COMPACT, CONTRACTIBLE n-MANIFOLDS AND THEIR BOUNDARIES

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The purpose of this paper is to show that for $n \geq 6$ the function, assigning M to ∂M, from the set of compact, contractible n-manifolds to the set of homology (n - 1)-spheres is a bijection. (A similar result has been announced by M. Kato, see [5].) We also show that for M as above, the group of concordance classes of homeomorphisms of M onto itself is isomorphic to the group of concordance classes of homeomorphisms of ∂M onto itself.

We consider both PL and topological manifolds and maps in this paper. The term manifold allows the possibility that the boundary is not empty. We use ∂M to denote the boundary of a manifold M and int M to denote the interior of M. We use D^n and S^n to denote the standard n-cell and n-sphere. By the term disk we mean a 2-cell. We use $A * B$ to denote the join of spaces A and B. If M is a manifold and P a subpolyhedron of M, then N(P, M) denotes a regular neighborhood of P in M, see J. F. P. Hudson and E. C. Zeeman [4]. Finally, if M and N are manifolds and $h: \partial M \to \partial N$ is a homeomorphism, we denote by $M \cup_h N$ the manifold obtained by identifying $x \in \partial M$ with $h(x) \in \partial N$. We let ρ_M and ρ_N denote the inclusions of M and N, respectively, into $M \cup_h N$. Hence

$$\rho_N^{-1} \circ \rho_M \mid \partial M = h.$$

Furthermore, if $C \subseteq M$ and $C' \subseteq N$, then $C \cup_h C'$ denotes $\rho_M(C) \cup \rho_N(C')$. Although C and C' may also be manifolds with boundary, no confusion should arise between the two uses of the notation $\cup_h$. We shall also not distinguish between $A \subseteq M$ and $\rho_M(A) \subseteq M \cup_h N$ when no confusion can arise.

LEMMA 1. *Let* M *and* N *be contractible* PL *n-manifolds* $(n \geq 5)$. *Let* $h: \partial M \to \partial N$ *be a homeomorphism,* $J \subseteq \partial M$ *a simple closed curve,* $D \subseteq M$ *a disk such that* $D \cap \partial M = J$. *Suppose* T *is a regular neighborhood of* J *in* ∂M, *and let* C *be a regular neighborhood of* $D \cup T$ *in* M, *relative to* $c\ell(\partial M - T)$. *Moreover,* D' *denotes a disk in* N *such that* $D' \cap \partial N = h(J)$, *and* C' *denotes a regular neighborhood of* $D' \cup h(T)$ *in* N, *relative to* $c\ell(\partial N - h(T))$. *Then there exists a* PL *homeomorphism*

$$f: (C \cup_h C', T, D \cup_h D') \to (S^2 \times D^{n-2}, S^1 \times D^{n-2}, S^2),$$

where $C \cup_h C'$ *and* $D \cup_h D'$ *are subsets of* $M \cup_h N$.

Proof. Consider $S = M \cup_h N$. By Van Kampen's theorem, S is simply connected. By the Mayer-Vietoris sequence, S has the same homology groups as ∂M. From the Lefschetz duality theorem [2], we obtain the following diagram:

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$$\begin{array}{ccccccccc}
\longrightarrow H^{q-1}(M, \partial M) & \longrightarrow & H^{q-1}(M) & \longrightarrow & H^{q-1}(\partial M) & \longrightarrow & H^{q}(M, \partial M) \longrightarrow \\
\downarrow \approx & & \downarrow \approx & & \downarrow \approx & & \downarrow \approx \\
\longrightarrow H_{n-q+1}(M) & \longrightarrow & H_{n-q+1}(M, \partial M) & \longrightarrow & H_{n-q}(\partial M) & \longrightarrow & H_{n-q}(M) \longrightarrow
\end{array}$$

Since

$$H_i(M) = \begin{cases} 0 & \text{if } i \neq 0, \\ Z & \text{if } i = 0 \end{cases} \quad \text{and} \quad H^i(M) = \begin{cases} 0 & \text{if } i \neq 0, \\ Z & \text{if } i = 0, \end{cases}$$

we see that S has the same homology groups as S^n. Hence, by the generalized Poincaré theorem (see J. B. Wagoner [9, Theorem 5.1] for a PL version) or the h-cobordism theorem (see J. Milnor [7] or J. F. P. Hudson [3]), there exists a PL homeomorphism $\phi: S \to S^n$. Let $K = D \cup_h D'$. Then $K \approx S^2$, and by Zeeman's unknotting theorem [10], we can assume that $\phi(D) = v * S^1$ and $\phi(D') = w * S^1$, where $S^2 = \{v, w\} * S^1$. Furthermore, we assume that $S^n = S^2 * S^{n-3}$ and that $S^2 \times D^{n-2}$ is canonically embedded in this join structure. Since $\phi(C)$ is a regular neighborhood of $\phi(D) = v * S^1$ (mod $w * S^1$), there exists a homeomorphism $\psi: S^n \to S^n$ such that $\psi \mid S^2$ is the identity and

$$\psi \circ \phi(C) = (v * S^1) \times D^{n-2}.$$

Since $\psi \circ \phi(T)$ is a regular neighborhood in $\partial((v * S^1) \times D^{n-2})$ of $\psi \circ \phi(J) = S^1$, there exists a homeomorphism θ of $\partial((v * S^1) \times D^{n-2})$ onto itself, taking $\psi \circ \phi(T)$ onto $S^1 \times D^{n-2}$ such that $\theta \mid S^1$ is the identity. Extend θ to all of S^n, keeping it fixed on S^2. We now see that $\theta \circ \psi \circ \phi$ is a homeomorphism taking (C, T) onto $((v * S^1) \times D^{n-2}, S^1 \times D^{n-2})$ and that $\theta \circ \psi \circ \phi(D') = w * S^1$. Since $\theta \circ \psi \circ \phi(C')$ is a regular neighborhood in S^n of

