NECESSARY AND SUFFICIENT CONDITIONS FOR WEYL'S THEOREM

Karl Gustafson

1. INTRODUCTION

H. Weyl [26] showed that if T is a bounded self-adjoint operator, then the essential spectrum $\sigma_e(T)$ consists of the points remaining in the spectrum $\sigma(T)$ after one excludes all eigenvalues of finite multiplicity that are isolated points of $\sigma(T)$; that is,

$$(*) \qquad \sigma_e(T) = \sigma(T) - \pi_{00}(T).$$

Using current terminology, one says that Weyl's theorem holds for the operators T that satisfy (*); these are the operators, then, whose essential spectra have the same character, in the sense (*), as in the self-adjoint case. That (*) remains true for normal operators, by the spectral theorem, was observed by J. Schwartz in [23]; L. A. Coburn [5] showed that (*) holds for hyponormal operators and for Toeplitz operators; V. Istrăţescu [13] showed (condition (β) of [1]) that if each point of the spectrum $\sigma(T)$ is a bare point of $\sigma(T)$, that is, if it lies on the circumference of some closed disc that contains $\sigma(T)$, and if (condition α of [1]) the restriction T_M of T to each of its invariant subspaces M has a normaloid resolvent, that is, if $\|(\lambda - T_M)^{-1}\|^{-1} = d(\lambda, \sigma(T_M))$ for all λ not in $\sigma(T_M)$, then (*) holds for T. S. K. Berberian [1] showed that sufficient for (*) to hold is that

(1) (condition (β') of [1]) each eigenvalue of finite multiplicity is a semibare point of $\sigma(T)$, that is, it lies on the circumference of some closed disc containing no other point of $\sigma(T)$, and

(1') (condition (α') of [1]) the restriction T_M of T to each of its reducing subspaces M has a normaloid resolvent. In addition (see [1, Examples 6, 3, Lemma 2, Corollary 1]) it has been observed that (*) holds for seminormal operators. Moreover, in [1] Berberian proves a general theorem including many of the previous results, namely, that (*) holds for a bounded operator T on a Hilbert space if

(i) T is reduced by each of its finite-dimensional eigenspaces, in other words, $N(T - \lambda) \subset N((T - \lambda)^*)$ for all λ, and

(ii) the restriction T_M of T to each of its reducing subspaces M has the property that every isolated point of $\sigma(T_M)$ is an eigenvalue of T_M (this is condition (α''') of [1]).

The purpose of this paper is to determine rather completely when (*) holds for an arbitrary operator T, by giving several conditions that are both necessary and sufficient for (*); the first ((1 i) of Theorem 1) reflects the viewpoint implicit in Berberian's condition (i), namely, that of a certain type of (partial) reducing behavior by the eigenspaces of T. Then, in (2), (3), (4), and (5) of Theorem 1, we present

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several other interesting viewpoints, each yielding conditions that are necessary and sufficient for Weyl's theorem to hold; namely, in terms of

(2) discontinuities of the minimum modulus function,

(3) finite ascents and descents,

(4) algebraic versus geometric multiplicities and eigenspace gaps, and

(5) the boundary $\partial\sigma$ of the spectrum and poles of the resolvent operator.

We give some other related conditions in the corollaries that follow; also, by means of examples we indicate the sharpness of the various conditions for (*), and, in particular, we show that among the classes of normal-like operators the sufficiency of (*) does not extend significantly beyond seminormal operators.

While the previous investigations of (*) were concerned with bounded operators T on a (complex) Hilbert space, we shall consider (*) for arbitrary closed operators T (if T is not closed, (*) holds trivially) that are densely defined in a (complex) Banach space X; except where we specify the contrary, this will be our setting. In particular (see Corollary 4), we show that (*) holds for *unbounded* formally seminormal operators in a Hilbert space; this interesting class of operators includes the formally normal operators studied by Coddington [6] and others.

We use Fredholm theory extensively (where we give no other reference, see Kato [15], for example). For further information on (*), we refer the reader to Berberian [2], Schechter [21], Gustafson and Weidmann [11], and the references therein.

