\{\sum n_i a^i\right\}$ of Γ_q for which $\sum n_i \equiv 0 \pmod q$. Therefore there is an exact sequence

$$0 \to \pi_2(P_q^1) \to \Gamma_q \xrightarrow{A} Z_q \to 0,$$

where the augmentation ring homomorphism A is given by

$$A\left(\left\{\sum n_i a^i\right\}\right) = \sum n_i \pmod q.$$

We have the following important consequence:

(3) Each element γ of the group U_q of units of Γ_q can be written uniquely in the form $\{s+\alpha\}$, where s belongs to the group Z_q^* of units of Z_q, and α belongs to $\pi_2(P_q^1)$. In fact, $s = A(\gamma)$ and $\alpha = \gamma - A(\gamma)$.

A second fact we need is that the Puppe action of $\alpha \in [S^2, P_q^1]$ on $f \in [P_q^1, P_q^1]$ has the following crucial property ([2, Theorem 15.4]):

(4) Two maps f, g: $P_q^1 \to P_q^1$ induce the same homomorphism $f_\# = g_\#: \pi_1 \to \pi_1$ if and only if there exists an $\alpha \in [S^2, P_q^1]$ with $f^\alpha \cong g$.

Finally, for each integer s with $0 < s < q$ and $(s, q) = 1$, the map of the unit disc determined by the correspondence $(r, \theta) \to (r, s\theta)$ passes under the identifications $(1, \theta) \equiv (1, \theta + 2\pi/q)$ to give a map $f_s: P_q^1 \to P_q^1$ with $f_{s\#}(a) = a^s$ on π_1. We give the suspensions of these maps the same labels.

Within this framework we now formulate the isomorphism between $\mathcal{E}(P_q^1)$ and E_q, considered as the group of units of the ring Γ_q with the modified multiplication $\gamma \circ \gamma' = \gamma\theta_s(\gamma')$ for $s = A(\gamma)$, where $\theta_s\left(\left\{\sum n_i a^i\right\}\right) = \left\{\sum n_i a^{is}\right\}$.

THEOREM 5. *The correspondence of* $\gamma = \{s+\alpha\} \in E_q$ *with* $f_s^\alpha: P_q^1 \to P_q^1$ *determines an isomorphism* $V: E_q \to \mathcal{E}(P_q^1)$. *Moreover,* V *is compatible with the homomorphisms*

$$A: E_q \to Z_q^* \quad \text{and} \quad \#: \mathcal{E}(P_q^1) \to \operatorname{Aut} Z_q$$

and the canonical isomorphism $Z_q^* \approx \operatorname{Aut} Z_q$.

To save the reader the effort of translating this theorem of [6] into the current notation, we give a self-contained proof in the next section. Olum points out that the two multiplications on the group U_q of units of Γ_q coincide on the subgroup U_q^1 of units of augmentation 1, so that U_q^1 may be regarded as a subgroup of E_q. Moreover, Olum gives the following description of E_q.

THEOREM 6. *The group* E_q *is a split extension*

$$1 \to U_q^1 \xrightarrow{i} E_q \xrightarrow{A} Z_q^* \to 1,$$

where i *and* A *denote injection and augmentation, and where the result of the operation of* $s \in Z_q^*$ *on* $u \in U_q^1$ *is* $\theta_s(u)$. *A splitting is provided by the mapping* $B: Z_q^* \to E_q$ *defined by* $B(s) = 1 + a + \cdots + a^{s-1}$.

Notice that the suspension homomorphism $\Sigma: \mathcal{E}(P_q^1) \to \mathcal{E}(P_q^2)$ determines a morphism

$$\begin{array}{ccccccccc}
1 & \longrightarrow & U^1_q & \xrightarrow{\ i\ } & E_q & \xrightarrow{\ A\ } & Z^*_q & \longrightarrow & 1 \\
 & & \Big\downarrow & & \Big\downarrow V & & \Big\downarrow \approx & & \\
 & & \Big\downarrow & & \mathcal{E}(P^1_q) & \xrightarrow{\ \#\ } & \mathrm{Aut}\, Z_q & & \\
 & & \Big\downarrow & & \Big\downarrow \Sigma & & \Big\downarrow = & & \\
0 & \longrightarrow & Z_q & \xrightarrow{\ \overline{\omega}\ } & \mathcal{E}(P^2_q) & \xrightarrow{\ \#\ } & \mathrm{Aut}\, Z_q & \longrightarrow & 1\,.
\end{array}$$

