APPROXIMATIONS OF DOUBLY SUBSTOCHASTIC OPERATORS

C. W. Kim

1. INTRODUCTION AND PRELIMINARIES

We prove three approximation theorems for positive contractions T on $L_1[0, \infty)$ having at least one of the two properties $T1 = 1$ and $T^*1 = 1$. The main results concern approximations of such operators by convex combinations of those operators induced by invertible measure-preserving maps on $[0, \infty)$ in well-known operator topologies.

Let X, $\mathfrak{F}$, and μ denote the nonnegative real half-line, the class of Lebesgue measurable sets, and Lebesgue measure. On X, we shall consider only $\mathfrak{F}$-measurable real functions (modulo μ-equivalence), and by a set on X we shall always mean an element of $\mathfrak{F}$. We shall omit the phrase "almost everywhere," it being understood wherever applicable. We assume $1 \leq p \leq \infty$. Let $L_p = L_p(X, \mathfrak{F}, \mu)$, and let $[L_p]$ be the Banach space of bounded linear operators from L_p into itself. We say that T is a *positive contraction* on L_p if $T \in [L_p]$, $Tf \geq 0$ for each f $(0 \leq f \in L_p)$, and $\|T\|_p \leq 1$.

For each positive contraction T on L_1, the adjoint T^* determined by the equation $\int_X (Tf) g \, d\mu = \int_X f T^* g \, d\mu$ for $f \in L_1$ and $g \in L_\infty$ is a positive contraction on L_∞. The operators T and T^* can be extended uniquely to positive linear operators on the cone of nonnegative numerical functions u and v as follows:

$$Tu = \lim_n Tf_n, \qquad \text{where } 0 \leq f_n \in L_1, \; f_n \uparrow u,$$

$$T^*v = \lim_n T^*g_n, \qquad \text{where } 0 \leq g_n \in L_\infty, \; g_n \uparrow v.$$

The extensions satisfy the equation $\int_X (Tu) v \, d\mu = \int_X u T^* v \, d\mu$. In particular, suppose $T1 \leq 1$. This condition is equivalent to the condition that

$$\int_X T^* g \, d\mu \leq \int_X g \, d\mu$$

for $0 \leq g \in L_1 \cap L_\infty$, and thus T^* is uniquely extended to a positive contraction on L_1. The extended operator will also be denoted by T^*. Observe that in this case T is also a positive contraction on L_∞. We shall always assume that T and T^* represent the extended operators.

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Let $\mathfrak{D}_s$ be the set of positive contractions T on L_1 such that $T1 \leq 1$. The elements of $\mathfrak{D}_s$ are called *doubly substochastic* (d.s.s.) operators. Note that T is d.s.s. if and only if T^* is d.s.s. By the Riesz convexity theorem, d.s.s. operators are also positive contractions on L_p $(1 < p < \infty)$. A d.s.s. operator T is called *weakly doubly stochastic* (w.d.s.) or *weak* doubly stochastic* (w*.d.s.) according as $T1 = 1$ or $T^*1 = 1$. Let $\mathfrak{D}_w$ and $\mathfrak{D}_w^*$ denote the set of w.d.s. operators and the set of w*.d.s. operators. A d.s.s. operator T such that $T1 = 1$ and $T^*1 = 1$ is called *doubly stochastic* (d.s.). If we denote by $\mathfrak{D}$ the set of d.s. operators, then $\mathfrak{D} = \mathfrak{D}_w \cap \mathfrak{D}_w^*$. It follows readily that the sets of operators mentioned above are convex semigroups under multiplication. In particular, $\mathfrak{D}_s$ and $\mathfrak{D}$ are self-adjoint, that is, $\mathfrak{D}_s = \mathfrak{D}_s^*$ and $\mathfrak{D} = \mathfrak{D}^*$. Let Φ be the class of measure-preserving maps ϕ from $(X, \mathfrak{F}, \mu)$ onto itself, and let Φ_1 be the class of maps $\phi \in \Phi$ that are invertible and measure-preserving. Then each $\phi \in \Phi$ gives rise to a d.s. operator T_ϕ that is defined by the equation $T_\phi f(x) = f(\phi(x))$. For brevity, we also write Φ for $\{T_\phi: \phi \in \Phi\}$ and Φ_1 for $\{T_\phi: \phi \in \Phi_1\}$.

The contraction operators defined above have valuable discrete analogs. The idea is to replace L_p by ℓ_p. Then it follows readily that there is a bijection between the set of d.s.s. operators on ℓ_1 and the set of d.s.s. matrices. Similar remarks apply to w.d.s. (w*.d.s., d.s.) operators on ℓ_1. By a d.s.s. matrix (t_{ij}) we mean an infinite matrix with nonnegative entries t_{ij} such that $\sum_j t_{ij} \leq 1$ for each i and $\sum_i t_{ij} \leq 1$ for each j. A d.s.s. matrix (t_{ij}) is called w.d.s. or w*.d.s. according as $\sum_j t_{ij} = 1$ for each i or $\sum_i t_{ij} = 1$ for each j. By a d.s. matrix we mean a d.s.s. matrix having the row sums and the column sums all equal to 1. A d.s. matrix in which there is exactly one entry 1 in each row and column is called a *permutation matrix.*

Since $\mathfrak{D}_s \subset \bigcap_{1 \leq p \leq \infty} [L_p]$, we may topologize $\mathfrak{D}_s$ by various operator topologies for $[L_p]$ $(1 \leq p \leq \infty)$. By the L_p-weak (strong, norm) operator topology for $\mathfrak{D}_s$, we mean the weak (strong, norm) operator topology for $[L_p]$ restricted to $\mathfrak{D}_s$. J. R. Brown [1, Theorem 4, p. 370] showed that $\mathfrak{D}_s$ is the closure of Φ_1 in the L_2-weak operator topology, and that it is the closed convex hull of Φ_1 in the L_2-strong operator topology.