$$\theta \circ \psi \circ \phi(D' \cup h(T)) = w * S^1 \cup S^1 \times D^{n-2} \ (\text{mod } \theta \circ \psi \circ \phi(\overline{\partial N - h(T)}))$$

and hence a regular neighborhood in $\overline{S^n - (v * S^1) \times D^{n-2}}$ of

$$(w * S^1) \cup S^1 \times D^{n-2} \ (\text{mod } \partial(\overline{(v * S^1) \times D^{n-2} - S^1 \times D^{n-2}})),$$

there exists a homeomorphism η of S^n onto itself taking $\theta \circ \psi \circ \phi(C')$ onto $(w * S^1) \times D^{n-2}$ such that η is fixed on $w * S^1 \cup (v * S^1) \times D^{n-2}$. Hence $\eta \circ \theta \circ \psi \circ \phi \mid C \cup_h C'$ is the required homeomorphism. #

THEOREM 1. *Let* M *and* N *denote compact, contractible* PL n-*manifolds* ($n \geq 5$). *Let* h *be a* PL *homeomorphism of* ∂M *onto* ∂N. *Then there exist neighborhoods* W *and* Y *of* ∂M *and* ∂N, *respectively, and a* PL *homeomorphism* $H: W \to Y$ *such that*

i) W *is* 1-*connected, and*

ii) $H \mid \partial M = h$.

Proof. Let $U \approx (\partial M) \times [0, 1]$ and $V \approx (\partial N) \times [0, 1]$ be collars of ∂M and ∂N, respectively. Extend h to take U onto V. Let $J_1, J_2, \cdots, J_k$ be a collection of loops in $(\partial M) \times 1$ such that $[J_1], [J_2], \cdots, [J_k]$ generate $\pi_1(\partial M \times 1)$. Since $n \geq 5$, we can assume that each J_i is a simple closed curve and that the J_i are pairwise disjoint. Since closure (M - U) is homeomorphic to M, each J_i can be shrunk to a point in closure (M - U). Since $n \geq 5$, we can assume that each J_i bounds a disk D_i

such that $D_i \cap U = J_i$ and the D_i are pairwise disjoint. The neighborhood W is obtained by thickening up the D_i and adding them to U; that is, $W = U \cup \bigcup_{i=1}^k C_i$, where $C_i = N(D_i, c\ell(M - U))$ is a regular neighborhood of D_i in $c\ell(M - U)$ that intersects $(\partial M) \times 1$ in a regular neighborhood of $\partial D_i = J_i$. Now repeat the same process in N, use the curves $h(J_i)$ to obtain disks E_i such that $E_i \cap V = h(J_i)$, and thicken up the E_i to $F_i = N(E_i, c\ell(N - V))$, a regular neighborhood of E_i in $c\ell(N - V)$ that intersects $(\partial N) \times 1$ in $h(C_i \cap (\partial M \times 1))$. Let $Y = V \cup \bigcup_{i=1}^k F_i$.

We consider $C_i \cup_h F_i$ as a subset of $c\ell(M - U) \cup_h c\ell(N - V)$. By Lemma 1, there exists a homeomorphism

$$g_i: C_i \cup_h F_i \to S^2 \times D^{n-2}$$

taking C_i onto $(v * S^1) \times D^{n-2}$, F_i onto $(w * S^1) \times D^{n-2}$, and $C_i \cap F_i$ onto $S^1 \times D^{n-2}$. Let $\theta: S^2 \to S^2$ be a homeomorphism with the properties that

$$\theta \mid S^1 = \text{id}, \quad \theta(v * S^1) = w * S^1, \quad \text{and} \quad \theta(w * S^1) = v * S^1.$$

Let $\phi: S^2 \times D^{n-2} \to S^2 \times D^{n-2}$ be defined by $\phi = \theta \times \text{id}$. Then $g_i^{-1} \circ \phi \circ g_i$ is a homeomorphism of $C_i \cup_h F_i$ onto itself, which is the identity on

$$C_i \cap F_i = C_i \cap \partial M \times 1 = F_i \cap \partial N \times 1,$$

and $g_i^{-1} \circ \phi \circ g_i$ takes C_i onto F_i. We extend $h: U \to V$ to $H: W \to Y$ by the relation

$$H(x) = \begin{cases} h(x) & \text{if } x \in U, \\ \rho_N^{-1} \circ g_i^{-1} \circ \phi \circ g_i \circ \rho_M(x) & \text{if } x \in C_i, \end{cases}$$

where ρ_M and ρ_N are the inclusions of $c\ell(M - U)$ and $c\ell(N - V)$, respectively, into $c\ell(M - U) \cup_h c\ell(N - V)$. #

LEMMA 2. *Let* N *be a contractible, open* PL n-*manifold* ($n \geq 5$), *and let* S *be a* PL *bicollared* (n - 1)-*sphere in* N. *Then the closed interior of* S *is a* PL n-*cell.*

Proof. Since S is PL bicollared, it follows that C, the closed interior of S, is a PL n-manifold. By Van Kampen's theorem, the Mayer-Vietoris sequence, and the Hurewicz theorem, C has the homotopy groups of an n-cell. Hence it follows from the generalized Poincaré theorem that C is a PL n-cell.