2. NECESSARY AND SUFFICIENT CONDITIONS

The essential spectrum $\sigma_e(T)$ in the present paper is identical with the set $\sigma_e^4(T)$ of [11], with the set $\sigma_{em}(T)$ of [21], and with the set $\omega(T) = \bigcap \sigma(T+B)$, the intersection being taken over all compact operators B; that is, the essential spectrum $\sigma_e(T)$ is the complement in the (complex) scalars of the set

$$\Delta_4 = \{\lambda \mid \lambda - T \text{ is a Fredholm operator with index } i(T) = 0\} .$$

We recall that a closed operator T is said to be a Fredholm operator if R(T) is closed, $\alpha(T) \equiv \dim N(T)$ is finite, and $\beta(T) \equiv \operatorname{codim} R(T)$ is finite; if T and R(T) are closed and if at least one of $\alpha(T)$ and $\beta(T)$ is finite, T is said to be a semi-Fredholm operator with index $i(T) = \alpha(T) - \beta(T)$. In (*), π_{00} denotes the set of values λ that are isolated points of the spectrum $\beta(T)$ such that $0 < \alpha(\lambda - T) < \infty$; in other words, π_{00} is the set of isolated eigenvalues of finite geometric multiplicity.

It will be convenient to introduce the notation $\Delta_4^s \equiv \Delta_4^{\text{singular}}$ for the values λ in Δ_4 that are also eigenvalues (that is, for which $\alpha(\lambda - T) > 0$); since $\Delta_4^{\text{nonsingular}}$ is the resolvent set $\rho(T)$, we see that for each operator T the scalars constitute the disjoint union of ρ, Δ_4^s, and σ_e, and that Weyl's theorem (*) holds for T if and only if $\Delta_4^s = \pi_{00}$.

We now replace Berberian's condition (i) with the following requirement, which we denote by (λ).

(λ) In the case of a Hilbert space, the condition (λ) is said to be satisfied at a particular λ if

$$N(T - \lambda) \cap N[((T - \lambda)^*)^n]$$

is nontrivial, for some positive integer n, which may depend on λ; in the case of a Banach space, the condition (λ) is said to be satisfied at a particular λ if

$$\{N(T - \lambda)\} - \{R(T - \lambda)^n\}$$

is nontrivial, for some positive integer n, which may depend on λ.

One might convey the meaning of (λ) in words by saying that T *is not iteratively eventually anti-reduced by* $N(T - \lambda)$, or (less accurately), that $N(T - \lambda)$ *partially reduces* T. Alternately, as will become clear in the demonstration of Theorem 1 below, (λ) corresponds to a known decomposition of X into "invariant pairs" of subspaces (see [15, p. 240] and the references therein).

We recall the *minimum modulus* of T, namely,

$$\gamma(T) = \inf \|Tx\|/d(x, N(T)) \qquad (x \in D(T),\ x \notin N(T)),$$

and for convenience let us denote by $\gamma(\lambda)$ the minimum modulus function $\gamma(\lambda - T)$ for an operator T under consideration. Also we recall (see for example Taylor [25]) that the *ascent* of T at λ_0 is the smallest n such that

$$N((\lambda_0 - T)^n) = N((\lambda_0 - T)^{n+1}),$$

and the *descent* of T at λ_0 is the smallest n such that

$$R((\lambda_0 - T)^n) = R((\lambda_0 - T)^{n+1}).$$

Also, each λ_0 in π_{00} has an algebraic multiplicity dim P(X), where P is the *algebraic eigenprojection* $P = (2\pi i)^{-1} \int_\Gamma (\lambda - T)^{-1} d\lambda$; here Γ denotes any rectifiable simple closed curve containing λ_0 in its interior and the rest of the spectrum $\sigma(T)$ in its exterior. Finally, let

$$\delta(\lambda, \lambda_0) \equiv \delta(N(T - \lambda), N(T - \lambda_0)) \equiv \sup d(x_\lambda, N(T - \lambda_0)) \qquad (x_\lambda \in N(T - \lambda),\ \|x_\lambda\| = 1)$$

denote the *gap* (see [15, p. 197]) between the subspaces $N(T - \lambda)$ and $N(T - \lambda_0)$, and let us say that T satisfies the *eigenspace gap* (lower bound) *condition* at an eigenvalue λ_0 if there exists a sequence $\{\lambda_n\}$ such that $\lambda_n \to \lambda_0$ and either $|\lambda_n - \lambda_0| = o(\delta(\lambda_n, \lambda_0))$, or $N(T - \lambda_n) = \{0\}$.

THEOREM 1. *Each of the following is a necessary and sufficient condition for Weyl's theorem* (*) *to hold for* T.

