IDEALS OF COMPACT OPERATORS ON HILBERT SPACE

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This paper is dedicated in sorrow to the memory
of our late colleague, David Topping.

1. INTRODUCTION

Let $\mathscr{H}$ be a separable, infinite-dimensional Hilbert space, and let $\mathscr{L}(\mathscr{H})$ denote the ring of bounded linear operators on $\mathscr{H}$. The two-sided ideals in $\mathscr{L}(\mathscr{H})$ were originally described by J. von Neumann (see, for instance, [2, Section 1]), and since then a good deal of attention has been devoted to a special class of such ideals, namely, the *norm ideals* of R. Schatten [4] (also called *s. n. ideals* by I. C. Gohberg and M. G. Kreĭn [3]). Very little seems to be known, however, about the general ideal structure of $\mathscr{L}(\mathscr{H})$. For example, it is well known (and easy to prove) that every proper ideal in $\mathscr{L}(\mathscr{H})$ is contained in the ideal $\mathfrak{C}$ of compact operators; consequently, if T is an operator on $\mathscr{H}$ and if there exists a proper ideal $\mathfrak{J}$ in $\mathscr{L}(\mathscr{H})$ such that T belongs to $\mathfrak{J}$, then T is certainly compact. But, given a compact operator, can we assert that there exist ideals other than $\mathfrak{C}$ that contain it? The purpose of this note is to make a modest beginning toward a general theory of ideals. In particular, we answer the preceding question in the affirmative.

In the sequel, all Hilbert spaces are understood to be complex and separable, and all operators are bounded and linear. Moreover, the term *ideal* will always be used to mean two-sided ideal. We remind the reader that if $\mathfrak{J}$ is an ideal, and if T belongs to $\mathfrak{J}$, then T^* and $|T| = (T^*T)^{1/2}$ also belong to $\mathfrak{J}$; moreover, every proper ideal $\mathfrak{J}$ is not only contained in the ideal $\mathfrak{C}$ of all compact operators, but also contains the ideal $\mathfrak{F}$ of all operators of finite rank. The von Neumann-Calkin characterization of the ideals in $\mathscr{L}(\mathscr{H})$ goes as follows. Let C denote the collection of all the nonnegative real sequences $\{\lambda_n\}_{n=1}^{\infty}$ that tend to zero as n tends to infinity. Following Calkin, we call a subset J of C an *ideal set* if it satisfies the following conditions.

(i) If $\{\lambda_n\} \in J$ and if π denotes any permutation of the positive integers, then $\{\lambda_{\pi(n)}\} \in J$.

(ii) If $\{\lambda_n\}$ and $\{\mu_n\}$ both belong to J, then so does $\{\lambda_n + \mu_n\}$.

(iii) If $\{\lambda_n\} \in J$, and if $\{\mu_n\}$ is any sequence in C such that $\mu_n \leq \lambda_n$ for every n, then $\{\mu_n\} \in J$.

Now let $\mathfrak{J}$ be any proper ideal in $\mathscr{L}(\mathscr{H})$. If T belongs to $\mathfrak{J}$, then so does $|T|$, and since $|T|$ is compact, it has an orthonormal basis of eigenvectors. If the corresponding eigenvalues are arranged into a sequence (counting multiplicities), that sequence belongs to C. In this way a sequence in C, determined up to permutation, is associated with each operator T in $\mathfrak{J}$. It turns out that the set of all sequences so

Received February 5, 1971.
The research of the first two authors was partially supported by the National Science Foundation.

Michigan Math. J. 18 (1971).

obtained from the various operators in $\mathfrak{J}$ forms an ideal set J, called the *ideal set of* $\mathfrak{J}$. Conversely, if J is any ideal set in C and T is a compact operator on $\mathscr{H}$, we say that T *belongs to* J provided the sequence of eigenvalues of $|T|$ belongs to J. The set of operators belonging to J in this sense forms an ideal $\mathfrak{J}$ of which J is obviously the ideal set. Note that this correspondence between ideals and ideal sets is one-to-one and inclusion-preserving, and that the whole set C is the ideal set of the maximal ideal $\mathfrak{C}$. At the opposite extreme, the ideal set of the minimal ideal $\mathfrak{F}$ is the set F of all finitely nonzero sequences. The following lemma summarizes some basic information concerning ideal sets that will be needed later.

LEMMA 1.1. *Let* $\{\lambda_n\}$ *be a sequence belonging to* C *but not belonging to* F, *and let* $\{\lambda_{k_n}\}$ *denote the subsequence of all the nonzero terms of* $\{\lambda_n\}$. *Then* $\{\lambda_n\}$ *belongs to a given ideal set* J *if and only if* $\{\lambda_{k_n}\}$ *does so. If* $\{\lambda_n\}$ *belongs to an ideal set* J, *then so does every subsequence* $\{\lambda_{m_n}\}$. *On the other hand, if* J *contains some tail* $\{\lambda_{p+n}\}_{n=1}^{\infty}$ *of* $\{\lambda_n\}$, *then* J *also contains* $\{\lambda_n\}$.

The first assertion is precisely the content of Lemmas 1.1 and 1.2 of [2]. The other two assertions are more or less obvious consequences of the first.