In Section 2, we prove (Theorem 2.1) that $\mathfrak{D}_w^*$ is the closed convex hull of Φ_1 in the L_1-strong operator topology. It is also shown (Theorem 2.2) that $\mathfrak{D}$ is the closed convex hull of Φ_1 in the L_1-strong* operator topology, and that the convex hull of Φ_1 is a dense subset of $\mathfrak{D}$ in the L_2-strong* operator topology (see Section 2 for definitions of strong* operator topologies). The first part of Theorem 2.2 is a continuous analogue of a theorem of B. A. Rattray and J. E. L. Peck [7] as well as an analogue of a theorem of Peck [6] for d.s. operators. The latter part of the theorem is a sharper form of Brown's strong approximation for d.s. operators. In Section 3, we prove the L_2-norm approximation theorem for w.d.s. operators of Hilbert-Schmidt type. Our results hold also when the underlying space X is the real line.

2. STRONG APPROXIMATIONS

For each positive integer n, let $\mathfrak{F}_n$ be the σ-algebra generated by dyadic intervals $D_i^n = [(i-1)/2^n, i/2^n)$ $(i = 1, 2, \cdots)$, and let U_n be a conditional expectation operator defined by the equation $U_n f = E(f \mid \mathfrak{F}_n)$ for $f \in L_1 \cap L_\infty$, or equivalently by the equation

$$U_n f = \sum_{i=1}^{\infty} \left(2^n \int_{D_i^n} f\, d\mu \right) \chi(\,\cdot\,; D_i^n) \qquad \text{for } f \in L_1 \cap L_\infty .$$

We shall denote the indicator (characteristic) function of a set A by $\chi(\,\cdot\,; A)$. In particular, we shall write $e_i^n = \chi(\,\cdot\,; D_i^n)$. It is easy to see that $U_n \in \mathfrak{D}$, $U_n = U_n^*$, and $U_n U_m = U_m U_n = U_n$ if $n \leq m$. Observe that U_n is a projection on L_2. Denote the identity operator by I. It follows essentially from the martingale convergence theorem [2, Theorem 4.1, p. 319] that $U_n \to I$ as $n \to \infty$ in the L_p-strong operator topology (L_p-s.o.t.), where $1 \leq p < \infty$. Hence TU_n and $U_n T U_n$ converge to T in the L_p-strong operator topology $(1 \leq p < \infty)$.

Let n be a fixed positive integer. If we set $\mathfrak{H}_{n,p} = U_n(L_p)$ $(1 \leq p \leq \infty)$, that is, $\mathfrak{H}_{n,p} = \{U_n f\colon f \in L_p\}$, then $\mathfrak{H}_{n,p}$ is a closed subspace of L_p that is isomorphic to ℓ_p. Each $T \in \mathfrak{D}_s$ induces in a natural way a positive contraction $T_n\colon \mathfrak{H}_{n,2} \to \mathfrak{H}_{n,2}$ defined by the condition

$$T_n h = U_n T h \qquad \text{for } h \in \mathfrak{H}_{n,2} ,$$

or equivalently by the condition

$$T_n U_n f = U_n T U_n f \qquad \text{for } f \in L_2 .$$

Following P. R. Halmos [3, p. 118], we call T_n the *compression* of T to $\mathfrak{H}_{n,2}$, and T a *dilation* of T_n to L_2. Since $\mathfrak{H}_{n,1} \subset \mathfrak{H}_{n,2}$, we have also the relation $T_n U_n f = U_n T U_n f$ for $f \in L_1$. In terms of the basis $\{e_i^n\}_i$ for $\mathfrak{H}_{n,2}$, the compression T_n is uniquely represented by the d.s.s. matrix (t_{ij}^n) [where $t_{ij}^n = 2^n(Te_j^n, e_i^n)$] as follows:

$$T_n(U_n f) = \sum_i \left\{ \sum_j t_{ij}^n 2^n (f, e_j^n) \right\} e_i^n \qquad (f \in L_2) . \tag{2.1}$$

We write $(f, g) = \int_X fg\, d\mu$. It is a straightforward exercise to show that T_n^* is the compression of T^* to $\mathfrak{H}_{n,2}$ and

$$T_n^*(U_n f) = \sum_i \left\{ \sum_j t_{ji}^n 2^n (f, e_j^n) \right\} e_i^n \qquad (f \in L_2). \tag{2.2}$$

Clearly, (2.1) and (2.2) hold for each $f \in L_1$. On the other hand, for each d.s.s. matrix (s_{ij}), we define a positive contraction $S_n\colon \mathfrak{H}_{n,1} \to \mathfrak{H}_{n,1}$ by the condition

$$S_n(U_n f) = \sum_i \left\{ \sum_j s_{ij} 2^n (f, e_j^n) \right\} e_i^n \qquad (f \in L_1) .$$