(1 i) *Every* λ *in* Δ_4^s *satisfies* (λ), *and*

(1 ii) *every* λ *in* π_{00} *satisfies the condition* $\gamma(\lambda) > 0$.

(2) $\gamma(\lambda)$ *is discontinuous at every* λ *in* $\Delta_4^s \cup \pi_{00}$.

(3 i) T *has finite ascent at every* λ *in* Δ_4^s, *and*

(3 ii) T *has finite descent at every* λ *in* π_{00}.

(4 i) *Every* λ *in* Δ_4^s *satisfies the eigenspace gap condition, and*

(4 ii) *every* λ *in* π_{00} *also has finite algebraic multiplicity.*

(5 i) $\Delta_4^s \subset \partial\sigma$, *and*

(5 ii) *every* λ *in* π_{00} *is a pole of the resolvent operator.*

Proof. If $\lambda \in \Delta_4^s$, then $N[((T - \lambda)^*)^n] = R((T - \lambda)^n)$ (see [9]), and consequently the Hilbert-space version and the Banach-space version of condition (λ) coincide; therefore we shall consider only the Banach-space version. Let λ_0 be in Δ_4^s; then all λ in a small neighborhood of λ_0 are also in Δ_4^s, and moreover one knows (see [15, pp. 241-242]) that $\alpha(\lambda - T)$ is constant for all λ in a small deleted neighborhood of λ_0, and that $\alpha(\lambda - T) < \alpha(\lambda_0 - T)$ in that neighborhood if and only if (λ_0) holds. Supposing then that every λ in Δ_4^s satisfies (λ), pick some $\lambda_1 \neq \lambda_0$ within this deleted neighborhood, and suppose $\alpha(\lambda_1 - T) \neq 0$; then, for all λ in a sufficiently small deleted neighborhood of λ_1, it follows from (λ_1) that $\alpha(\lambda - T) < \alpha(\lambda_1 - T)$, and this contradicts the constantness of $\alpha(\lambda - T)$ in the original deleted neighborhood. Conversely, if (*) holds, then $\Delta_4^s \subset \pi_{00}$, and by the same reasoning, (λ_0) holds for each λ_0 in Δ_4^s. Thus (1 i) holds if and only if $\Delta_4^s \subset \pi_{00}$. Again by the same reasoning, we see that $\pi_{00} \subset \Delta_4^s$ if and only if (1 ii) holds; for if $\gamma(\pi_{00}) > 0$, then $i(\lambda_0 - T) = i(\lambda - T) = 0$ for each λ_0 in π_{00} and for all nearby λ, so that $\pi_{00} \subset \Delta_4^s$; and since always $\gamma(\Delta_4^s) > 0$, $\pi_{00} \subset \Delta_4^s$ implies that $\gamma(\pi_{00}) > 0$.

Concerning (2), suppose $\gamma(\lambda)$ is discontinuous at every λ in Δ_4^s, and, in particular, at λ_0 in Δ_4^s. Since $\gamma(\lambda - T) > 0$ for all λ near λ_0, it follows from [10, Corollary 5.74] that $\alpha(\lambda - T) < \alpha(\lambda_0 - T)$; for otherwise, γ would be continuous at λ_0. Since all nearby values λ are also in Δ_4^s, the discontinuity of $\gamma(\lambda)$ requires that $\alpha(\lambda - T) = 0$ in Δ_4^s, as in the proof of (1 i) above; hence $\Delta_4^s \subset \pi_{00}$. To see that $\pi_{00} \subset \Delta_4^s$ if and only if $\gamma(\lambda)$ is discontinuous at each λ_0 in π_{00}, we note that if λ_0 is in π_{00}, then necessarily $\gamma(\lambda) = \|(\lambda - T)^{-1}\|^{-1} \to 0$ as $\lambda \to \lambda_0$; hence the discontinuity of γ on π_{00} is equivalent to the condition $\gamma(\pi_{00}) > 0$, which by (1 ii) is equivalent to the inclusion $\pi_{00} \subset \Delta_4^s$. Similarly, if $\Delta_4^s \subset \pi_{00}$, then necessarily $\gamma(\lambda_0) > 0$ and $\gamma(\lambda - T) \to 0$ as $\lambda \to \lambda_0$ for each λ_0 in Δ_4^s; this completes the argument for (2).