2. ADMISSIBLE FUNCTIONS

We shall consider nondecreasing, real-valued functions f, defined on the half-line $[0, +\infty)$ and satisfying the condition

1) $f(0) = 0$.

For any such function f we shall denote by M(f) the collection of all sequences $\{\lambda_n\}$ in C that are *summed by* f in the sense that they satisfy the condition

$$\sum_n f(\lambda_n) < +\infty .$$

If f vanishes in a neighborhood of 0, then every sequence in C is summed by f: M(f) = C. On the other hand, if $\lim_{t\to 0+} f(t) \neq 0$, then only the finitely nonzero sequences are summed by f: M(f) = F. It is desirable to rule out both of these trivial cases. Accordingly, we restrict our attention to functions f that also satisfy the further conditions

2) $\lim_{t\to 0+} f(t) = 0$

and

3) $f(t) > 0$ for $t > 0$.

For brevity's sake, let us call a nondecreasing function on $[0, +\infty)$ that satisfies 1), 2), and 3) an *admissible function.* The following two elementary lemmas are central to our purposes.

LEMMA 2.1. *If* $\{\lambda_n\}$ *is any sequence in* C, *there exist admissible functions* f *that sum* $\{\lambda_n\}$. *On the other hand, if* $\{\lambda_n\}$ *is any sequence in* $C\setminus F$, *there exist admissible functions* f *that do not sum* $\{\lambda_n\}$.

Proof. We may suppose that $\{\lambda_n\}$ has infinitely many distinct, nonzero terms. Let $\{\nu_n\}$ denote the result of arranging these distinct terms in decreasing order, so that

$$\nu_1 > \nu_2 > \cdots > \nu_n > \cdots > 0,$$

and for each n, let p_m denote the (finite but positive) number of times ν_m is repeated in the sequence $\{\lambda_n\}$. Then, if $\{\phi_n\}$ is any monotone sequence in C, there clearly exist admissible functions f satisfying the condition $f(\nu_n) = \phi_n$ for all n, and for any such f,

$$\sum_n f(\lambda_n) = \sum_n p_n \phi_n .$$

Thus, in order to construct an admissible function f that does not sum $\{\lambda_n\}$, we need only choose for $\{\phi_n\}$ any monotone sequence in C such that $\sum_n \phi_n = +\infty$. On the other hand, in order to obtain an admissible function f that *does* sum $\{\lambda_n\}$, we may start with an arbitrary summable sequence $\{\alpha_n\}$ of positive numbers, set $\gamma_n = \alpha_n / p_n$, and then define

$$\phi_n = \gamma_1 \wedge \cdots \wedge \gamma_n \qquad (n = 1, 2, \cdots).$$

LEMMA 2.2. *If* f *is an admissible function, there exist sequences in* $C \setminus F$ *that are summed by* f. *Indeed, each sequence* $\{\lambda_n\}$ *in* $C \setminus F$ *has a subsequence that belongs to* M(f). *On the other hand, there also exist sequences in* C *that are not summed by* f. *Indeed, each sequence* $\{\lambda_n\}$ *in* C *is a subsequence of a sequence that does not belong to* M(f).

Proof. Since $\lim \lambda_n = 0$ and $\lim_{t \to 0+} f(t) = 0$, it is clear that $\{\lambda_n\}$ has subsequences $\{\lambda_{k_n}\}$ such that $f(\lambda_{k_n})$ tends to zero with any desired rapidity. Thus the first half of the lemma is easily disposed of. The second half is equally easy. To begin with, we may assume that $\{\lambda_n\}$ contains no zeros (for zeros can always be interpolated as needed; see Lemma 1.1). But then all that is necessary is to construct a new sequence in which each λ_n is repeated p_n times, where p_n is a positive integer chosen so that $p_n f(\lambda_n) \geq 1$.

3. AN ORDERING

It is clear that M(f) depends only on the behavior of f in an arbitrarily small interval $[0, \varepsilon)$. Hence, the natural ordering on the set of admissible functions is the following.

Definition. If f and g are admissible functions, then f *is dominated by* g ($f \prec g$) provided there exist positive numbers M and ε such that

$$f(t) \leq M g(t) \qquad (0 \leq t < \varepsilon). \tag{1}$$

It is readily seen that $f \prec g$ implies $M(g) \subset M(f)$. As it turns out, the converse is also valid.

THEOREM 3.1. *If* f *and* g *are admissible functions and* f *is not dominated by* g, *then there exist sequences* $\{\lambda_n\}$ *that are summed by* g *but not by* f.