THEOREM 2. *Let* M *and* N *be compact, contractible* PL n-*manifolds* ($n \geq 5$). *Let* U *and* V *be neighborhoods of* ∂M *and* ∂N, *respectively, such that* U *and* V *are* 1-*connected. Suppose further that* $h: U \to V$ *is a* PL *homeomorphism with* $h(\partial M) = \partial N$. *Then there exists a* PL *homeomorphism* $H: M \to N$ *such that* $H \mid M - C = h \mid M - C$, *for some* n-*cell* $C \subseteq \text{int } M$. (*In particular,* $H \mid \partial M = h \mid \partial M$.)

Proof. Let T' be a triangulation of M. Let T be a triangulation of int M obtained from T' by subdivision and having the property that for every $\sigma \in T$, the inequality $\text{dia } \sigma < \rho(\sigma, \partial M)$ holds. Let T^2 denote the 2-skeleton of T.

Since M is contractible and U is 1-connected, it follows from the homotopy exact sequence for a pair that (M, U) is 2-connected. Furthermore, the dimension of T^2 is 2, and $T^2 - U$ is compact. Therefore we can apply Stalling's Engulfing Theorem [8] to obtain a compact set $F \leq \text{int } M$ and a homeomorphism $g_1: M \to M$ such that g_1 is the identity except on F and such that $T^2 \subseteq g_1(U)$.

Let S be the polyhedron obtained by adding to T^2 all closed simplexes of T that intersect neither M - U nor F. Then $S \subseteq g_1(U)$. Let S' be the complementary skeleton to S (that is, S' is the union of all simplexes of the first barycentric subdivision of T that do not intersect S). Then S' is a compact polyhedron whose dimension is at most n - 3.

Let C be a piecewise-linear n-cell in int M, and let int C denote the interior of C. Since both int M and int C are contractible, the homotopy exact sequence for a pair says that the pair (int M, int C) is (n - 3)-connected. Applying the Engulfing Theorem again, we obtain a homeomorphism g_2: int M → int M, which is the identity except on a compact set contained in int M, and for which $S' \subseteq g_2(\text{int } C)$.

Since S and S' are complementary skeletons, we can apply a theorem like Theorem 8.1 of [8] to get a homeomorphism g_3: int M → int M such that

$$\text{int } M \subseteq g_1(U) \cup g_3 \circ g_2(\text{int } C) .$$

Hence $\text{int } M \subseteq U \cup g_1^{-1} \circ g_3 \circ g_2(C)$, and the boundary of C is a piecewise-linearly bicollared (n - 1)-sphere in U. Thus h(Bd C) is a piecewise-linearly bicollared (n - 1)-sphere in $V \subseteq N$, and by Lemma 2 the closed interior of h(Bd C) is an n-cell D. Since h takes the closed exterior of C onto the closed exterior of D, $h \mid M - \text{int } C$ can be extended to a homeomorphism H, taking M onto N. #

The referee has pointed out that the triangulation theorems due to R. C. Kirby and L. C. Siebenmann [6] imply that any homology n-sphere $(n \geq 5)$ and any compact, contractible n-manifold $(n \geq 6)$ [if we assume the boundary has a unique PL structure, $n \geq 5$ is enough] has a unique PL structure. Hence the assumption of a PL structure in the results above is generally redundant. We shall continue to assume explicitly a PL structure in the statements of lemmas, but in the statements of the major results we shall state what can be proved using the above-mentioned triangulation theorems.

COROLLARY 2.1. *Let* M *and* N *be compact, contractible* PL *n-manifolds* $(n \geq 5)$. *If* ∂M *and* ∂N *are* PL *homeomorphic, then* M *and* N *are* PL *homeomorphic. In fact, if* h: $\partial M \to \partial N$ *is a* PL *homeomorphism, then* h *extends to a* PL *homeomorphism* H: M → N. *Furthermore, if* $n \geq 6$, *we need not assume the* PL *structure on* M *and* N, *and it follows that any homeomorphism* h: $\partial M \to \partial N$, *not necessarily* PL, *extends to a homeomorphism* H: M → N.

Proof. If h: $\partial M \to \partial N$ is PL, then, by Theorem 1, h extends to h': W → Y, where W is a 1-connected neighborhood of ∂M. By Theorem 2, there exists a PL homeomorphism H: M → N that agrees with h' near ∂M. If $n \geq 6$, then M and N have unique PL structures [6], and furthermore h: $\partial M \to \partial N$ is isotopic to a PL homeomorphism g: $\partial M \to \partial N$. Let U and V be closed PL collar neighborhoods of ∂M and ∂N in M and N, respectively. Use the isotopy between h and g to extend h to H' from U onto V such that H' is PL on the inner boundary component of U. Now apply the first part of the corollary to closure M - U and closure N - V. #

COROLLARY 2.2. *Let* M *be a compact, contractible topological n-manifold, locally flatly embedded in* S^n $(n \geq 5)$ (*if* n = 5, *we assume further that* M *is a* PL *manifold and is* PL *embedded in* S^n). *If* N = closure $(S^n - M)$ *is simply connected, then* N *is homeomorphic to* M.