By the Schur test [3, p. 23], S_n extends uniquely to a positive contraction on $\mathfrak{H}_{n,2}$ (denoted also by S_n) so that

$$S_n(U_n f) = \sum_i \left\{ \sum_j s_{ij} 2^n (f, e_j^n) \right\} e_i^n \qquad (f \in L_2) . \tag{2.3}$$

We have also the relation

$$(2.4)\qquad S_n^*(U_n f) = \sum_i \left\{ \sum_j s_{ji} 2^n (f, e_j^n) \right\} e_i^n \quad (f \in L_2).$$

We shall show that if (s_{ij}) is a permutation matrix, then the contraction S_n on $\mathfrak{H}_{n,2}$ has a unitary dilation that is induced by ϕ in Φ_1. By an operator S_n on $\mathfrak{H}_{n,2}$ induced by a d.s.s. matrix (s_{ij}) we shall mean the operator S_n determined by (2.3).

LEMMA 2.1. *Let* S_n *be the operator on* $\mathfrak{H}_{n,2}$ *induced by a permutation matrix* (s_{ij}). *Then there is a* $\phi \in \Phi_1$ *such that*

$$S_n U_n f = T_\phi U_n f \quad \textit{and} \quad S_n^* U_n f = T_\phi^* U_n f \quad \textit{for } f \in L_p \ (p = 1, 2).$$

Proof. Since (s_{ij}) is a permutation matrix, there exists a bijection σ from the set of positive integers onto itself such that $s_{i\sigma(i)} = 1$ for each i (therefore $s_{ij} = 0$ whenever $j \neq \sigma(i)$). Note that

$$X = \bigcup_{i=1}^{\infty} D_i^n = \bigcup_{i=1}^{\infty} D_{\sigma(i)}^n .$$

Let ϕ: $X \to X$ be a point map of the form $\phi(x) = x + b_i$ for $x \in D_i^n$ such that $\phi(D_i^n) = D_{\sigma(i)}^n$ $(i = 1, 2, \cdots)$. Clearly, $\phi \in \Phi_1$ and T_ϕ satisfies the conditions

$$T_\phi e_i^n = e_{\sigma^{-1}(i)}^n \quad \text{and} \quad T_\phi^* e_i^n = T_{\phi^{-1}} e_i^n = e_{\sigma(i)}^n .$$

Recall that $e_i^n = \chi(\,\cdot\,; D_i^n)$ $(i = 1, 2, \cdots)$. It follows that for each $f \in L_p$ $(p = 1, 2)$,

$$T_\phi U_n f = \sum_i 2^n (f, e_i^n) T_\phi e_i^n = \sum_i 2^n (f, e_i^n) e_{\sigma^{-1}(i)}^n = S_n U_n f .$$

Similarly, $T_\phi^* U_n f = S_n^* U_n f$.

Clearly, T_ϕ in Lemma 2.1 is a unitary operator on L_2. Moreover, since $\mathfrak{H}_{n,2}$ is invariant under T_ϕ, that is, since $T_\phi(\mathfrak{H}_{n,2}) \subset \mathfrak{H}_{n,2}$, we see that $U_n T_\phi U_n f = T_\phi U_n f = S_n U_n f$ for $f \in L_2$. Hence the unitary dilation T_ϕ is indeed an extension of S_n to L_2. The convex hull of $\Delta \subset \mathfrak{D}_s$ will be denoted by ch(Δ). From Lemma 2.1, we have at once the following proposition.

LEMMA 2.2. *Let* S_n *be the operator on* $\mathfrak{H}_{n,2}$ *induced by a* d.s. *matrix* (s_{ij}) *that belongs to the convex hull of permutation matrices. Then* S_n *has a dilation (extension)* S *in* ch(Φ_1) *such that*

$$S_n U_n f = S U_n f, \quad S_n^* U_n f = S^* U_n f \quad \textit{for } f \in L_p \ (p = 1, 2).$$

When $\mathfrak{D}_s$ is endowed with a topology $\mathcal{T}$, the closed convex hull of Φ_1 will be denoted by cch(Φ_1: $\mathcal{T}$). We shall prove the following approximation theorem.

THEOREM 2.1. $\mathfrak{D}_w^* = \text{cch}(\Phi_1 : L_1 - \text{s.o.t.})$

LEMMA 2.3. *For each* w*.d.s. *matrix* (t_{ij}), *for each* $\varepsilon > 0$, *and for each positive integer* N, *there is a* d.s. *matrix* (s_{ij}) *in the convex hull of permutation matrices such that*

$$\sum_i |t_{ij} - s_{ij}| < \varepsilon \qquad (j = 1, 2, \cdots, N).$$

Proof. Suppose N is a positive integer and $0 < \varepsilon < 1/2$. Since all column sums of (t_{ij}) are 1 by definition, there exists a positive integer $n = n(\varepsilon, N)$ such that $\sum_{i>n} t_{ij} < \varepsilon/4$ for each j $(1 \le j \le N)$. Choose a positive integer m such that $n/m < \varepsilon/4$. Let p_{ij} be a nonnegative integer such that

$$p_{ij} < mt_{ij} \le p_{ij} + 1 \text{ when } t_{ij} > 0 \quad \text{ and } \quad p_{ij} = 0 \text{ when } t_{ij} = 0.$$