Proof. Using the fact that (1) does not hold for any positive M and ε, we can easily construct a strictly decreasing sequence $\{\kappa_n\}$ satisfying the conditions

$$g(\kappa_n) \leq n^{-2}, \quad f(\kappa_n) \geq n\,g(\kappa_n) \qquad (n = 1, 2, \cdots).$$

For each n, let p_n denote the smallest positive integer p for which $p\,g(\kappa_n) \geq n^{-2}$, so that

$$n^{-2} \leq p_n g(\kappa_n) \leq 2n^{-2},$$

and take for $\{\lambda_n\}$ the sequence

$$\underbrace{\kappa_1, \cdots, \kappa_1}_{p_1}, \underbrace{\kappa_2, \cdots, \kappa_2}_{p_2}, \cdots, \underbrace{\kappa_n, \cdots, \kappa_n}_{p_n}, \cdots.$$

Then

$$\sum_m g(\lambda_m) = \sum_n p_n g(\kappa_n) \leq 2 \sum_n n^{-2} < +\infty,$$

while

$$\sum_m f(\lambda_m) \geq \sum_n n p_n g(\kappa_n) \geq \sum_n 1/n = +\infty.$$

It may be noted that a more compact way of characterizing the relation $f \prec g$ is to say that

$$\limsup \frac{f(t)}{g(t)} < +\infty.$$

We close the discussion of the ordering of admissible functions by giving a simple example of a pair of functions f and g neither of which dominates the other. Indeed, if

$$f(t) = \begin{cases} 1 & (1/2 < t), \\ 1/(2n+1)! & (1/(2n+2)! < t \leq 1/(2n)!), \end{cases}$$

while

$$g(t) = \begin{cases} 1 & (1 < t), \\ 1/(2n)! & (1/(2n+1)! < t \leq 1/(2n-1)!) \end{cases}$$

for $n = 1, 2, \cdots$, and if $\{\lambda_n\}$ and $\{\mu_n\}$ denote the sequences

$$\frac{1}{2}, \underbrace{1/4!, \cdots, 1/4!}_{3!}, \cdots, \underbrace{1/(2n)!, \cdots, 1/(2n)!}_{(2n-1)!}, \cdots$$

and

$$1, 1/3!, 1/3!, \underbrace{1/5!, \cdots, 1/5!}_{4!}, \cdots, \underbrace{1/(2n+1)!, \cdots, 1/(2n+1)!}_{(2n)!}, \cdots,$$

then it may readily be verified that $\{\lambda_n\}$ is summed by f but not by g, and similarly that $\{\mu_n\}$ is summed by g but not by f.

4. THE MAIN CONSTRUCTION

While the set M(f) of sequences summed by an admissible function f need not be an ideal set in the sense of Section 1, it is clear that M(f) does possess both properties (i) and (iii) of ideal sets. Moreover, if $\{\lambda_n\}$ and $\{\mu_n\}$ are two sequences belonging to M(f), then $f(\lambda_n \vee \mu_n) = f(\lambda_n) \vee f(\mu_n)$; this shows that $\{\lambda_n \vee \mu_n\}$ also belongs to M(f). But then so does $\{(\lambda_n + \mu_n)/2\}$. Thus we see that a necessary and sufficient condition for M(f) to be an ideal set is that $\{2\lambda_n\}$ should belong to M(f) along with $\{\lambda_n\}$, or equivalently, that 2M(f) should coincide with M(f). On the other hand, it is clear that if $M(f) = \alpha M(f)$ for a single $\alpha \neq 1$, then $M(f) = \alpha M(f)$ for every α $(0 < \alpha < +\infty)$. Thus we have the following result.

LEMMA 4.1. *A necessary and sufficient condition for* M(f) *to be an ideal set is that there should exist a positive number* α *different from* 1 *such that* $\alpha M(f) = M(f)$.

Using Theorem 3.1, we can easily translate this condition on M(f) into a condition on f itself.

THEOREM 4.2. *Let* f *be an admissible function. Then a necessary and sufficient condition for* M(f) *to be an ideal set is that there should exist some* $\alpha > 1$ *such that*

$$f(\alpha t) \prec f(t). \tag{2}$$

Proof. The set M(g) summed by the function $g(t) = f(\alpha t)$ coincides with $(1/\alpha) M(f)$.

All of the better known ideals in $\mathscr{L}(\mathscr{H})$ (with the exception of $\mathfrak{F}$ and $\mathfrak{C}$ themselves) correspond to ideal sets that are of the form M(f) for some f satisfying (2). In particular, of course, the so-called p-ideals of Schatten [4] are the ideals corresponding to the sets $M(f_p)$, where $f_p(t) = t^p$ $(1 \leq p < +\infty)$. In connection with ideal sets of the form M(f), where f is an admissible function, two interesting questions arise that we have not been able to resolve. The first is whether the collection of ideal sets $\{M(f)\text{: f is admissible}\}$ is linearly ordered under inclusion. (Later, we show that the collection of all ideal sets is not linearly ordered.) The second is whether every ideal set of the form M(f) arises from a continuous function, in the sense that there exists an admissible function f' continuous in a neighborhood of the origin such that M(f) = M(f').

The main observation of this paper is that even though the sets M(f) may not be ideal sets themselves, they can always be used to construct ideal sets.

THEOREM 4.3. *If* f *is an admissible function, then the union*

$$S(f) = \bigcup_{\alpha > 0} \alpha M(f)$$

and the intersection

$$D(f) = \bigcap_{\alpha > 0} \alpha M(f)$$

are both ideal sets.