Proof. By the Mayer-Vietoris sequence, N has the homology of a point. Since we assumed N was simply connected, it has the homotopy of a point and hence is contractible. Since N is a manifold and since $\partial M = \partial N$, Corollary 2.1 implies that M is homeomorphic to N. #

Let $H_{TOP}(M)$ denote the group of topological homeomorphisms of M onto itself, and let $H_{PL}(M)$ denote the subgroup of PL homeomorphisms of M onto itself. Let $C_{TOP}(M)$ denote the group of concordance classes of topological homeomorphisms of M onto itself and $C_{PL}(M)$ denote the group of PL concordance classes of PL homeomorphisms of M onto itself.

COROLLARY 2.3. *Let* M *be a compact, contractible* n-*manifold. Let* $\phi: H_i(M) \to H_i(\partial M)$ (i = TOP *or* i = PL) *be the restriction-induced homomorphism, that is,* $\phi(h) = h \mid \partial M$. *If* $n \geq 5$ *and if* M *is a* PL *manifold, then*

$$\phi: H_{PL}(M) \to H_{PL}(\partial M)$$

is onto, and if $n \geq 6$, *then* $\phi: H_{TOP}(M) \to H_{TOP}(\partial M)$ *is onto.*

Proof. If $g: \partial M \to \partial M$ is a homeomorphism, apply Corollary 2.1 to get a homeomorphism $G: M \to M$ extending g. #

Clearly, ϕ will not be one-to-one, in other words, the extension of g is not unique; but it is unique up to concordance, as the next corollary shows.

COROLLARY 2.4. *Let* M *be a compact, contractible* n-*manifold. Let* $\psi: C_i(M) \to C_i(\partial M)$ (i = TOP *or* i = PL) *be the restriction-induced homomorphism, that is,* $\psi([h]) = [h \mid \partial M]$. *If* $n \geq 5$ *and if* M *is a* PL *manifold, then*

$$\psi: C_{PL}(M) \to C_{PL}(\partial M)$$

is an isomorphism. If $n \geq 6$, *then each of the homomorphisms in the following diagram is an isomorphism:*

$$\begin{array}{ccc} C_{PL}(M) & \xrightarrow{\psi} & C_{PL}(\partial M) \\ \downarrow \subseteq & & \downarrow \subseteq \\ C_{TOP}(M) & \xrightarrow{\psi} & C_{TOP}(\partial M) \end{array}$$

Proof. It follows from Corollary 2.3 that ψ is onto. To show that ψ is one-to-one, let $h: M \to M$ be a homeomorphism. If $h \mid \partial M$ is PL concordant to the identity, then there exists a PL homeomorphism $G: (\partial M) \times I \to (\partial M) \times I$ such that

$$G \mid (\partial M) \times \{0\} = \text{identity} \quad \text{and} \quad G \mid (\partial M) \times \{1\} = h \mid \partial M .$$

It follows from the Generalized Poincaré Theorem that $M \times I \approx I^{n+1}$. Of course,

$$\partial(M \times I) = M \times \{0\} \cup M \times \{1\} \cup (\partial M) \times I .$$

Hence we can define $H': \partial(M \times I) \to \partial(M \times I)$ by the conditions

$$H' \mid M \times \{0\} = \text{identity}, \quad H'(M \times \{1\}) = h, \quad \text{and } H' \mid (\partial M) \times I = G .$$

Since $M \times I \approx I^{n+1}$, H' extends to $H: M \times I \to M \times I$, and therefore h is concordant to identity. Hence $\psi: C_{PL}(M) \to C_{PL}(\partial M)$ is one-to-one.

If $n \geq 6$, then M has a unique PL structure [6]. Furthermore, if h is a homeomorphism from either M onto M or ∂M onto ∂M, then h is isotopic to a PL homeomorphism [6]. Hence $C_{PL}(M) \xrightarrow{\subseteq} C_{TOP}(M)$ and $C_{PL}(\partial M) \xrightarrow{\subseteq} C_{TOP}(\partial M)$ are both onto. Again appealing to [6], we see that if H: $(\partial M) \times I \to (\partial M) \times I$ is a concordance and $H \mid \partial M \times \{0, 1\}$ is a PL homeomorphism, then H is isotopic, staying fixed on $\partial M \times \{0, 1\}$, to a PL homeomorphism. Hence $C_{PL}(\partial M) \xrightarrow{\subseteq} C_{TOP}(\partial M)$ is one-to-one. This last statement, together with commutativity, implies that $C_{PL}(M) \xrightarrow{\subseteq} C_{TOP}(M)$ is also one-to-one. Hence all the homomorphisms are isomorphisms. #

One interpretation of Corollary 2.1 is that each homology n-sphere $(n \geq 4)$ bounds at most one contractible $(n+1)$-manifold. We show next (Theorem 3) that each homology n-sphere $(n \geq 5)$ bounds at least one contractible $(n+1)$-manifold. To do this, we need some facts about regular neighborhoods of 2-manifolds. The following results are included in an unpublished paper of this author. More elegant proofs will appear in R. Dieffenbach's dissertation [1].

THEOREM. *Let* T *be a compact, orientable* 2-*manifold,* PL *embedded in an orientable* PL n-*manifold* $(n \geq 5)$. *Let* R *be a regular neighborhood of* T *relative to* ∂T.

1) *If* $\partial T \neq \emptyset$, *then* (R, T) *is* PL *homeomorphic to* $(T \times D^{n-2}, T \times 0)$.