Define the matrix (r_{ij}) by the formula $r_{ij} = p_{ij}/m$. It is easy to show that $\sum_j r_{ij} < 1$ for each i, $\sum_i r_{ij} < 1$ for each j, and $0 < 1 - \varepsilon/2 < \sum_i r_{ij} < 1$ for each $j \le N$. We can construct a d.s. matrix (s_{ij}) of the form $s_{ij} = q_{ij}/m$, where q_{ij} is an integer $(0 \le q_{ij} \le m)$, such that

$$0 \le s_{ij} - r_{ij} \le 1/m \qquad \text{for } i, j = 1, 2, \cdots.$$

Since the entries s_{ij} take only finitely many distinct rationals, it follows from a theorem of J. R. Isbell [4, Proposition 2, p. 3] that (s_{ij}) belongs to the convex hull of permutation matrices. For each $j \le N$,

$$\sum_i |t_{ij} - r_{ij}| \le \sum_{i \le n} (t_{ij} - r_{ij}) + \sum_{i>n} t_{ij} < n/m + \varepsilon/4 < \varepsilon/2,$$

and

$$\sum_i |r_{ij} - s_{ij}| = \sum_i s_{ij} - \sum_i r_{ij} < 1 - (1 - \varepsilon/2) = \varepsilon/2,$$

so that $\sum_i |t_{ij} - s_{ij}| < \varepsilon$.

Remark. Since there is a bijection between the set of w*.d.s. matrices and the set of w*.d.s. operators on ℓ_1, we may topologize w*.d.s. matrices by the strong operator topology for $[\ell_1]$, called the ℓ_1-s.o.t. It is easy to show that the set of w*.d.s. matrices is closed in the ℓ_1-s.o.t., so that by Lemma 2.3, it constitutes the closed convex hull of permutation matrices in the ℓ_1-s.o.t.

Proof of Theorem 2.1. Note that both $\mathfrak{D}_s$ and $\mathfrak{D}_w^*$ are closed in the L_1-s.o.t. It suffices therefore to prove that for each $T \in \mathfrak{D}_w^*$, each $\varepsilon > 0$, and each $f \in L_1$, there is an $S \in ch(\Phi_1)$ such that $\|Tf - Sf\|_1 < \varepsilon$. We may assume without loss of generality that f vanishes outside an interval [0, N], where N is a positive integer. Choose an n sufficiently large so that

$$\|Tf - U_n T U_n f\|_1 < \varepsilon/3, \qquad \|f - U_n f\|_1 < \varepsilon/3.$$

Set $n_1 = 2^n N$. Let T_n be the compression of T to $\mathfrak{H}_{n,2}$. Let (t_{ij}^n) be the w*.d.s. matrix defined by $t_{ij}^n = 2^n (Te_j^n, e_i^n)$. Then

$$U_n T U_n f = T_n U_n f = \sum_i \left\{ \sum_{j \le n_1} t_{ij}^n 2^n (f, e_j^n) \right\} e_i^n.$$

It follows from Lemma 2.3 that there is a d.s. matrix (s_{ij}) in the convex hull of permutation matrices such that

$$\sum_i |t_{ij} - s_{ij}| < \varepsilon/3 \, \|f\|_1 \qquad \text{for } 1 \leq j \leq n_1 .$$

Let S_n be the operator on $\mathfrak{H}_{n,2}$ that is induced by the matrix (s_{ij}). By Lemma 2.2, S_n has a dilation $S \in \operatorname{ch}(\Phi_1)$ such that

$$S U_n f = S_n U_n f = \sum_i \left\{ \sum_{j \leq n_1} s_{ij} 2^n (f, e_j^n) \right\} e_i^n .$$

It follows that

$$\|U_n T U_n f - S U_n f\|_1 = \|T_n U_n f - S_n U_n f\| \leq \sum_i \left\{ \sum_{j \leq n_1} |t_{ij}^n - s_{ij}| (|f|, e_j^n) \right\} < \varepsilon/3 .$$

On the other hand, $\|S U_n f - S f\|_1 < \varepsilon/3$, and hence $\|T f - S f\|_1 < \varepsilon$.

The example below shows that $\mathfrak{D}_w$ is not closed in the L_1-s.o.t.

Example. Let S be an operator on L_1 defined by the equation

$$Sf = \frac{1}{2}\int_0^2 f\,d\mu \cdot \chi_{[1,2)} + \frac{1}{3}\int_2^5 f\,d\mu \cdot \chi_{[2,3)} + \frac{1}{4}\int_5^9 f\,d\mu \cdot \chi_{[3,4)} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{1}{n+1} \int_{a_n}^{b_n} f\,d\mu \cdot \chi_{[n,n+1)} ,$$

where $b_n = \frac{1}{2}n(n+3)$, $a_n = \frac{1}{2}(n+2)(n-1)$. Then $S \in \mathfrak{D}_s - \mathfrak{D}_w$.