Proof. It is easy to verify that both S(f) and D(f) have the properties (i) and (iii) of ideal sets. To see that S(f) also has property (ii), suppose that both $\{\lambda_n\}$ and $\{\mu_n\}$ belong to S(f), and choose α and β so that $\{\lambda_n\} \in \alpha M(f)$ and $\{\mu_n\} \in \beta M(f)$. If $\gamma = \alpha \vee \beta$, then both $\{\lambda_n\}$ and $\{\mu_n\}$ belong to $\gamma M(f)$. But then, as we noted above, $\{(\lambda_n + \mu_n)/2\}$ also belongs to $\gamma M(f)$, so that

$$\{\lambda_n + \mu_n\} \in 2\gamma M(f).$$

Finally, to see that D(f) also has property (ii), let both $\{\lambda_n\}$ and $\{\mu_n\}$ belong to D(f), and let α be positive. Since both $\{\lambda_n\}$ and $\{\mu_n\}$ belong to $(\alpha/2)M(f)$, it follows as before that $\{(\lambda_n + \mu_n)/2\}$ belongs to $(\alpha/2)M(f)$, and therefore $\{\lambda_n + \mu_n\}$ belongs to $\alpha M(f)$.

Since the family $\{\alpha M(f)\}_{\alpha=0}^{\infty}$ is increasing with α, it is clear that in forming S(f) we need only consider large values of α. In particular, $S(f) = \bigcup_{\alpha > 1} \alpha M(f)$. Similarly, $D(f) = \bigcap_{0<\alpha<1} \alpha M(f)$. It may be worthwhile to note that S(f) consists of all sequences $\{\lambda_n\}$ in C with the property that $\{\alpha \lambda_n\}$ is summed by f for some sufficiently small α, while D(f) consists of all sequences $\{\lambda_n\}$ in C with the property that $\{\alpha\lambda_n\}$ is summed by f for every $\alpha > 0$.

Theorem 4.3, together with the pair of admissible functions f and g constructed at the end of Section 3, yields immediately the following corollary, which we believe to be new.

COROLLARY 4.4. *The ideals in $\mathscr{L}(\mathscr{H})$ are not linearly ordered. In particular, there exist admissible functions* f *and* g *such that neither* $S(f) \supset S(g)$ *nor* $S(g) \supset S(f)$.

Proof. It clearly suffices to prove the second assertion of the corollary. Let f and g be the admissible functions constructed at the end of Section 3, and let $\{\lambda_n\}$ and $\{\mu_n\}$ be the sequences exhibited in Section 3 after the definition of f and g. We have already seen that

$$\{\lambda_n\} \in M(f), \quad \{\lambda_n\} \notin M(g), \quad \{\mu_n\} \in M(g), \quad \text{and } \{\mu_n\} \notin M(f).$$

A calculation shows that for fixed α $(0 < \alpha < 1)$, $f(\alpha\mu_n) = f(\mu_n)$ and $g(\alpha\lambda_n) = g(\lambda_n)$ for sufficiently large n. This shows that $\{\lambda_n\} \notin (1/\alpha)M(g)$ and that

$$\{\mu_n\} \notin (1/\alpha)M(f).$$

Hence, by definition, $\{\lambda_n\} \notin S(g)$ and $\{\mu_n\} \notin S(f)$. Since $\{\lambda_n\} \in S(f)$ and $\{\mu_n\} \in S(g)$, the proof is complete.

Before stating the next theorem, we need to strengthen Lemma 2.2.

LEMMA 4.5. *For each* c o u n t a b l e *collection* Φ *of admissible functions, there exist sequences in* $C\setminus F$ *that are summed by every* $f \in \Phi$. *Indeed, each sequence* $\{\lambda_n\}$ *in* $C\setminus F$ *has a subsequence with this property. On the other hand, there also exist sequences in* C *that are not summed by any* $f \in \Phi$. *Indeed, each sequence in* C *is a subsequence of such a sequence.*

Proof. We may assume Φ to be arranged into a sequence $\{f_n\}_{n=1}^{\infty}$. The latter assertion of the lemma may be proved by a simple modification of the construction used in Lemma 2.2 in the case of a single f. We assume the given sequence $\{\lambda_n\}$ to be free of zeros, and we construct a new sequence with the desired property by first repeating λ_1 p_1 times, where $p_1 \cdot f_1(\lambda_1) \geq 1$, then repeating λ_2 p_2 times, where $p_2 \cdot f_1(\lambda)$ and $p_2 \cdot f_2(\lambda_2)$ both exceed 1, and so forth. To prove the first part of the lemma, we take recourse to the familiar diagonal procedure. The sequence $\{\lambda_n\}$ has a subsequence $\{\lambda_n^{(1)}\}$ that is summed by f_1. This sequence, in turn, has a subsequence $\{\lambda_n^{(2)}\}$ that is also summed by f_2. Continuing in this way, we obtain an infinite sequence $\{\lambda_n^{(k)}\}$ of sequences such that each $\{\lambda_n^{(k)}\}_{n=1}^{\infty}$ is summed by $f_1, f_2, \cdots, f_k$, and such that (for $k > 1$), each is a subsequence of its predecessor. But then the diagonal sequence $\{\lambda_n^{(n)}\}_{n=1}^{\infty}$ is summed by every f_k.

THEOREM 4.6. *If $\{\lambda_n\}$ is any sequence in $C \setminus F$, then there exist ideal sets* J *and* K *such that*

$$F \neq J \subset K \neq C$$

and such that $\{\lambda_n\}$ belongs to $K \setminus J$.