2) *If* $\partial T = \emptyset$, *then there are exactly two possibilities:*

a) (R, T) *is* PL *homeomorphic to* $(T \times D^{n-2}, T \times 0)$, *or*

b) (R, T) *is* PL *homeomorphic to*

$$(T_1 \times D^{n-2}, T_1 \times 0) \cup_\gamma (T_2 \times D^{n-2}, T_2 \times 0),$$

where T_1 *and* T_2 *are submanifolds of* T *with* $T_1 \cup T_2 = T$, *where* $T_1 \cap T_2 = \partial T_1 = \partial T_2$ *is a simple closed curve, and where*

$$\gamma: (\partial T_1) \times D^{n-2} \to (\partial T_2) \times D^{n-2}$$

is the homeomorphism generated by the nontrivial element of

$$\pi_1(SO(n-2)) \approx Z_2 .$$

Since the two possibilities in case b) are distinct and since the regular neighborhoods of T in S^n must be a product, we obtain the following corollary.

COROLLARY. *Let* S *and* T *be compact, orientable* 2-*manifolds,* PL *properly embedded in* D^n $(n \geq 5)$, *that is,* $\partial S \subseteq \partial D^n$ *and* $\text{int}\, S \subseteq \text{int}\, D^n$. *Suppose that* $\partial S = \partial T$ *is a simple closed curve. Let* h: $(\partial S) \times D^{n-2} \to \partial D^n$ *be a* PL *embedding. Then* h *extends to* H: $S \times D^{n-2} \to D^n$ *if and only if* h *extends to* G: $T \times D^{n-2} \to D^n$.

LEMMA 3. *Let* M *be a* PL n-*manifold* $(n \geq 5)$ *such that* $\pi_1(M) = 1$. *Suppose* T *is a closed, orientable* PL 2-*manifold contained in* M *with a product neighborhood* N; *that is, there exists a* PL *homeomorphism* h: $(T \times D^{n-2}, T \times \{0\}) \to (N, T)$ *with the property that* $h(t, 0) = t$ *for every* $t \in T$. *Under these hypotheses there exists a* PL 2-*sphere* $S \subseteq M$ *such that*

i) S *is homologous to* T, *and*

ii) *a regular neighborhood of* S *is* PL *homeomorphic to* $S \times D^{n-2}$.

Proof. The proof is by induction on $\chi(T)$, the Euler characteristic of T. If $\chi(T) = 2$, we are done. If $\chi(T) < 2$, we show that we can find a surface T' satisfying the hypothesis of the lemma and the inequality $\chi(T') > \chi(T)$.

Suppose $\chi(T) < 2$. There exists a simple closed curve J_1 on T such that J_1 does not separate T. Hence there exists a simple closed curve J_2 on T such that $J_1 \cap J_2$ is a single point p and J_1 and J_2 cross at p. Since $\pi_1(M) = 1$ and since the dimension of M is at least 5, there exist disks D_1 and D_2 such that

$$\partial D_1 = J_1, \quad \partial D_2 = J_2, \quad D_1 \cap D_2 = p, \quad D_1 \cap T = J_1, \quad \text{and} \quad D_2 \cap T = J_2.$$

Since $D_1 \cap D_2 = p$, it follows that $D_1 \cup D_2$ is collapsible. Let R be a regular neighborhood of $D_1 \cup D_2$ that intersects T in a regular neighborhood of $J_1 \cup J_2$. We assume further that $R \cap N = h((R \cap T) \times D^{n-2})$. In particular, we have that $R \cap T$ is a punctured torus and $h(\partial(R \cap T) \times D^{n-2}) \subseteq \partial R$. Of course, R is an n-cell. Let D be a disk in R such that $\partial D = D \cap \partial R = \partial(R \cap T)$. Since $h \mid \partial(R \cap T) \times D^{n-2}$ extends to $(R \cap T) \times D^{n-2}$, it follows from the above-mentioned corollary that $h \mid \partial(R \cap T) \times D^{n-2}$ extends to take $D \times D^{n-2}$ into R also. Hence the desired 2-manifold is $T' = (T - (R \cap T)) \cup D$. Clearly, T' is a closed, orientable 2-manifold in M, and if N' is a regular neighborhood of T', then $(T' \times D^{n-2}, T \times 0)$ is homeomorphic to (N', T'). Finally, since T and T' differ only inside the cell R, it follows that T is homologous to T'. #

LEMMA 4. *Let* M *be a* PL n*-manifold with the homology groups of* S^n $(n \geq 5)$. *Let* $J_1, \cdots, J_k$ *be disjoint* PL *simple closed curves in* $M \times \{1\} \subseteq M \times [0, 1]$ *whose homotopy classes generate* $\pi_1(M \times \{1\})$. *Then we can add* 2*-handles* $h_1, \cdots, h_k$ *to* $M \times [0, 1]$ *along* $J_1, \cdots, J_k$, *respectively, so that if*

$$M' = M \times [0, 1] \cup h_1 \cup \cdots \cup h_k \quad \text{and} \quad K = \partial M' - M \times \{0\},$$

then

a) $\pi_1(M') \approx \pi_1(K) \approx 1$,

b) $H_i(M', Z) \approx 0$ $(i \neq 0, 2, n)$ *and* $H_0(M', Z) \approx H_n(M', Z) \approx Z$,

c) $H_2(M', Z) \approx Z \oplus \cdots \oplus Z$ (k *times*), *and*

d) *there exist disjoint* PL 2*-spheres* $S_1, \cdots, S_k$ *in* K *with the properties that the homology classes of the* S_i *generate* $H_2(M', Z)$ *and each* S_i *has a neighborhood* U_i *such that* $(U_i, S_i) \overset{PL}{\approx} (S_i \times D^{n-2}, S_i \times \{0\})$.