From a result of M. Schechter [21, Theorem 1.1] it follows that $\Delta_4^s \subset \pi_{00}$ if and only if (3 i) holds (observe that in the notation of [21], $\rho(\lambda_0) < \infty$ if and only if T has finite ascent at λ_0). From a theorem of Lay [16] (see also Taylor [25] and the references therein), it follows that if T has finite descent at λ_0 in π_{00}, then λ_0 is a pole of the resolvent operator, and hence (see the discussion concerning (5) below) $\pi_{00} \subset \Delta_4^s$ if and only if (3 ii) holds.

The eigenspace gap condition of (4 i) is motivated by a result given by R. Bouldin [3] (Corollary 6 below). If some λ_0 in Δ_4^s is not in π_{00}, then all λ in some small neighborhood of λ_0 are also eigenvalues of T, and by the gap order condition there exist sequences $\{\lambda_n\}$ and $\{x_n\}$ such that $\lambda_n \to \lambda_0$, x_n lies in $N(T - \lambda_n)$, and

$$\|(T - \lambda_0)x_n\| / d(x_n, N(T - \lambda_0)) = |\lambda_n - \lambda_0| / \delta(\lambda_n, \lambda_0) \to 0;$$

this contradicts the fact that $\gamma(T - \lambda_0) > 0$. The necessity was artificially taken care of by the second condition $N(T - \lambda_n) = \{0\}$. The main point of (4) is to observe that $\pi_{00} \subset \Delta_4^s$ if and only if (4 ii) each λ_0 in π_{00} also has finite algebraic multiplicity. The sufficiency of this assertion follows from the well-known fact that when $\dim P < \infty$, then $\lambda_0 - T$ is a Fredholm operator, and then $i(\lambda_0 - T) = 0$, because of the nearby resolvent points. To establish the necessity, we note that if $\dim P = \infty$, then the approximate nullity $\alpha'(\lambda_0 - T)$ is also infinite [15, p. 239]; but the closed range $R(\lambda_0 - T)$ implies that $\alpha'(\lambda_0 - T) = \alpha(\lambda_0 - T) < \infty$.

By the arguments used above, (5 i) holds if and only if $\Delta_4^s \subset \pi_{00}$ (for sufficient conditions for $\Delta_4^s \subset \partial\sigma$, see Corollary 8). It is well-known that if (5 ii) is satisfied and λ_0 in π_{00} is a pole of order n, then $R(P) = N((\lambda_0 - T)^n)$. Since $N((\lambda_0 - T)^n)$ is finite-dimensional, $\pi_{00} \subset \Delta_4^s$, as in (4 ii). Conversely, if $\pi_{00} \subset \Delta_4^s$, then $\dim P < \infty$ for each λ_0 in π_{00}; it follows that $R(P) = N((\lambda_0 - T)^n)$ for some $n = n(\lambda_0)$, for each λ_0 in π_{00}, so that $(\lambda_0 - T)^m P = 0$ for all $m \geq n$; therefore λ_0 is a pole.

We remark that the closed-range theorems provide other ways of verifying the basic condition $\gamma(\pi_{00}) > 0$. For example, each of the following is a sufficient condition for $\gamma(\lambda_0) > 0$: $R(\lambda_0 - T)$ is closed, $R((\lambda_0 - T)^*)$ is closed, $\gamma((\lambda_0 - T)^*) > 0$, $R(\lambda_0 - T) = {}^{\perp}N((\lambda_0 - T)^*)$, $R((\lambda_0 - T)^*) = N(\lambda_0 - T)^{\perp}$, $\alpha'(\lambda_0 - T)$ is finite, $\beta(\lambda_0 - T)$ is finite.

3. COROLLARIES AND EXAMPLES

Earlier investigations of (*) have proceeded (roughly) along two different lines of development, namely, the consideration of the various classes of normal-like operators, and the placing of special conditions on the spectrum, for example, the (α), (α'), (α''), (α'''), (β), (β'), (G_1), (G_1'), and reducibility conditions of [1], [2], [3], [13].

Let B(H) denote the set of all bounded operators on a Hilbert space H. As concerns the first of the two directions mentioned above, the principal classes of normal-like operators on a Hilbert space are (in order of increasing generality) the classes of normal, quasi-normal, subnormal, hyponormal, seminormal, linearly normaloid, normaloid or convexoid, and spectraloid operators (see [8] for a brief summary). We recall that T ($T \in B(H)$) is *hyponormal* if $\|Tx\| \geq \|T^*x\|$, *seminormal* if either T or T^* is hyponormal, *linearly normaloid* if $\alpha T + \beta$ is normaloid for all scalars α and β, *normaloid* if the numerical radius $|W(T)|$ is equal to the operator norm $\|T\|$, *convexoid* if the convex hull of $\sigma(T)$ is the closure of the numerical range, and *spectraloid* if the spectral radius $|\sigma(T)|$ is equal to the numerical radius $|W(T)|$.