Proof. According to Lemma 2.1, there exist admissible functions f and g such that f sums $\{\lambda_n\}$ and g does not. By replacing f by $f \wedge g$ if necessary, we may arrange matters so that $f \leq g$. We then define $J = D(g)$ and $K = S(f)$. Since $\{\lambda_n\} \in M(f) \setminus M(g)$, it is clear that $\{\lambda_n\} \in K \setminus J$, and it is also clear that J is contained in K. In order to verify that K is distinct from all of C, we employ Lemma 4.5. Indeed, if $\{\alpha_n\}$ is a sequence of positive numbers that tends to infinity (for example, if $\alpha_n = n$), then

$$S(f) = \bigcup_{n=1}^{\infty} \alpha_n M(f),$$

so that S(f) is the set of sequences in C that are summed by at least one function in a countable family of functions. Similarly, if $\{\alpha_n\}$ is a sequence of positive numbers tending to zero (for example, if $\alpha_n = 1/n$), then

$$D(g) = \bigcap_{n=1}^{\infty} \alpha_n M(g),$$

so that D(g) coincides with the set of all sequences in C that are summed by every function in a countable family of admissible functions, and consequently $J \neq F$, by Lemma 4.5.

THEOREM 4.6'. *If* T *is a compact operator of infinite rank on $\mathscr{H}$, then there exist ideals $\mathfrak{J}$ and $\mathfrak{K}$ in $\mathscr{L}(\mathscr{H})$ such that*

$$\mathfrak{F} \neq \mathfrak{J} \subset \mathfrak{K} \neq \mathfrak{C}$$

and such that $T \in \mathfrak{K} \setminus \mathfrak{J}$.

It is now clear that the question posed in the introduction is to be answered in the affirmative.

COROLLARY 4.7. *The ideal $\mathfrak{C}$ is the union of the set of all ideals $\mathfrak{J} \subsetneq \mathfrak{C}$. Dually, the ideal $\mathfrak{F}$ is the intersection of the set of all ideals $\mathfrak{J} \supsetneq \mathfrak{F}$.*

The argument used to prove Theorem 4.6 can be made to yield more. The following sharper result, which says roughly that we may at will adjoin or delete one operator at a time, is useful for various purposes. (See [1, Theorem 3.2] for example.)

THEOREM 4.8. *Let* J *be any ideal set in* C *distinct from* F, *and let* $\{\lambda_n\}$ *be any one sequence belonging to* $J \setminus F$. *Then there exists an ideal set* J' *such that*

$$F \subsetneqq J' \subset J$$

and such that $\{\lambda_n\}$ *does not belong to* J'. *Dually, if* J *is any ideal set distinct from* C *and if* $\{\lambda_n\}$ *is a sequence belonging to* $C \setminus F$, *then there exists an ideal set* J'' *such that*

$$J \subset J'' \subsetneqq C$$

and such that $\{\lambda_n\}$ *belongs to* J''.

Proof. The first of the two assertions turns out to be the simpler. Let g be any admissible function that does not sum $\{\lambda_n\}$ (Lemma 2.1), and set

$$J' = J \cap D(g).$$

Clearly, J' is an ideal set that is contained in J and does not contain $\{\lambda_n\}$. On the other hand, by Lemma 4.4, $\{\lambda_n\}$ has a subsequence $\{\lambda_n'\}$ that does not belong to F but is summed by every function $g_m(t) = g(mt)$ $(m = 1, 2, \cdots)$. Since this subsequence belongs to J (by Lemma 1.1) as well as to D(g), it belongs to J', and therefore $J' \neq F$.

For the proof of the other half of the theorem, we shall need a lemma.

LEMMA 4.9. *Let* J *and* K *be ideal sets. Then the smallest ideal set* $J \vee K$ *containing both* J *and* K *coincides with the set* L *of all sequences of the form* $\{\lambda_n \vee \mu_n\}$, *where* $\{\lambda_n\} \in J$, $\{\mu_n\} \in K$.

Proof. It is simple to verify that L contains both J and K, and that it satisfies conditions (i) and (iii) of ideal sets. Moreover, it is clear that any ideal set containing both J and K must also contain L. Thus the lemma will be proved if we verify that L satisfies (ii). To see this, let $\{\lambda_n\}$ and $\{\mu_n\}$ belong to J and K, respectively. Then $\{2\lambda_n\}$ and $\{2\mu_n\}$ do so too; consequently, $\{2(\lambda_n \vee \mu_n)\}$ belongs to L, and therefore, by (iii), so does $\{\lambda_n + \mu_n\}$. But now, if $\{\lambda_n'\}$ and $\{\mu_n'\}$ are two additional sequences belonging to J and K, respectively, then $\{\lambda_n + \lambda_n'\} \in J$ and $\{\mu_n + \mu_n'\} \in K$; by what has just been shown, $\{\lambda_n + \lambda_n' + \mu_n + \mu_n'\} \in L$, and the lemma follows.

Remark. It is implicit in the proof above that $J \vee K$ also coincides with the set $J + K$ of all sequences $\{\lambda_n + \mu_n\}$ $(\{\lambda_n\} \in J,\ \{\mu_n\} \in K)$.