Proof. Since $H_1(M, Z) = 0$, there exist pairwise disjoint, connected, orientable surfaces $T_1, T_2, \cdots, T_k$ in $M \times \{1\}$ such that $\partial T_i = J_i$ $(1 \leq i \leq k)$. Let N_i be a regular neighborhood of T_i in $M \times \{1\}$ relative to ∂T_i. Then, by the theorem on regular neighborhoods of 2-manifolds, there exists a homeomorphism

$$g_i: (T_i \times D^{n-2}, T_i \times 0) \to (N_i, T_i)$$

such that $g_i(t, 0) = t$ for every $t \in T_i$. Let D_i be a 2-disk, and let W_i be the closed, orientable 2-manifold obtained by identifying ∂D_i with $J_i = \partial T_i$. Let h_i denote the 2-handle $D_i \times D^{n-1}$ with attaching homeomorphism $g_i \mid J_i \times D^{n-1}$. Then, clearly, g_i can be extended to an embedding

$$G_i: W_i \times D^{n-1} \to N_i \cup h_i \subseteq M \times [0, 1] \cup h_i.$$

Let $M_0 = M \times [0, 1]$, and let $M_i = M_0 \cup h_1 \cup \cdots \cup h_i$ $(1 \leq i \leq k)$.

It follows easily from Van Kampen's theorem that $\pi_1(M_k) = 1$. Furthermore, if γ is a loop in $K = (\partial M_k) - M \times \{0\}$, then γ is trivial in M_k and hence in

$M \times \{1\} \cup h_1 \cup \cdots \cup h_k$. But the h_i are 2-handles of dimension $n + 1 \geq 6$; hence, by general position, γ is trivial missing the cores of the h_i (that is, missing $G_i(D_i \times \{0\}) \subseteq h_i$). But $h_i - G_i(D_i \times \{0\})$ deformation retracts to

$$h_i - G_i(D_i \times \operatorname{int} D^{n-1}).$$

Hence γ is trivial in

$$M \times \{1\} \cup (h_1 - G_1(D_1 \times \operatorname{int} D^{n-1})) \cup \cdots \cup (h_k - G_k(D_k \times \operatorname{int} D^{n-1})) = K,$$

that is, $\pi_1(K) = 1$.

The Mayer-Vietoris sequence shows that $H_i(M_k, Z) = 0$ if $i \neq 0, 2, n$ and that $H_0(M_k, Z) \approx H_n(M_k, Z) \approx Z$. Furthermore, it follows from the Mayer-Vietoris sequence that $H_2(M_k, Z)$ is generated by the homology classes of the surfaces W_i $(1 \leq i \leq k)$. Since each W_i has a product neighborhood in K, it follows from Lemma 3 that there exist 2-spheres $S_1, \cdots, S_k$ in K such that the homology classes of the S_i generate $H_2(M, Z)$ and such that each S_i has a product neighborhood. Hence $M' = M_k$ is the desired manifold. #

THEOREM 3. *Let* M *be a topological* n-*manifold with the homology groups of* S^n $(n \geq 5)$. *Then there exists a compact, contractible* $(n+1)$-*manifold* N *such that* $\partial N \approx M$.

Proof. By [6], M has a unique PL structure. We start with $M \times I$, add 2-handles along $M \times \{1\}$ to kill the fundamental group, then add 3-handles along the altered $M \times \{1\}$ to kill the second homotopy group. It now follows that the boundary component obtained from $M \times \{1\}$ is an n-sphere; therefore we can complete the construction by attaching an $(n+1)$-cell.

Let $J_1, \cdots, J_k$ denote pairwise disjoint, PL simple closed curves in $M \times \{1\}$ whose homotopy classes generate $\pi_1(M \times \{1\})$. We add handles $d_1, d_2, \cdots, d_k$ to $M \times I$ to obtain an $(n+1)$-manifold M' with the properties stated in Lemma 4. By conclusion d) of Lemma 4, there exist pairwise disjoint 2-spheres $S_1, S_2, \cdots, S_k$ in $K = \partial(M') - M \times \{0\}$ whose homology classes generate $H_2(M', Z)$ and that have product neighborhoods. Hence we can attach 3-handles $C_1, C_2, \cdots, C_k$ to M' along $S_1, \cdots, S_k$ to obtain an $(n+1)$-manifold M''. Let $N_0 = M'$ and $N_i = N_{i-1} \cup C_i$ $(0 < i \leq k)$. Then $M'' = N_k$. We can use the Mayer-Vietoris sequence to show that, for $1 \leq i \leq k$,

$$H_j(N_i, Z) \approx \begin{cases} 0 & \text{if } j \neq 0, 2, n, \\ Z & \text{if } j = 0 \text{ or } j = n, \\ Z \oplus Z \oplus \cdots \oplus Z \text{ (k - i times)} & \text{if } j = 2. \end{cases}$$

Furthermore, Van Kampen's theorem shows that $\pi_1(N_i) = 1$ for $1 \leq i \leq k$. We now have that

$$H_j(M'', Z) \approx \begin{cases} 0 & \text{if } j \neq 0, n, \\ Z & \text{if } j = 0 \text{ or } j = n. \end{cases}$$

By the Universal Coefficient Theorem for cohomology, we have the relations

$$H^j(M'', Z) = \begin{cases} 0 & \text{if } j \neq 0, n, \\ Z & \text{if } j = 0 \text{ or } j = n. \end{cases}$$