The following general result, mentioned in the introduction, combines both directions of development.

COROLLARY 1 (Berberian [1]). *If* $T \in B(H)$ *is a bounded operator on a Hilbert space satisfying* (i) *and* (ii), *then Weyl's theorem* (3) *holds for* T.

Proof. Condition (i) implies (1 i) of Theorem 1, and (i) and (ii) together imply (1 ii) of Theorem 1; for if λ_0 is in π_{00}, then

$$\gamma(\lambda_0) = \inf_{m \in N(\lambda_0 - T)^{\perp}} \|(\lambda_0 - T)m\| / \|m\| > 0 ,$$

since $\lambda_0 - T$ restricted to $N(\lambda_0 - T)^{\perp}$ is invertible, by (ii).

COROLLARY 2 (Coburn [5]). *If* $T \in B(H)$ *is hyponormal, then* (*) *holds for* T.

Proof. As in [1], this follows from Corollary 1, since hyponormal operators are known to satisfy (i) and (ii) (see Stampfli [24]). For later reference, we now give another proof.

Since $T - \lambda$ is also hyponormal, we have the inclusion $N(T - \lambda) \subset N(T - \lambda)^*$, and hence $\Delta_4^s \subset \pi_{00}$, by (1 i); let us note that $\|(T - \lambda)x\|^2 \leq \|(T - \lambda)^2 x\|$, so that a hyponormal operator has ascent 1 at every eigenvalue, and that therefore (3 i) also applies. If $\lambda_0 \in \pi_{00}$ and $\Gamma \equiv C$ is a sufficiently small circle of radius r and center λ_0, then $\|(\lambda - T)^{-1}\|^{-1} = r$ for λ in C (see [24]), and therefore

$$\|(T - \lambda_0)P\| = \frac{1}{2\pi} \left\| \int_C (\lambda - \lambda_0)(\lambda - T)^{-1} d\lambda \right\| \leq r .$$

Hence $R(P) = N(T - \lambda_0)$, and consequently $\pi_{00} \subset \Delta_4^s$ by (4 ii).

COROLLARY 3 (Berberian, Schechter, see [1]). *If* $T \in B(H)$ *is seminormal, then* (*) *holds for* T.

Proof. As in [1], this follows immediately from Corollary 2 after it is established [1, Examples 6 and 3, and Lemma 2] that $\pi_{00}(T^*) = \overline{\pi_{00}(T)}$, since the symmetry condition always implies (see [10], for example) that $\sigma_e(T^*) = \overline{\sigma_e(T)}$. An alternate proof, to be used later, is as follows.

Since by Corollary 2 above, (*) already holds for hyponormal T, let T^* be hyponormal. Each λ_0 in the set

$$\Delta_4^s(T) = \overline{\Delta_4^s(T^*)} = \overline{\pi_{00}(T^*)}$$

is an isolated point of the spectrum, with $\alpha(\lambda_0 - T) = \alpha((\lambda_0 - T)^*)$; hence $\Delta_4^s(T) \subset \pi_{00}(T)$. If $\lambda_0 \in \pi_{00}$, then

$$\|(T - \lambda_0)P\| \leq r^2 \max_{\lambda \in C} \|(\lambda - T)^{-1}\| = r^2 \max_{\lambda \in C} \|((\lambda - T)^*)^{-1}\| = r ,$$

so that $R(P) = N(T - \lambda_0)$, and hence $\pi_{00} \subset \Delta_4^s$, by (4 ii).