Define w.d.s. operators T_n $(n = 1, 2, \cdots)$ by the relation

$$T_n f = \frac{1}{n+1} \int_0^{n+1} f\,d\mu \cdot \chi_{[0,1)} + Sf .$$

Then

$$S^* f = \sum_{n=1}^{\infty} \frac{1}{n+1} \int_n^{n+1} f\,d\mu \cdot \chi_{[a_n, b_n)} ,$$

$$T_n^* f = \frac{1}{n+1} \int_0^1 f\,d\mu \cdot \chi_{[0,n+1)} + S^* f \qquad (n = 1, 2, \cdots).$$

Note that $S^* \in \mathfrak{D}_s - \mathfrak{D}_w^*$ and $T_n^* \in \mathfrak{D}_w^*$. A simple calculation yields the bound

$$\|T_n - S\|_1 \leq 1/(n+1) \qquad (n = 1, 2, \cdots) .$$

On the other hand,

$$\|T_n^* \chi_{[0,1)} - S^* \chi_{[0,1)}\|_1 = 1 \qquad (n = 1, 2, \cdots).$$

Let p be fixed (p = 1, 2). Following J. T. Schwartz [9, p. 39], we define the L_p-strong* operator topology (s*.o.t.) for $\mathfrak{D}_s$ as the topology induced by ε-neighborhoods, an ε-neighborhood of T being defined as the set

$$\{ S: \|(T - S) f_i\|_p < \varepsilon, \ \|(T^* - S^*) g_i\|_p < \varepsilon, \ i = 1, \cdots, n\},$$

where $f_i, \cdots, f_n$ and $g_i, \cdots, g_n$ are arbitrary elements in L_p. We can easily show that both $\mathfrak{D}_s$ and $\mathfrak{D}$ are closed in the L_1-s*.o.t. In the L_2-s*.o.t., $\mathfrak{D}_s$ is closed; but it is not clear whether $\mathfrak{D}$ is also closed. The L_1-s*.o.t. for $\mathfrak{D}_s$ has a discrete version, called the ℓ_1-s*.o.t. for the set of d.s.s. matrices. In this topology, a neighborhood base at a d.s.s. matrix (t_{ij}) consists of sets of the form

$$\left\{ (s_{ij}): \sum_i |t_{ik} - s_{ik}| < \varepsilon, \ \sum_j |t_{kj} - s_{kj}| < \varepsilon, \ k = 1, 2, \cdots, n \right\}.$$

We state without proof a theorem of Rattray and Peck [7].

LEMMA 2.4 (Rattray and Peck). *The set of* d.s. *matrices is the closed convex hull of permutation matrices in the* ℓ_1-s*.o.t.

THEOREM 2.2. $\mathfrak{D} = \mathrm{cch}(\Phi_1: L_1\text{-s*.o.t.}) \subset \mathrm{cch}(\Phi_1: L_2\text{-s*.o.t.})$.

Proof. The proof is similar to that of Theorem 2.1. Let p be fixed (p = 1, 2). Let f be a continuous function with compact support [0, N], where N is a positive integer. Given $T \in \mathfrak{D}$ and $\varepsilon > 0$, we choose a positive integer n sufficiently large so that

$$\|Tf - U_n T U_n f\|_p < \varepsilon/3, \qquad \|T^* f - U_n T^* U_n f\|_p < \varepsilon/3, \qquad \|f - U_n f\|_p < \varepsilon/3.$$

Set $n_1 = 2^n N$. Let T_n be the compression of T to $\mathfrak{H}_{n,2}$, and let (t_{ij}^n) be the d.s. matrix that represents T_n as (2.1). Then T_n^* is represented by the transpose of (t_{ij}^n) as (2.2). It follows from Lemma 2.4 that there is a d.s. matrix (s_{ij}) in the convex hull of permutation matrices such that

$$\sum_i |t_{ik}^n - s_{ik}| < \varepsilon/3c, \qquad \sum_j |t_{kj}^n - s_{kj}| < \varepsilon/3c, \tag{2.5}$$

where $k = 1, 2, \cdots, n_1$ and $c = n_1 \|f\|_\infty$. Let S_n be the operator on $\mathfrak{H}_{n,2}$ induced by the matrix (s_{ij}). By Lemma 2.2, S_n has a dilation S in $\mathrm{ch}(\Phi_1)$. Using (2.5) and the inequalities $\|f\|_1 \leq \|f\|_2 \sqrt{N} \leq \|f\|_\infty N$, we obtain the bounds

$$\|U_n T U_n f - S U_n f\|_1 \leq \varepsilon/3 \cdot 2^n, \qquad \|U_n T U_n f - S U_n f\|_2 \leq \varepsilon/3\sqrt{2^n},$$

and thus

$$\|U_n T U_n f - S U_n f\|_p < \varepsilon/3.$$

Similarly, it follows that

$$\|U_n T^* U_n f - S^* U_n f\|_p < \varepsilon/3 .$$

We see at once that

$$\|Tf - Sf\|_p < \varepsilon , \qquad \|T^* f - S^* f\|_p < \varepsilon ,$$

so that $\mathfrak{D} \subset \mathrm{cch}(\Phi_1 \colon L_p\text{-s}^*.\text{o.t.})$. Since $\mathfrak{D}$ is closed in the L_1-s*.o.t., it follows that $\mathfrak{D} = \mathrm{cch}(\Phi_1 \colon L_1\text{-s}^*.\text{o.t.})$. This completes the proof.