Hence from the Lefschetz Duality theorem [2] we get the diagram

$$\begin{array}{ccccccccc} \longrightarrow H^{q-1}(M'', Z) & \longrightarrow & H^{q-1}(\partial M'', Z) & \longrightarrow & H^q(M'', \partial M'', Z) & \longrightarrow & H^q(M'', Z) & \longrightarrow \\ \downarrow \approx & & \downarrow \approx & & \downarrow \approx & & \downarrow \approx & \\ \rightarrow H_{n-q+2}(M'', \partial M'', Z) & \rightarrow & H_{n-q+1}(\partial M'', Z) & \rightarrow & H_{n-q+1}(M'', Z) & \rightarrow & H_{n-q+1}(M'', \partial M'', Z) & \rightarrow . \end{array}$$

Thus

$$H_j(\partial M'', Z) \approx \begin{cases} 0 & \text{is } j \neq 0, n, \\ Z \oplus Z & \text{if } j = 0 \text{ or } j = n. \end{cases}$$

Finally the Mayer-Vietoris sequence, applied to $\partial M''$, gives us the relations

$$H_j(\partial M'' - M \times \{0\}, Z) \approx \begin{cases} 0 & \text{if } j \neq 0, n, \\ Z & \text{if } j = 0 \text{ or } j = n. \end{cases}$$

Thus $\partial M'' - M \times \{0\}$ is a homology sphere. To show that $\partial M'' - M \times \{0\}$ is a homotopy sphere, we need only prove that $\pi_1(\partial M'' - M \times \{0\}) = 1$. Recall that $K = \partial M' - M \times \{0\}$ and that $\pi_1(K) = 1$. Furthermore,

$$\partial M'' - M \times \{0\} = \left[K \cup \bigcup_{i=1}^{k} \partial C_i \right] - \bigcup_{i=1}^{k} \operatorname{int}(C_i \cap K).$$

It follows from Van Kampen's theorem that $\pi_1\left(K \cup \bigcup_{i=1}^k C_i\right) = 1$; and since each C_i is a cell of dimension at least 6 and since it is a 3-handle, it follows from general position arguments that every loop in $\partial M'' - M \times \{0\}$ can be shrunk in $K \cup \bigcup_{i=1}^k C_i$, missing the core of each C_i. Since C_i minus its core deformation retracts to $C_i - \operatorname{int}(C_i \cap K)$, it follows that $\pi_1(\partial M'' - M \times \{0\}) = 1$.

Thus $\partial M'' - M \times \{0\}$ is a homotopy sphere, and by the generalized Poincaré theorem, it is a sphere. Hence we complete the construction of the manifold N by attaching an $(n+1)$-ball to M'' along $\partial M'' - M \times \{0\}$ (so that $N = M'' \cup B^{n+1}$). Van Kampen's theorem shows that N is simply connected. The Mayer-Vietoris sequence shows that N is a homology cell. The Hurewicz theorem says that N is a homotopy cell, and hence N is contractible. #

COROLLARY 3.1. *Let* M *be a homology* n-*sphere. If* $n \geq 5$, *then* M *bounds a unique contractible* $(n+1)$-*manifold. If* $n = 4$, *then* M *bounds at most one contractible* $(n+1)$-*manifold.*

COROLLARY 3.2. *Let* M *be a homology* n-*sphere* $(n \geq 5)$. *Then* M *can be embedded in* S^{n+1} *so that the complement of the image will be simply connected.*

Proof. By Theorem 3, there exists a contractible $(n+1)$-manifold N such that $M \approx \partial N$. Since $N \cup_{\partial N} N \approx S^{n+1}$, the result follows.

Contractible open manifolds do not seem to submit so easily to the techniques of this paper. In view of Corollary 2.1, it seems reasonable to conjecture that if X and Y are contractible open manifolds that have homeomorphic neighborhoods of ∞, then X and Y are homeomorphic. The following partial solution of this conjecture was suggested by the referee.

THEOREM 4. *Let* X *and* Y *be contractible, open,* PL n-*manifolds* ($n \geq 6$). *Let* ε *denote the end of* X. *Suppose that* π_1 *is stable at* ε, $\pi_1(\varepsilon)$ *is finitely presented,* $H_1(\varepsilon) = 0$, *and* ε *is tame (that is, there exists a connected open neighborhood* U *of* ε *such that* U *is dominated by a finite* CW *complex and the natural map* $i: \pi_1(\varepsilon) \to \pi_1(U)$ *has a left inverse). If* $W \subseteq X$ *and* $V \subseteq Y$ *are closed neighborhoods of the ends of* X *and* Y, *respectively, and if* $h: W \to V$ *is a* PL *homeomorphism, then for some neighborhood* $W_1 \subseteq W$ *of* ε, $h \mid W_1$ *extends to a* PL *homeomorphism* $H: X \to Y$.

Proof. By Theorem 4.5 of L. C. Siebenmann's thesis, there exists a neighborhood W_0 of ε such that $W_0 \subseteq W$, W_0 is a connected PL n-manifold, $\pi_1(\varepsilon) \approx \pi_1(W_0)$, and $\pi_i(W_0, \partial W_0) = 0$ for $0 \leq i \leq n - 3$.