To show the triviality of the homomorphism $U^1_q \to Z_q$, consider in U^1_q an arbitrary element $\gamma = \{1 + \alpha\}$ as in (3). We see that

$$\Sigma(V(\gamma)) = \Sigma(1^{\alpha}) = 1^{\Sigma\alpha} = \overline{\omega}(\{\Sigma\alpha\}),$$

where $\{\Sigma\alpha\} \in \pi_3(P^2_q)/q\pi_3(P^2_q) = \mathrm{Ext}(Z_q, \pi_3(P^2_q))$. Since Σ maps $\pi_2(P^1_q)$ into $q\pi_3(P^2_q)$, by Lemma 1, it follows that $\{\Sigma\alpha\} = 0$. Therefore $\Sigma(V(\gamma)) = 0$ for $\gamma \in U^1_q$.

Note that since U^1_q is the kernel of ΣV, two self-equivalences of P^1_q have the same suspension if and only if they induce the same automorphism of π_1.

The splitting $B\colon Z^*_q \to E_q$ given in Theorem 6 provides splittings

$$B^n\colon Z^*_q \to \mathcal{E}(P^n_q) \qquad (n = 1, 2, 3)$$

defined by the relations

$$B^1(s) = V(B(s)) = f_s^{B(s)-s}, \qquad B^2(s) = \Sigma(B^1(s)) = f_s, \qquad B^3(s) = \Sigma(B^2(s)) = f_s.$$

By Theorem 6, the group extension

$$1 \to U^1_q \to \mathcal{E}(P^1_q) \xrightarrow{\#} Z^*_q \to 1$$

for $\mathcal{E}(P^1_q)$ has operators $\theta\colon Z^*_q \to \mathrm{Aut}\, U^1_q$, given by the relation

$$\theta(s)\left(\left\{\sum n_i a^i\right\}\right) = \left\{\sum n_i a^{is}\right\}.$$

Hence, the splitting $B^1\colon Z^*_q \to \mathcal{E}(P^1_q)$ provides an isomorphism of $\mathcal{E}(P^1_q)$ with the semidirect product $U^1_q \times_\theta Z_q$.

The splitting $B^2\colon Z^*_q \to \mathcal{E}(P^2_q)$ yields an isomorphism of $\mathcal{E}(P^2_q)$ with the semidirect product $Z_q \times_\phi Z_q$, where $\phi\colon Z^*_q \to \mathrm{Aut}\, Z_q$ denotes the operators in the group extension

$$0 \to Z_q \xrightarrow{\overline{\omega}} \mathcal{E}(P^2_q) \xrightarrow{\#} Z^*_q \to 1.$$

To verify that the homomorphism $\phi\colon Z^*_q \to \mathrm{Aut}\, Z_q$ is the canonical isomorphism, we observe that the definition of operators ([4, p. 108]) implies the relation

$$\overline{\omega}(\phi(s)\{\alpha\}) = f_s \circ \overline{\omega}(\{\alpha\}) \circ f_s^{-1}$$

for $\{\alpha\} \in \pi_3(P^2_q)/q\pi_3(P^2_q) \approx Z_q$. Moreover, the triple composite on the right can be written as $f_s \circ \alpha \circ k \circ f_s^{-1} + 1$, since $\overline{\omega}(\{\alpha\}) = \alpha \circ k + 1$, and both distributive laws hold for suspended maps. Now, calculations employing the Hopf invariant show that if the maps $d_s\colon S^3 \to S^3$ and $e_s\colon S^2 \to S^2$ are of degree s, and if $\alpha \in \pi_3(S^2)$, then $e_s \circ \alpha \simeq s\alpha \circ d_s$. Consequently, for $\alpha \in \pi_3(P_q^2)$, we have the relation $f_s \circ \alpha \circ k \simeq s\alpha \circ k \circ f_s$, or equivalently, $f_s \circ \alpha \circ k \circ f_s^{-1} \simeq s\alpha \circ k$. Thus, $\overline{\omega}(\phi(s)\{\alpha\}) = s\alpha \circ k + 1 = \overline{\omega}(s\{\alpha\})$; hence, $\phi(s)\{\alpha\} = s\{\alpha\}$, as we claimed.