In the present context (of unbounded operators) we should ask whether (*) holds for some unbounded analogues of the seminormal operators. We shall say that a closed operator T in a Hilbert space is *formally hyponormal* if $\|Tx\| \geq \|T^* x\|$ on $D(T) \subset D(T^*)$, and we shall say that an operator T defined on a dense subspace in a Hilbert space is *formally seminormal* if either T or T^* is formally hyponormal. We note that the formally hyponormal operators include closed operators T for which $D \equiv D(T^* T) = D(TT^*)$ and for which $T^* T - TT^* \geq 0$ on D; the verification of this is similar to that for normal operators: since D is a core for both D(T) and $D(T^*)$, since T dominates T^* on D, and since T^* is closed, it follows that $D(T) \subset D(T^*)$ and $\|Tx\| \geq \|T^* x\|$ on D(T) as well. The formally seminormal operators constitute a nontrivial class, and in particular, the class of formally hyponormal operators includes the class of formally normal operators ($\|Tx\| = \|T^* x\|$ on $D(T) \subset D(T^*)$; see E. A. Coddington [6] and E. Nelson [17], and the references therein). The class of formally normal operators in turn includes the unbounded self-adjoint, normal, and closed symmetric operators; that (*) holds for a closed, symmetric, nonself-adjoint operator T can be seen directly from the deficiency index theory, from which it is easily verified that both $\pi_{00}(T)$ and $\Delta_4^s(T)$ are empty.

COROLLARY 4. *If* T *is a formally seminormal operator in a Hilbert space, then* (*) *holds for* T.

Proof. That (*) holds for formally seminormal operators follows exactly as in the proof of Corollary 3 given above, once we know that (*) holds for formally hyponormal operators and that $(\lambda - T)^{-1}$ is hyponormal for each λ in the resolvent set near $\lambda_0 \in \pi_{00}(T)$, for a formally hyponormal operator T. We first establish the latter fact (our proof differs from that given for bounded T in [24]). Because formal hyponormality is preserved under translation, it suffices to consider the case where T is formally hyponormal and $T^{-1} \in B(H)$; letting $A = (T^{-1})^* T$, we see that the hyponormality of T^{-1} is equivalent to the condition $\|A\| \leq 1$. The operator A is bounded, for $A^* = T^* T^{-1}$ has domain $D(A^*) = H$, since $D(T) \subset D(T^*)$; and $\|A\| = \|A^*\| \leq 1$, because $\|T^* T^{-1} x\| \leq \|x\|$ for all x. That (*) holds for formally hyponormal operators follows exactly as in the proof of Corollary 2 given above.

Another useful version of σ_e is $\sigma_\ell(T)$; $\sigma_\ell(T)$ is $\sigma_e(T)$ plus all limit points of the spectrum, and hence we may, as Weyl did, regard $\sigma_\ell(T)$ as the set of limit points of the spectrum, if we include all eigenvalues of infinite algebraic multiplicity. Let us note that $\sigma_\ell(T)$ is identical with the set $\sigma_e^5(T)$ of [11], with the set $\sigma_{eb}(T)$ of [21], and with the set σ_e of [4].

Since $\sigma_\ell - \sigma_e$ consists of the values λ in Δ_4^s that are not isolated points of the spectrum, we can state the following result; the second part generalizes [2, Propositions 5.3 and 5.5]. The verification is straightforward but tedious, and is therefore omitted.

COROLLARY 5. $\sigma_e = \sigma_\ell$ *whenever* (*) *holds for either* T *or* T^*, *and if and only if* $\Delta_4^s \subset \pi_{00}$. *Let* τ *and* τ' *be defined as in* [2] *for those* $T \in B(H)$ *that are reduced by their finite-dimensional eigenspaces; then* $\sigma_\ell = \sigma_e \supset \tau \supset \tau'$; *and if* (*) *holds, then* $\sigma_\ell = \tau'$.

Thus one can use any of the five criteria in Theorem 1 to determine whether $\sigma_e = \sigma_\ell$. For example, Bouldin [3] defines $N(T - \lambda)$ to be "not an asymptotic eigenspace" provided there exists some $\delta < 1$ such that $|(f, g)| \leq \delta$ whenever $f \in N(T - \lambda)$, $\|f\| = 1 = \|g\|$, and g is an eigenvector for some eigenvalue distinct from λ, and he gives the following result.

COROLLARY 6 (Bouldin [3]). *If* T *is an operator on a Hilbert space, and if* T *possesses no finite-dimensional asymptotic eigenspaces, then* $\sigma_e(T) = \sigma_\ell(T)$.

Proof. T satisfies the eigenspace gap condition (4 i); for if some λ_0 in Δ_4^s is not in π_{00}, then for all λ near λ_0 and for all x_λ in $N(\lambda - T)$ with $\|x_\lambda\| = 1$, we have the uniform lower bound $\delta(\lambda, \lambda_0) \geq d(x_\lambda, N(T - \lambda_0)) \geq (1 - \delta)^{1/2}$.