In particular, $\pi_1(\partial W_0) \xrightarrow{\subseteq} \pi_1(W_0)$ is an isomorphism; hence, by Van Kampen's theorem, $\pi_1(X - \text{int } W_0) \approx \pi_1(X) \approx 1$. Similarly,

$$\pi_1(Y - h(\text{int } W_0)) \approx \pi_1(Y) \approx 1.$$

We now proceed as in the proof of Theorem 1. Let $J_1, J_2, \cdots, J_k$ be simple closed curves in ∂W_0 whose homotopy classes generate $\pi_1(\partial W_0) \approx \pi_1(W_0)$. Since $n \geq 6$ and since $\pi_1(X - \text{int } W_0) \approx 1$, the J_i bound pairwise disjoint disks D_i with $\text{int } D_i \subseteq X - W_0$. Let C_i be a regular neighborhood of D_i in $X - \text{int } W_0$ such that $C_i \cap \partial W_0$ is a regular neighborhood of J_i in ∂W_0. Let E_i be a disk in $Y - h(\text{int } W_0)$ bounded by $h(J_i)$, and let $\text{int } E_i \subseteq Y - h(W_0)$. Finally, let F_i be a regular neighborhood of E_i, in $Y - h(\text{int } W_0)$, with $F_i \cap h(\partial W_0) = h(C_i \cap \partial W_0)$.

We now extend $h \mid W_0$ to take $W_0 \cup \bigcup_{i=1}^k C_i$ homeomorphically onto $h(W_0) \cup \bigcup_{i=1}^k F_i$. To accomplish this, we need a result analogous to Lemma 1. Let

$$Z = (X - \text{int } W_0) \cup_{h \mid \partial W_0} (Y - h(\text{int } W_0)),$$

and let $S_i = D_i \cup E_i \subseteq Z$. Then S_i is a 2-sphere, and $h \mid W_0$ will extend to take C_i onto F_i if and only if $C_i \cup_h F_i$ is homeomorphic to $S_i \times D^{n-2}$. This will be the case if S_i is contained in an n-cell in Z, and we can engulf S_i with an n-cell if Z is 2-connected. Since

$$\pi_1(X - \text{int } W_0) \approx \pi_1(Y - h(\text{int } W_0)) \approx 1,$$

Van Kampen's theorem implies $\pi_1(Z) \approx 1$. Hence, to prove that $\pi_2(Z) \approx 0$, it suffices to prove $H_2(Z) \approx 0$. Since $\pi_i(W_0, \partial W_0) \approx 0$ for $i \leq n - 3$, we see that $H_i(W_0, \partial W_0) \approx 0$ for $i \leq n - 3$. Since $n \geq 6$, we obtain the relations

$$0 \approx H_3(W_0, \partial W_0) \to H_2(\partial W_0) \to H_2(W_0) \to H_2(W_0, \partial W_0) \approx 0.$$

The fact that $H_2(\partial W_0) \to H_2(W_0)$ is an isomorphism implies the relations

$$0 \approx H_3(X) \to H_2(\partial W_0) \to H_2(X - \text{int } W_0) \oplus H_2(W_0) \to H_2(X) \approx 0,$$

and hence $H_2(X - \text{int } W_0) \approx 0$. Similarly $H_2(Y - h(\text{int } W_0)) \approx 0$. Thus we see from

$$0 \approx H_2(X - \text{int } W_0) \oplus H_2(Y - h(\text{int } W_0)) \to H_2(Z) \to H_1(\partial W_0)$$
$$\to H_1(X - \text{int } W_0) \oplus H_1(Y - h(\text{int } W_0)) \approx 0$$

that $H_2(Z) \approx H_1(\partial W_0)$. But the map $\pi_1(\partial W_0) \xrightarrow{\approx} \pi_1(W_0)$ shows that

$$H_1(\partial W_0) \approx H_1(W_0).$$

Since $\pi_1(W_0) \approx \pi_1(\varepsilon)$ and since $H_1(\varepsilon) = 0$, we obtain $H_1(\partial W_0) \approx 0$. Hence Z is 2-connected and $h \mid W_0$ extends to $W_0 \cup \bigcup_{i=1}^k C_i$. The proof now concludes exactly as the proof of Theorem 2, because we have shown that X and Y have homeomorphic 1-connected neighborhoods of ∞.

REFERENCES

1. R. Dieffenbach, *Regular neighborhoods of orientable* 3-*manifolds*. Dissertation, Univ. of Iowa, 1970.
2. M. J. Greenberg, *Lectures on algebraic topology*. W. A. Benjamin, New York, 1967.
3. J. F. P. Hudson, *Piecewise linear topology*. W. A. Benjamin, New York, 1969.
4. J. F. P. Hudson and E. C. Zeeman, *On regular neighbourhoods*. Proc. London Math. Soc. (3) 14 (1964), 719-745.
5. M. Kato, *Embedding spheres and balls in codimension* ≤ 2. Bull. Amer. Math. Soc. 75 (1969), 1260-1265.
6. R. C. Kirby and L. C. Siebenmann, *On the triangulation of manifolds and the Hauptvermutung*. Bull. Amer. Math. Soc. 75 (1969), 742-749.
7. J. Milnor, *Lectures on the* h-*cobordism theorem*. Notes by L. Siebenmann and J. Sondow. Princeton Univ. Press, Princeton, N.J., 1965.
8. J. R. Stallings, *On topologically unknotted spheres*. Ann. of Math. (2) 77 (1963), 490-503.
9. J. B. Wagoner, *Surgery on a map and an approach to the Hauptvermutung*. Dissertation, Princeton Univ., 1966.
10. E. C. Zeeman, *Unknotting combinatorial balls*. Ann. of Math. (2) 78 (1963), 501-526.

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