3. NORM APPROXIMATION

By a d.s.s. operator T with kernel $t(x, y)$ we mean a d.s.s. operator T of the form

$$Tf(x) = \int_X t(x, y) f(y)\, d\mu(y) ,$$

where $t(x, y)$ is a nonnegative measurable function on the product space $X^2 = X \times X$, and where $\int_X t(x, y)\, d\mu(y) \leq 1$ and $\int_X t(x, y)\, d\mu(x) \leq 1$. Furthermore, we say that the operator T is of *Hilbert-Schmidt type* if its kernel $t(x, y)$ belongs to $L_2(X^2, \mu^2)$, where $\mu^2 = \mu \times \mu$. Given such a $T \in \mathfrak{D}_s$, we see readily that $U_n T U_n$ has the kernel $t_n(x, y)$ of the form

$$t_n(x, y) = \sum_i \sum_j 2^n t_{ij}^n e_i^n(x) e_j^n(y) ,$$

where $e_i^n = \chi(\,\cdot\,; D_i^n)$ and

$$t_{ij}^n = 2^n (Te_j^n , e_i^n) = 2^n \int_{X^2} t(x, y) e_i^n(x) e_j^n(y)\, d\mu^2(x, y) .$$

Moreover, we have the relations

$$\text{(3.1)} \qquad \sum_i \sum_j (t_{ij}^n)^2 = \int_{X^2} t_n^2(x, y)\, d\mu^2(x, y) \leq \int_{X^2} t^2(x, y)\, d\mu^2(x, y) < \infty$$

and

$$\text{(3.2)} \qquad \begin{aligned} \|T - U_n T U_n\|_2 &= \|T^* - U_n T^* U_n\|_2 \\ &\leq \left(\int_{X^2} |t(x, y) - t_n(x, y)|^2\, d\mu^2 \right)^{1/2} \to 0 \quad (n \to \infty) . \end{aligned}$$

If $Y \subset X$ and ψ_α is a measure-preserving map from Y onto a subset $Z_\alpha \subset X$, then ψ_α induces a d.s.s. operator T_{ψ_α} defined by the equation

(3.3) $$T_{\psi_\alpha} f(x) = \begin{cases} f(\psi_\alpha(x)) & \text{on } X, \\ 0 & \text{on } X - Y. \end{cases}$$

The set of such d.s.s. operators will be denoted by $\Phi(Y)$. Let $\Phi_1(Y)$ be the set of operators $T_{\psi_\alpha} \in \Phi(Y)$ induced by invertible measure-preserving maps $\psi_\alpha: Y \to Z_\alpha$. It is easy to see that $T^*_{\psi_\alpha} = T_{\psi_\alpha^{-1}} \in \Phi_1(Z_\alpha)$ when $T_{\psi_\alpha} \in \Phi_1(Y)$. For each $A \subset X$, we define I_A by the equation $I_A f(x) = \chi_A(x) f(x)$. Then I_A is a self-adjoint d.s.s. operator: $I_A = I^*_A$. We state the following L_2-norm approximation theorem.

THEOREM 3.1. *Suppose* T *is a* w.d.s. *operator of Hilbert-Schmidt type, and* $\varepsilon > 0$. *Then there exist a positive integer* M *and a* d.s.s. *operator* $V = \sum_{i=1}^s d_i T_{\sigma_i} \in \mathrm{ch}(\Phi[0, M])$ *such that*

$$\|T - V\|_2 < \varepsilon .$$

Moreover, there exist $T_{\xi_i} \in \Phi(Z_i)$ *with* $\mu(Z_i) = M$ $(1 \le i \le s)$ *such that*

$$\left\| T^* - \sum_{i=1}^{s} d_i T_{\xi_i} \right\|_2 < \varepsilon .$$

The proof of the theorem follows readily from the following four lemmas. We begin with some notions on w.d.s. matrices [8, p. 188]. A d.s.s. matrix (t_{ij}) will be called an (m, ∞)-w.d.s. matrix if $\sum_j t_{ij} = 1$ for $1 \le i \le m$ and $t_{ij} = 0$ for $i > m$ and $j \ge 1$. By an (m, n)-w.d.s. matrix (t_{ij}) we mean an (m, ∞)-w.d.s. matrix (t_{ij}) such that $t_{ij} = 0$ for $1 \le i \le m$ and $j > n$. It is easy to show that $n \ge m$ for each (m, n)-w.d.s. matrix. An (m, n)-w.d.s. matrix having only 0 and 1 as its entries is called an (m, n)-weak permutation (w.p.) matrix. The following result follows from the proof of Lemma 9 of [8, pp. 192-194].

LEMMA 3.1 (P. Révész). *Let* (s_{ij}) *be an* (m, ∞)-w.d.s. *matrix. For each* $\varepsilon > 0$, *there exist a positive integer* $k_0 = k_0(\varepsilon) \ge m$ *and* (m, k)-w.d.s. *matrices* $\rho = (r_{ij})$ *in the convex hull of* (m, k)-w.p. *matrices for which*

$$\sum_{i=1}^{m} \sum_{j} |s_{ij} - r_{ij}| < \varepsilon \quad \textit{whenever } k \ge k_0 .$$

We shall establish an analogue of Lemma 2.1 for an (m, m')-w.p. matrix. In the following lemmas, we assume that n is a fixed positive integer.