Finally, since the automorphism group of $Z_{(2,q)}$ is trivial, the group extension

$$0 \to Z_{(2,q)} \overset{\overline{\omega}}{\to} \mathcal{E}(P_q^2) \overset{\#}{\to} Z_q^* \to 1$$

for $\mathcal{E}(P_q^3)$ has trivial operators, and therefore the splitting $B^3\colon Z_q^* \to \mathcal{E}(P_q^3)$ provides an isomorphism of $\mathcal{E}(P_q^3)$ with the direct product $Z_{(2,q)} \times Z_q$.

Since the suspension morphisms relating the three group extensions are compatible with the three splittings, it follows that the suspension homomorphisms

$$\Sigma\colon \mathcal{E}(P_q^1) \to \mathcal{E}(P_q^2) \quad \text{and} \quad \Sigma\colon \mathcal{E}(P_q^2) \to \mathcal{E}(P_q^3)$$

can be described in terms of the (semi)direct products, as in Theorem 3.

5. A PROOF OF THEOREM 5

Throughout this section, we consider only maps $f\colon P_q^1 \to P_q^1$ for which $f_\#\colon \pi_1 \to \pi_1$ is an automorphism, say, $f_\#(a) = a^s$ with $(s, q) = 1$. If

$$N\colon (P_q^1, S^1) \to (P_q^1, S^1)$$

is a representation of $f\colon P_q^1 \to P_q^1$, then the diagram

$$\text{(5)} \qquad \begin{array}{ccccccccccc} 0 & \longrightarrow & \pi_2(P_q^1) & \longrightarrow & \pi_2(P_q^1, S^1) & \overset{\partial}{\longrightarrow} & \pi_1(S^1) & \longrightarrow & \pi_1(P_q^1) & \longrightarrow & 1 \\ & & \downarrow f_\# & & \downarrow N_\# & & \downarrow N_\# & & \downarrow f_\# & & \\ 0 & \longrightarrow & \pi_2(P_q^1) & \longrightarrow & \pi_2(P_q^1, S^1) & \overset{\partial}{\longrightarrow} & \pi_1(S^1) & \longrightarrow & \pi_1(P_q^1) & \longrightarrow & 1 \end{array}$$

is a commutative ladder of homomorphisms with the last three rungs given by, say

$$N_\#(1) = \sum n_i a^i, \qquad N_\#(\rho) = \rho^t, \qquad f_\#(a) = a^s,$$

respectively. One can easily verify the relations

$$\text{(6)} \qquad t \equiv s \pmod q, \qquad t = \sum n_i, \qquad N_\#(a^j) = a^{js} N_\#(1).$$

The ideal I in $Z[\pi_1]$ generated by the element $1 + a + \cdots + a^{q-1}$ consists of the elements $\sum m_i a^i$ with equal coefficients $m_0 = m_1 = \cdots = m_{q-1}$. Since the element $N_\#\left(\sum a^i\right) = \sum a^{is} N_\#(1)$ belongs to the ideal I as $(s, q) = 1$, the homomorphism $N_\#\colon Z[\pi_1] \to Z[\pi_1]$ passes to the quotient ring $\Gamma_q = Z[\pi_1]/I$ to determine the middle rung of the short commutative ladder

$$
(7)\qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_2(P^1_q) & \longrightarrow & \Gamma_q & \xrightarrow{A} & Z_q & \longrightarrow & 0 \\
 & & \downarrow f_\# & & \downarrow f_* & & \downarrow f_\# & & \\
0 & \longrightarrow & \pi_2(P^1_q) & \longrightarrow & \Gamma_q & \xrightarrow{A} & Z_q & \longrightarrow & 0\,,
\end{array}
$$

in which commutativity is guaranteed by (5) and (6).

If $M: (P^1_q, S^1) \to (P^1_q, S^1)$ is another representation of $f: P^1_q \to P^1_q$, then the endomorphisms $N_\#$ and $M_\#$ of $\pi_2(P^1_q, S^1)$ need coincide only on the subgroup $\pi_2(P^1_q)$. Nevertheless, they determine the same homomorphism $f_*: \Gamma_q \to \Gamma_q$, since the elements $N_\#(1)$ and $M_\#(1)$ of $Z[\pi_1]$ determine the same coset in Γ_q. To prove this, let $M_\#(1) = \sum m_i a^i$. Since $1 - a^j$ is in the image of the map

$$\pi_2(P^1_q) \to \pi_2(P^1_q, S^1),$$

we see that $M_\#(1 - a^j) = N_\#(1 - a^j)$. Choose j so that $sj \equiv 1 \pmod q$. Then

$$
\begin{aligned}
(1 - a)\sum n_i a^i &= (1 - a^{js})N_\#(1) = N_\#(1 - a^j) = M_\#(1 - a^j) \\
&= (1 - a^{js})M_\#(1) = (1 - a)\sum m_i a^i .
\end{aligned}
$$