COROLLARY 7. *If* $\text{meas}(\sigma(T)) = 0$, *then* $\sigma_e = \sigma_\ell$, *and then* (*) *holds if and only if* $\gamma(\pi_{00}) > 0$.

Proof. Whenever $\text{meas}(\Delta_4^s) = 0$, then $\Delta_4^s \subset \pi_{00}$.

Concerning the second direction of investigation of (*), we note that the weakest pair of special conditions on the spectrum mentioned in [1] are (α''') and (β'), and, as Berberian observed in [1, after Corollary 2], (*) holds for an operator satisfying (α'''), (β'), and (G_1); for the meanings of (α''') and (β'), see the introductory comments; T is said to satisfy (G_1) if $\|(\lambda - T)^{-1}\|^{-1} = d(\lambda, \sigma(T))$ for all λ in the resolvent set for T. We observe next (Corollary 8, below) that a generalization of (β') and a variant of (α''') are necessary and sufficient for (*) to hold. Consider the conditions

(β'') each eigenvalue of T of finite (geometric) multiplicity lies in the boundary of the spectrum $\partial\sigma(T)$, and

(β''') $\Delta_4^s(T) \subset \partial\sigma(T)$;

We have the sufficient conditions (β') $\Rightarrow$ (β'') $\Rightarrow$ (β'''), and as we saw in (5 i), (β''') holds if and only if $\Delta_4^s \subset \pi_{00}$.

On the other hand, (1 ii) implies that $\pi_{00} \subset \Delta_4^s$ if and only if T satisfies the condition

(α_G''') $\gamma(\lambda_0 - T) > 0$ for each λ_0 in π_{00};

in the case of a bounded operator T on a Hilbert space, this is equivalent to the condition that the restriction of $T - \lambda_0$ to $N(T - \lambda_0)^\perp$ has a bounded inverse for each λ_0 in π_{00}. A stronger condition is

(α_G'') $\|(T - \lambda)^{-1}\| = O(r^{-n})$ for some n, where $r = d(\lambda, \sigma(T))$, for all λ sufficiently close to λ_0, for each λ_0 in π_{00};

stronger yet, but generalizing the behavior of the resolvent operator for a hyponormal operator, is the condition

(α_G') $(\lambda - T)^{-1}$ is spectraloid in a neighborhood of π_{00};

denote the previously mentioned condition G_1 by (α_G), that is,

(α_G) $(\lambda - T)^{-1}$ is normaloid for all λ in $\rho(T)$;

then $(\alpha_G) \Rightarrow (\alpha_G') \Rightarrow (\alpha_G'') \Rightarrow (\alpha_G''')$. To see the sufficiency of (α_G'') and (α_G'), recall that

$$\|(T - \lambda_0)^n P\| \leq r^{n+1} \max_C \|(\lambda - T)^{-1}\| ;$$

under (α_G''), we then have the bound $\|(T - \lambda_0)^n P\| \leq Mr$, and hence

$$R(P) = N((T - \lambda_0)^n) ,$$

which is finite-dimensional for λ_0 in π_{00}. In considering (α_G'), let k denote the equivalence ratio between the numerical radius and the operator norm; it is known that $2 \leq k \leq e$, the exact value of k depending on the particular Banach space and numerical range under consideration. Using the spectral mapping theorem, one obtains the relations

$$\|(T - \lambda_0)P\| \leq kr^2 \max_C |W((\lambda - T)^{-1})| = kr^2 \max_C |\sigma((\lambda - T)^{-1})| = kr ,$$

so that $R(P) = N(T - \lambda_0)$. We also note that since $(T - \lambda_0)^n P$ is quasi-nilpotent, and since clearly a quasi-nilpotent operator is zero if and only if it is spectraloid, one can replace (α_G'') by the weaker condition that some $(T - \lambda_0)^n P$ is spectraloid. Summarizing, we have the following proposition.

COROLLARY 8. *(*) holds for* T *if and only if* T *satisfies* (α_G''') *and* (β'''); *the other special conditions on the spectrum mentioned above are also sufficient for* (*).

By means of the following examples we indicate the sharpness of the various conditions for (*).

Let T_1 be any left-shift operator with nonzero weights of magnitude not greater than 1 and decreasing to zero, so that T_1 is quasi-nilpotent and

$$\sigma(T_1) = \sigma_e(T_1) = \pi_{00}(T_1) = \{0\} ;$$

for example, let $Te_1 = \{0\}$, $Te_n = n^{-1} e_{n-1}$ $(n = 1, 2, \cdots)$. Then (*) does not hold for T_1, even though T_1 (vacuously) satisfies both (α''') and (β'), the weakest pair of conditions mentioned in [1].