LEMMA 3.2. *Let* R_n *be the operator on* $\mathfrak{H}_{n,2}$ *induced by an* (m, m')-w.p. *matrix* (r_{ij}). *Then there exists an invertible measure-preserving map* ψ *from* $Y = \bigcup_{i=1}^m D_i^n$ *onto a subset* Z *of* X *such that for* $f \in L_2$, *the* d.s.s. *operator* T_ψ *defined by* (3.3) *satisfies the conditions*

$$T_\psi U_n f = R_n U_n f \quad \textit{and} \quad T^*_\psi U_n f = R^*_n U_n f .$$

Moreover, ψ *can be extended to* $\phi \in \Phi_1$, *with*

$$T_\psi = I_Y T_\phi = T_\phi I_Z \quad \textit{and} \quad T_\phi^* = I_Z T_\phi^* = T_\phi^* I_Y.$$

Proof. Since $m' \geq m$, there exists an injection

$$\sigma: \{1, 2, \cdots, m\} \to \{1, 2, \cdots, m'\}$$

such that $r_{i\sigma(i)} = 1$ for $1 \leq i \leq m$ and hence all other entries r_{ij} are 0. Let

$$Y = \bigcup_{i=1}^{m} D_i^n \quad \text{and} \quad Z = \bigcup_{i=1}^{m} D_{\sigma(i)}^n.$$

Let ψ: $Y \to Z$ be a point map of the form $\psi(x) = x + b_i$ for $x \in D_i^n$, with $\psi(D_i^n) = D_{\sigma(i)}^n$, where $1 \leq i \leq m$. Clearly, ψ is an invertible measure-preserving map from Y onto Z. Note that

$$T_\psi e_j^n = \begin{cases} e_i^n & \text{if } j = \sigma(i) \text{ for some } i \ (1 \leq i \leq m), \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$T_\psi U_n f = \sum_{i=1}^{m} 2^n (f, e_{\sigma(i)}^n) e_i^n = R_n U_n f.$$

Similarly, we prove the equation for T_ψ^*.

If we set $Y' = \bigcup_{i=1}^{m'} D_i^n$, then

$$Y' - Y = \bigcup_{i=m+1}^{m'} D_i^n \quad \text{and} \quad Y' - Z = \bigcup_{i=1}^{m'-m} D_{j_i}^n.$$

Let ϕ: $X \to X$ be such that

$$\phi(x) = \begin{cases} \psi(x) & (x \in Y), \\ x & (x \in X - Y'), \\ x + b_i & (x \in D_i^n \text{ and } \phi(D_i^n) = D_{j_i}^n, \text{ where } m+1 \leq i \leq m'). \end{cases}$$

Then ϕ belongs to Φ_1 and is an extension of ψ. Clearly, $T_\psi = I_Y T_\phi$. On the other hand,

$$T_\phi I_Z f(x) = T_\phi(\chi_Z(x) f(x)) = T_\phi \chi_Z(x) T_\phi f(x) = I_Y T_\phi f(x),$$

$$(I_Y T_\phi)^* = T_\phi^* I_Y^* = T_\phi^* I_Y, \quad \text{and} \quad (T_\phi I_Z)^* = I_Z T_\phi^*.$$

This completes the proof.

The following is an immediate corollary of Lemma 3.2.

LEMMA 3.3. *Let* R_n *be the operator on* $\mathfrak{H}_{n,2}$ *induced by an* (m, m')-w.d.s. *matrix* ρ. *Let* ρ *be a convex combination of* (m, m')-w.p. *matrices* π_t; *that is, let* $\rho = \sum_{i=1}^{t} c_i \pi_i$. *Then there exist invertible measure-preserving maps* ψ_i: $Y = \bigcup_{j=1}^{m} D_j^n \to Z_i$ $(1 \leq i \leq t)$ *such that*

$$\left(\sum_{i=1}^{t} c_i T_{\psi_i}\right) U_n f = R_n U_n f, \quad \left(\sum_{i=1}^{t} c_i T_{\psi_i}^*\right) U_n f = R_n^* U_n f \qquad (f \in L_2).$$

Moreover, there exist $\phi_1, \cdots, \phi_t$ *in* Φ_1 *such that* $\phi_i = \psi_i$ *on* Y *for* $1 \leq i \leq t$,

$$\sum_{i=1}^{t} c_i T_{\psi_i} = I_Y\left(\sum_{i=1}^{t} c_i T_{\phi_i}\right) = \sum_{i=1}^{t} c_i T_{\phi_i} I_{Z_i},$$

and

$$\sum_{i=1}^{t} c_i T_{\psi_i}^* = \left(\sum_{i=1}^{t} c_i T_{\phi_i}^*\right) I_Y = \sum_{i=1}^{t} c_i I_{Z_i} T_{\phi_i}^*.$$

The following approximation, proved earlier for the case where the underlying space is the unit interval [5, Lemma 2.5, p. 524], may easily be shown by a minor modification of the argument given in [5, pp. 524-525].

LEMMA 3.4. *There exist measure-preserving maps* θ_1 *and* θ_2 *from* X *onto itself such that*

$$\left\| \left\{ \frac{1}{2}(T_{\theta_1} + T_{\theta_2}) \right\}^{2k} - U_n \right\|_2 \leq 2^{-k} \qquad (k = 1, 2, \cdots).$$

Proof of Theorem 3.1. Let t(x, y) be the kernel for T. By (3.2), we can choose a positive integer n such that

$$\|T - U_n T U_n\|_2 = \|T^* - U_n T^* U_n\|_2 < \varepsilon/4.$$

Let T_n be the compression of T to $\mathfrak{H}_{n,2}$. Note that the w.d.s. matrix (t_{ij}^n) represents T_n and satisfies (3.1). We can therefore choose a positive integer M such that

$$\sum_{i > 2^n M} \sum_j (t_{ij}^n)^2 < (\varepsilon/4)^2.$$

Put $m = 2^n M$. Define the (m, ∞)-w.d.s. matrix (s_{ij}) by setting $s_{ij} = t_{ij}^n$ for $1 \leq i \leq m$ and $j = 1, 2, \cdots$, and $s_{ij} = 0$ elsewhere. Let S_n be the operator on $\mathfrak{H}_{n,2}$ induced by the matrix (s_{ij}). It can easily be seen that

$$\|U_n T U_n - S_n U_n\|_2 = \|T_n U_n - S_n U_n\|_2 \leq \left(\sum_{i > m} \sum_j (t_{ij}^n)^2\right)^{1/2} < \varepsilon/4.$$

Similarly, $\|U_n T^* U_n - S_n^* U_n\|_2 = \|T_n^* U_n - S_n^* U_n\|_2 < \varepsilon/4$.