Completion of the proof of Theorem 4.8. Let J be an ideal set, and let $\{\lambda_n\}$ be a sequence in C that does not belong to J. Since $\{\lambda_n\}$ is certainly not in F, it possesses infinitely many nonzero terms. Let $\{\lambda_n'\}$ denote the sequence of nonzero terms in $\{\lambda_n\}$, arranged in decreasing order, and let $\{\lambda_n''\}$ denote any strictly decreasing sequence such that $\lambda_n' \leq \lambda_n''$ for all n. By Lemma 1.1, it suffices to construct an ideal set J'' that contains J and $\{\lambda_n''\}$ and is distinct from C. Hence we may assume that $\lambda_n = \lambda_n''$, in other words, that $\{\lambda_n\}$ itself is a strictly decreasing sequence. Let f be any function that sums $\{\lambda_n\}$ (see Lemma 2.1), and set

$$J'' = J \vee S(f).$$

Clearly, J" is an ideal set containing both J and $\{\lambda_n\}$, and therefore the proof will be complete if we show that J" $\neq$ C. To see this, define $f_k(t) = f(t/k)$ for $k = 1, 2, \cdots$, and as in Lemma 4.4, construct the sequence

$$\underbrace{\lambda_1, \cdots, \lambda_1}_{p_1}, \underbrace{\lambda_2, \cdots, \lambda_2}_{p_2}, \cdots, \underbrace{\lambda_n, \cdots, \lambda_n}_{p_n}, \cdots$$

in which each λ_n is repeated p_n times, where p_n is chosen so that $p_n f_k(\lambda_n) \geq 1$ for $k = 1, 2, \cdots, n$. As it turns out, this sequence (call it $\{\mu_n\}$) does not belong to J". Indeed, suppose, on the contrary, that $\{\mu_n\} \in$ J". Then, by the preceding lemma, there exist sequences $\{\nu_n'\}$ and $\{\nu_n''\}$ in C such that $\mu_n = \nu_n' \vee \nu_n''$ for all n, where $\{\nu_n'\}$ belongs to J while $\{\nu_n''\}$ is summed by f_{k_0} for some k_0. Consider any fixed term λ_{n_0} of the original sequence, and suppose it does not appear in the sequence $\{\nu_n'\}$. Since λ_{n_0} appears p_{n_0} times in the sequence $\{\nu_n' \vee \nu_n''\}$, there must be p_{n_0} repetitions of λ_{n_0} in $\{\nu_n''\}$. If now there exists an infinite subsequence $\{\lambda_{n_k}\}_{k=1}^{\infty}$ of $\{\lambda_n\}$ such that no term of the subsequence appears in $\{\nu_n'\}$, then each λ_{n_k} would have to appear p_{n_k} times in $\{\nu_n''\}$, and since

$$p_{n_k} \cdot f_{k_0}(\lambda_{n_k}) \geq 1$$

for $n_k \geq k_0$, this would contradict the fact that f_{k_0} sums $\{\nu_n''\}$. Thus every infinite subsequence of $\{\lambda_n\}$ must have terms in common with $\{\nu_n'\}$; from this it follows at once that all of the terms of some tail $\{\lambda_n\}_{n=m_0}^{\infty}$ of $\{\lambda_n\}$ must appear in $\{\nu_n'\}$. In other words, for some m_0 the tail $\{\lambda_{m_0+n}\}$ of $\{\lambda_n\}$ is a permutation of a subsequence of $\{\nu_n'\}$. By Lemma 1.1, the tail itself (and with it the whole sequence $\{\lambda_n\}$) must belong to J. This contradiction shows that the hypothesis $\{\mu_n\} \in$ J" is untenable, and the proof is complete.

THEOREM 4.8'. *Let* $\mathfrak{J}$ *be any ideal in* $\mathcal{L}(\mathcal{H})$ *other than* $\mathfrak{F}$, *and let* T *be an arbitrary operator in* $\mathfrak{J} \setminus \mathfrak{F}$. *Then there exists an ideal* $\mathfrak{J}'$ *such that*

$$\mathfrak{F} \subsetneqq \mathfrak{J}' \subset \mathfrak{J}$$

and such that T *does not belong to* $\mathfrak{J}'$. *Dually, if* $\mathfrak{J} \neq \mathfrak{C}$ *and if* T *is a compact operator not belonging to* $\mathfrak{J}$, *then there exists an ideal* $\mathfrak{J}''$ *such that*

$$\mathfrak{J} \subset \mathfrak{J}'' \subsetneqq \mathfrak{C}$$

and such that T *belongs to* $\mathfrak{J}''$.

5. COMBINATIONS OF IDEAL SETS

Once some ideal sets have been constructed, it is always a simple matter to construct others. Thus, if Φ is any nonempty collection of admissible functions, then the intersections

$$\bigcap_{f \in \Phi} D(f) \quad \text{and} \quad \bigcap_{f \in \Phi} S(f)$$

are both ideal sets, which we may denote by $D_\delta(\Phi)$ and $S_\delta(\Phi)$, respectively. In the dual direction, the following result is of interest.