Hence $n_i - n_{i-1} = m_i - m_{i-1}$, that is, $n_i - m_i = n_{i-1} - m_{i-1}$ for $i \in Z_q$. Therefore $N_\#(1) - M_\#(1) = \sum (n_i - m_i)a^i$ belongs to the ideal I.

For a map $f: P^1_q \to P^1_q$ inducing an automorphism on π_1, we define the invariant $\gamma = f_*(1) \in \Gamma_q$, which we assert has the following properties:

(8) $A(\gamma_f) = s$ if $f_\#(a) = a^s$ on π_1.

(9) $f: P^1_q \to P^1_q$ is a homotopy equivalence if and only if the invariant $\gamma_f \in \Gamma_q$ is a unit.

(10) If $(s, q) = 1$ and $\alpha \in \pi_2(P^1_q)$, then the map $f^\alpha_s: P^1_q \to P^1_q$ has invariant $\{s + \alpha\} \in \Gamma_q$. Therefore, for each $\gamma \in \Gamma_q$ with $(A(\gamma), q) = 1$, there is a map $f: P^1_q \to P^1_q$ with invariant $\gamma_f = \gamma$.

(11) $f \cong g: P^1_q \to P^1_q$ if and only if $\gamma_f = \gamma_g$.

(12) $\gamma_{fg} = \gamma_f\, \theta_s(\gamma_g)$, where $s = A(\gamma_f)$ and $\theta_s\left(\left\{\sum m_i a^i\right\}\right) = \left\{\sum m_i a^{is}\right\}$.

It follows easily from these properties that the mapping $W: \mathcal{E}(P^1_q) \to E_q$, given by $W(f) = \gamma_f$, and the mapping $V: E_q \to \mathcal{E}(P^1_q)$, given by $V(\{s + \alpha\}) = f^\alpha_s$, are inverse functions preserving the group structures. We complete the proof of Theorem 5 by establishing properties (8) to (12).

Property (8) follows directly from (6). Property (9) requires use of the short commutative ladder (7), which shows that a map $f: P^1_q \to P^1_q$ induces isomorphisms of the homotopy groups if and only if the invariant $\gamma_f \in \Gamma_q$ is a unit. It is clear from the geometry of the construction of the Puppe action that the map $f^\alpha_s: P^1_q \to P^1_q$

is a map of pairs for which $f^{\alpha}_{s\#}: \pi_2(P^1_q, S^1) \to \pi_2(P^1_q, S^1)$ has value

$$f^{\alpha}_{s\#}(1) = f_{s\#}(1) + \alpha = s + \alpha \in Z[\pi_1].$$

This proves (10).

The implication $f \simeq g \Rightarrow \gamma_f = \gamma_g$ follows from the definition of the invariant γ. To prove the converse half of (11), suppose that $\gamma_f = \gamma_g$, with $A(\gamma_f) = s = A(\gamma_g)$. Then the maps f, g, and f_s induce the same automorphism of π_1, so that the crucial property (4) of the Puppe action provides two elements $\alpha, \beta \in \pi_2(P^1_q)$ for which $f \simeq f^{\alpha}_s$ and $g \simeq f^{\beta}_s$. From (10) and the first half of (11), we deduce that $\{s + \alpha\} = \{s + \beta\}$ in Γ_q; hence, $\alpha = \beta$ in $\pi_2(P^1_q)$. Therefore $f \simeq f^{\alpha}_s \simeq f^{\beta}_s \simeq g$.

Finally, let f, g: $P^1_q \to P^1_q$ be represented by M, N: $(P^1_q, S^1) \to (P^1_q, S^1)$, respectively. Let $f_{\#}(a) = a^s$. Using (6), we find that

$$N_{\#}(M_{\#}(1)) = N_{\#}\left(\sum m_i a^i\right) = \sum m_i N_{\#}(a^i) = \sum m_i a^{is} N_{\#}(1) = N_{\#}(1)\, \theta_s(M_{\#}(1)) .$$

This establishes (12).

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