By Lemma 3.1, we can choose a positive integer $M' > M$ and an (m, m')-w.d.s. matrix $\rho = (r_{ij})$, where $m' = 2^n M'$, such that $\sum_{i=1}^{m} \sum_j |s_{ij} - r_{ij}| < \varepsilon/4$. Here $\rho = \sum_{i=1}^{t} c_i \pi_i$, where $c_i > 0$, $\sum_{i=1}^{t} c_i = 1$, and the π_i are (m, m')-w.p. matrices. Let R_n denote the operator on $\mathfrak{H}_{n,2}$ induced by the matrix ρ. We have the inequalities

$$\|S_n U_n - R_n U_n\|_2 \leq \left(\sum_{i=1}^{m} \sum_j |s_{ij} - r_{ij}|^2 \right)^{1/2} \leq \sum_{i=1}^{m} \sum_j |s_{ij} - r_{ij}| < \varepsilon/4 .$$

Similarly, $\|S_n^* U_n - R_n^* U_n\|_2 < \varepsilon/4$. By Lemma 3.3, the operator R_n has a dilation (extension) Q of the form $Q = \sum_{i=1}^{t} c_i T_{\psi_i}$, where each ψ_i is an invertible measure-preserving map from $Y = [0, M]$ onto $Z_i \subset [0, M']$. Moreover, each ψ_i admits an extension $\phi_i \in \Phi_1$ with $Q = I_Y \left(\sum_{i=1}^{t} c_i T_{\phi_i} \right)$. Also,

$$Q^* = \sum_{i=1}^{t} c_i T_{\psi_i}^* = \sum_{i=1}^{t} c_i I_{Z_i} T_{\phi_i}^* ,$$

where $T_{\psi_i}^* = T_{\psi_i^{-1}}$ and $T_{\phi_i}^* = T_{\phi_i^{-1}}$. Note that

$$\|R_n U_n - Q U_n\|_2 = \|R_n^* U_n - Q^* U_n\|_2 = 0 .$$

By Lemma 3.4, we can choose a positive integer k and a d.s. operator $P = \frac{1}{2}(T_{\theta_1} + T_{\theta_2})$ with $\theta_1, \theta_2 \in \Phi$ such that $\|U_n - P^{2k}\|_2 < \varepsilon/4$ and thus

$$\|QU_n - QP^{2k}\|_2 < \varepsilon/4 , \qquad \|Q^* U_n - Q^* P^{2k}\|_2 < \varepsilon/4 .$$

Set $V = QP^{2k}$ and $W = Q^* P^{2k}$. From the inequalities above, it follows that

$$\|T - V\|_2 < \varepsilon , \qquad \|T^* - W\|_2 < \varepsilon .$$

Since $P^{2k} \in ch(\Phi)$, we may assume without loss of generality that

$$V = \sum_{i=1}^{s} d_i T_{\psi_i} T_{\theta_i} = I_Y \left(\sum_{i=1}^{s} d_i T_{\phi_i} T_{\theta_i} \right),$$

$$W = \sum_{i=1}^{s} d_i T_{\psi_i^{-1}} T_{\theta_i} = \sum_{i=1}^{s} d_i I_{Z_i} T_{\phi_i^{-1}} T_{\theta_i} ,$$

where $d_i > 0$, $\sum_{i=1}^{s} d_i = 1$, $\theta_i \in \Phi$, and $\phi_i \in \Phi_1$ is an extension of an invertible measure-preserving map ψ_i from Y onto a subset Z_i. If we set $\sigma_i = \theta_i \circ \psi_i$ and $\tau_i = \theta_i \circ \phi_i$, then $\tau_i \in \Phi$ is an extension of a measure-preserving map σ_i from Y onto a subset A_i. In this case, we have the relations $T_{\psi_i} T_{\theta_i} = T_{\sigma_i}$ and $T_{\phi_i} T_{\theta_i} = T_{\tau_i}$, so that

$$V = \sum_{i=1}^{s} d_i T_{\sigma_i} = I_Y\left(\sum_{i=1}^{s} d_i T_{\tau_i}\right).$$

Note that $\sum_{i=1}^{s} d_i T_{\tau_i} \in ch(\Phi)$. Similarly, by setting $\xi_i = \theta_i \circ \psi_i^{-1}$ and $\eta_i = \theta_i \circ \phi_i^{-1}$, we obtain the equation

$$W = \sum_{i=1}^{s} d_i T_{\xi_i} = \sum_{i=1}^{s} d_i I_{Z_i} T_{\eta_i},$$

where $\eta_i \in \Phi$ is an extension of a measure-preserving map ξ_i from Z_i onto a subset B_i. Recall that $Z_i = \psi_i(Y)$ and $\mu(Z_i) = M$. Thus the theorem is proved.

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Simon Fraser University
Burnaby, British Columbia, Canada