LEMMA 5.1. *The union of two ideal sets is an ideal set if and only if one of them is included in the other. A sufficient condition for the union of a collection of ideal sets to be an ideal set is that it should be directed upward with respect to inclusion.*

Proof. Let J_1 and J_2 be ideal sets, and suppose that neither is a subset of the other, so that there exist sequences $\{\lambda_n\}$ and $\{\mu_n\}$ in $J_1 \setminus J_2$ and $J_2 \setminus J_1$, respectively. Then, since $\{\lambda_n + \mu_n\}$ cannot belong to either J_1 or J_2, we have the relation $\{\lambda_n + \mu_n\} \notin J_1 \cup J_2$, so that $J_1 \cup J_2$ is not an ideal set. The final assertion of the lemma is obvious.

It is an immediate consequence of Lemma 5.1 that if Φ is a collection of admissible functions and Φ is directed downward with respect to $\prec$, then $\bigcup_{f \in \Phi} D(f)$ and $\bigcup_{f \in \Phi} S(f)$ are also ideal sets, which we may denote by $D_\sigma(\Phi)$ and $S_\sigma(\Phi)$. In this connection, the following simple result is of interest.

LEMMA 5.2. *If* f *and* g *are admissible functions, then* $M(f \vee g) = M(f) \cap M(g)$, *while* $M(f \wedge g)$ *coincides with the set of all sequences* $\{\lambda_n \vee \mu_n\}$, *where* $\{\lambda_n\} \in M(f)$, $\{\mu_n\} \in M(g)$.

Proof. For any sequence $\{\lambda_n\}$ in C, we construct sequences $\{\lambda_n^{(1)}\}$ and $\{\lambda_n^{(2)}\}$ by setting

$$\lambda_n^{(1)} = \begin{cases} \lambda_n & (f(\lambda_n) \leq g(\lambda_n)), \\ 0 & (f(\lambda_n) > g(\lambda_n)), \end{cases}$$

$$\lambda_n^{(2)} = \begin{cases} 0 & (f(\lambda_n) \leq g(\lambda_n)), \\ \lambda_n & (f(\lambda_n) > g(\lambda_n)) \end{cases}$$

for all n. Suppose now that $\{\lambda_n\}$ is summed by both f and g. Then in particular $\sum_n g(\lambda_n^{(1)})$ and $\sum_n f(\lambda_n^{(2)})$ are both finite, and since

$$(f \vee g)(\lambda_n) = g(\lambda_n^{(1)}) + f(\lambda_n^{(2)})$$

for every n, it follows that $\{\lambda_n\}$ is summed by $f \vee g$. This shows that $M(f) \cap M(g) \subset M(f \vee g)$, and since the reverse inclusion is obvious, the first assertion of the lemma is proved. To verify the second part, suppose $\{\lambda_n\}$ is summed by $f \wedge g$. Then, since $f(\lambda_n^{(1)}) \leq (f \wedge g)(\lambda_n)$ for all n, it follows that $\{\lambda_n^{(1)}\}$ is in M(f). Similarly, $\{\lambda_n^{(2)}\}$ is in M(g), and since $\lambda_n = \lambda_n^{(1)} \vee \lambda_n^{(2)}$, we have expressed $\{\lambda_n\}$ in the prescribed fashion. Since it is clear that every such sequence is summed by $f \wedge g$, the proof is complete.

THEOREM 5.3. *For any two admissible functions* f *and* g,

$$S(f \vee g) = S(f) \cap S(g) \quad \text{and} \quad S(f \wedge g) = S(f) \vee S(g).$$

Proof. If $\alpha\{\lambda_n\}$ is summed by f and $\beta\{\lambda_n\}$ is summed by g, then $\gamma\{\lambda_n\}$ is summed by both f and g for $\gamma = \alpha \wedge \beta$. But then, by the preceding lemma, $\gamma\{\lambda_n\}$ is summed by $f \vee g$, so that $\{\lambda_n\}$ belongs to $S(f \vee g)$. This shows that $S(f) \cap S(g) \subset S(f \vee g)$, and since the reverse inclusion is obvious, the first assertion of the theorem is proved. Similarly, for the second part, it suffices to show $S(f \wedge g) \subset S(f) \vee S(g)$. But if $\{\lambda_n\}$ belongs to $S(f \wedge g)$, then $\alpha\{\lambda_n\} \in M(f \wedge g)$ for some $\alpha > 0$, and hence, by Lemma 5.2, $\alpha\lambda_n = \mu_n \vee \nu_n$, where $\{\mu_n\} \in M(f)$ and $\{\nu_n\} \in M(g)$. But then

$$\lambda_n = (1/\alpha)\mu_n \vee (1/\alpha)\nu_n,$$

and this shows that $\{\lambda_n\}$ is in $S(f) \vee S(g)$.