Although it is necessary that (λ) hold for all λ in π_{00} in order that $\pi_{00} \subset \Delta_4^s$, it is not sufficient, as may be seen by letting T_2 be the direct sum of T_3 and T_4, where T_3 is a one-dimensional zero operator and T_4 is any right shift weighted in the same way as T_1 above; then $\sigma(T_2) = \sigma_e(T_2) = \pi_{00}(T_2) = \{0\}$, even though ($\{0\}$) holds. This explains the appearance of the stronger condition (1 ii). Similarly, T_2 shows that although it is necessary that T have finite ascents on π_{00} for (*) to hold, it is not sufficient, since $N(T_2) = N(T_2^2)$; for this reason, one needs the stronger (descent) condition (3 ii).

Finally, since (*) is true for all seminormal operators and all Toeplitz operators, and since these operators are convexoid and the convexoid property is translation-invariant, it is natural to ask whether (*) extends to all convexoid operators. The following example shows that this is not the case, and it also shows that among normal-like operators, (*) does not extend appreciably beyond the seminormal ones, since, by this example, (*) does not even extend to the linearly normaloid operators (nor, for that matter, to the class C_5 of [12]).

Let T_5 be an "annulus" normal operator with

$$\sigma(T_5) = \sigma_e(T_5) = A_a = \{\lambda \mid 0 < a \le |\lambda| \le 1\};$$

for example, take T_5 to be a diagonal operator whose diagonal elements constitute a dense countable subset of the annulus A_a, and let T_6 be the direct sum of T_2 and T_5. Then

$$\sigma(T_6) = \sigma_e(T_6) = A_a \cup \{0\},$$

while the closure of the numerical range $\overline{W(T_6)}$ is the unit disc. Clearly, T_6 is convexoid and normaloid. Moreover, T_6 is linearly normaloid; $\overline{W(T_6)}$ is a spectral set (in the von Neumann sense; see von Neumann [18] and [20, p. 437]) for T_6, and therefore T_6, being in the class C_5 of S. Hildebrandt [12], is linearly normaloid. But (*) does not hold for T_6, because $\{0\}$ is an isolated eigenvalue of finite (geometric) multiplicity.

C. R. Putnam [19] has recently shown that if meas $\sigma(T) = 0$ and T is seminormal, then T is normal; in that context, we remark that T_6 with $a = 1$ provides an example of a nonnormal but linearly normaloid C_5 operator with meas $\sigma(T) = 0$.

Additional Remark. The author would like to thank S. K. Berberian for pointing out two recent papers that also are concerned with unbounded seminormal operators. In [14], an operator T is called *hyponormal* if it is closed and densely defined, and if $D(T) = D(T^*)$ and $T^*T - TT^* \ge 0$. As was shown above Corollary 4, closed operators T for which $D \equiv D(T^*T) = D(TT^*)$ and for which $T^*T - TT^* \ge 0$ on D form a (proper) subclass of our formally hyponormal operators. In [7], an operator T is called seminormal if it is closed and densely defined, and if either T or T^* is hyponormal [T is called hyponormal if it is closed and densely defined, and if $D(T) = D(T^*)$ and $\|T^*x\| \le \|Tx\|$ on D(T)]. The following is an equivalent restatement of this definition of seminormality: An operator T is seminormal if it is closed and densely defined, and if $D(T) = D(T^*)$ and either $\|Tx\| \ge \|T^*x\|$ everywhere on D(T) or $\|Tx\| \le \|T^*x\|$ everywhere on D(T). This class of operators is clearly a (proper) subclass of our formally seminormal operators.

Recently, M. Schechter [22] has also obtained necessary and sufficient conditions for (*) in a setting of closed operators in a Banach space, as in the present paper. Although there is some overlap with our results (for example, Theorem 2.1 of [22] is our Corollary 8, by the closed-range theorem), in [22] the emphasis is on reducing subspaces, as in [1] and [2]. [22] also contains a proof of the fact that (*) holds if A is seminormal and $A \in B(H)$.

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University of Colorado
Boulder, Colorado 80302
and
Ecole Polytechnique Fédérale de Lausanne
Lausanne, Switzerland