COROLLARY 5.4. *For any collection Φ of admissible functions, the smallest ideal set $\bigvee_{f \in \Phi} S(f)$ containing all of the ideal sets $S(f)$ coincides with the union $S_\sigma(\Gamma)$, where Γ denotes the collection of all infima $f_1 \wedge \cdots \wedge f_k$ of finite collections of functions $f_1, \cdots, f_k$ selected from Φ.*

Proof. By an obvious extension of the foregoing theorem,

$$S(f_1 \wedge \cdots \wedge f_k) = S(f_1) \vee \cdots \vee S(f_k)$$

for any admissible functions $f_1, \cdots, f_k$. From this it is clear that the ideal $S_\sigma(\Gamma)$ contains every $S(f)$ $(f \in \Phi)$, and therefore it contains $\bigvee_{f \in \Phi} S(f)$ as well. On the other hand, the same equation shows that $S(g) \subset \bigvee_{f \in \Phi} S(f)$ for every g in Γ, and the proof is complete.

Remark. It is natural to inquire whether counterparts of these results also hold for the ideals D(f) and D(g). As regards $f \vee g$, there is no difficulty. That $D(f \vee g) = D(f) \cap D(g)$ for any two admissible functions f and g may be proved by a more or less obvious modification of the corresponding part of the proof of Theorem 5.3. As regards $f \wedge g$, however, things are unclear. It is evident that the inclusion $D(f) \vee D(g) \subset D(f \wedge g)$ always holds, but at the present we do not know whether there exist admissible functions f and g such that $D(f \wedge g)$ is properly larger than $D(f) \vee D(g)$.

6. OPEN QUESTIONS

This paper raises many more questions than it answers. An obvious example is the question just asked: can $D(f \wedge g)$ differ from $D(f) \vee D(g)$? This question is distinguished by having to do with the very apparatus of construction techniques under discussion—by being what might be called an "internal" question. A similar query is the following: can an ideal set S(f) coincide with an ideal set D(g) in a nontrivial way? To clarify the question, recall that if f satisfies condition (2) of Theorem 4.2, then $S(f) = D(f) = M(f)$. The question is: can $S(f) = D(g)$, where either f or g fails to satisfy (2)?

A different—and intrinsically more interesting—class of questions concerns the relations between the ideal sets constructible by the methods discussed above and the ideal structure at large. For example, is every ideal set of one of the forms $S_\sigma(\Phi)$, $S_\delta(\Phi)$, $D_\sigma(\Phi)$, $D_\delta(\Phi)$? (The p-ideal sets are of this form, for trivial reasons. On the other hand, $C = S_\sigma(\Phi)$, where Φ denotes the system of all admissible functions, and similarly $F = D_\delta(\Phi)$.) Can the ideal set corresponding to a norm ideal be nontrivially of the form $S_\sigma(\Phi)$ (or any of the other three forms) for a *countable* family Φ? We do not know. In a different direction, we may ask whether there exists a sequence $\{\lambda_n\}$ in $C \setminus F$ such that the principal ideal set $(\{\lambda_n\})$ generated by $\{\lambda_n\}$ coincides with the intersection of all the ideal sets S(f) such that f sums $\{\lambda_n\}$. Alternatively, does there exist a sequence $\{\lambda_n\}$ in $C \setminus F$ for which this is *not* the case? Again, we do not know. In a different direction, we may ask: for a given ideal set J, does there exist as admissible function f such that $J \subset M(f)$, or, alternatively, such that $M(f) \subset J$? In this connection, to sound faintly a positive note, we mention the following fragmentary result, which is to be compared with Theorem 4.6.

PROPOSITION 6.1. *For each sequence* $\{\lambda_n\}$ *in* $C \setminus F$, *there exist admissible functions* f *and* g *such that* $\{\lambda_n\}$ *belongs to* D(f) *but not to* S(g).

Proof. To establish the assertion concerning f, we first choose a sequence $\{\alpha_n\}$ that tends to infinity so slowly that $\mu_n = \alpha_n \cdot \lambda_n$ still belongs to C, and then we construct f so that $\{\mu_n\}$ is summed by f. To establish the assertion concerning g, choose any sequence $\{\alpha_n\}$ of positive numbers that tends to zero, and then construct g so that it does not sum the sequence $\{\alpha_n \cdot \lambda_n\}$.

Finally, there remain many open questions concerning the ideal structure of $\mathscr{L}(\mathscr{H})$ that have nothing to do with the constructions of this paper but are nevertheless suggested by them. For example: do there exist two ideals different from $\mathfrak{C}$ whose join is $\mathfrak{C}$? (If either of them corresponds to an ideal set of the form S(f), then the answer is no; this is intrinsic in the proof of Theorem 4.7.) Is $\mathfrak{C}$ the join of a countable family of strictly smaller ideals? How about the dual questions?

REFERENCES

1. A. Brown and C. Pearcy, *Compact restrictions of operators. II* (to appear).

2. J. W. Calkin, *Two-sided ideals and congruences in the ring of bounded operators in Hilbert space.* Ann. of Math. (2) 42 (1941), 839-873.

3. I. C. Gohberg and M. G. Kreĭn, *Introduction to the theory of linear nonselfadjoint operators.* Translated from the Russian by A. Feinstein. Translations of Mathematical Monographs, Vol. 18. American Mathematical Society, Providence, R.I., 1969.

4. R. Schatten, *Norm ideals of completely continuous operators.* Ergebnisse der Mathematik und ihrer Grenzgebiete. N.F., Heft 27. Springer-Verlag, Berlin, 1960.

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