THE QUALITATIVE BEHAVIOR OF THE SOLUTIONS OF A NONLINEAR VOLTERRA EQUATION

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1. INTRODUCTION

In this paper, we consider the equation

$$(1.1)\qquad x'(t) = \int_0^t b(t-\tau)g(x(\tau))\,d\tau + f(t) \qquad (0 \le t < \infty),$$

where $x(0)$ is a prescribed real number and $b(t)$, $f(t)$, $g(x)$ are prescribed real functions. The following is our main result.

THEOREM 1. *Let*

$$(1.2)\qquad b(t) \in L_1(0, 1),$$

$$(1.3)\qquad [-1]^k b^{(k)}(t) \le 0 \qquad (0 < t < \infty;\ k = 0, 1, 2),$$

$$(1.4)\qquad b(t) \not\equiv b(0+),$$

$$(1.5)\qquad g(x) \in C(-\infty, \infty),$$

$$(1.6)\qquad f(t) \in C[0, \infty) \cap L_1[0, \infty),$$

and let $x(t)$ *be a solution of* (1.1) *on* $0 \le t < \infty$ *such that*

$$(1.7)\qquad \sup_{0 \le t < \infty} |x(t)| < \infty .$$

Then $\lim_{t\to\infty} g(x(t))$ *exists and*

$$(1.8)\qquad \lim_{t\to\infty} g(x(t)) = 0 .$$

If, in addition,

$$(1.9)\qquad \lim_{t\to\infty} f(t) = 0,$$

then $\lim_{t\to\infty} x'(t) = 0$.

In (1.3), we assume that $b''(t)$ exists and is finite on $0 < t < \infty$. Theorem 1 obviously remains true if (1.3) is replaced by

$$[-1]^k b^{(k)}(t) \ge 0 \qquad (0 < t < \infty;\ k = 0, 1, 2).$$

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By way of comments, we note first that the existence of a bounded solution $x(t)$ on $0 \leq t < \infty$ is part of the hypothesis. Applying a result in [6], we see immediately that under the present assumptions, (1.1) has a local solution $x(t)$ (not necessarily unique). However, a somewhat different hypothesis is needed for a nonlocal existence proof. To see this, it suffices to take $f(t) \equiv 0$ and $g(x) = -x^{1+\delta}$ in (1.1), for some $\delta > 0$, and to apply Theorem 2 of [4]; the theorem implies that if $g(x)$ has this particular form, if (1.2), (1.3), and (1.4) are satisfied, and if $x(0)$ is large enough, then $x(t)$ has a finite escape time.

The asymptotic behavior of solutions of (1.1) has been considered in several papers, under hypotheses related to those of Theorem 1. See for example [1], [2], [3], [5].

In [5], J. J. Levin and J. A. Nohel analyze the equation (1.1) under the hypothesis that

$$b(t) \in C[0, \infty), \qquad [-1]^k b^{(k)}(t) \leq 0 \quad (k = 0, 1, 2, 3;\ 0 < t < \infty), \qquad b(t) \not\equiv b(0),$$

$$xg(x) > 0 \quad (x \neq 0), \qquad g(x) \in C(-\infty, \infty), \qquad G(x) = \int_0^x g(u)\,du \to \infty \quad (|x| \to \infty),$$

$$|g(x)| \leq K[1 + G(x)],$$

and they prove that $\lim_{t\to\infty} x^{(j)}(t) = 0$ $(j = 0, 1)$. (They also consider nonintegrable perturbations.) K. B. Hannsgen in [2] extends this result to a nonpositive, nondecreasing concave kernel $b(t)$ such that $b(t) \in C(0, \infty) \cap L_1(0, 1)$. The assumptions on $g(x)$ remain the same as in [5]. However, to obtain asymptotic results, Hannsgen also assumes that either $b(0+) > -\infty$ or $b(t) \in L_1(0, \infty)$.

In Theorem 1, we show that continuity is the only hypothesis on $g(x)$ needed in the proof that $\lim_{t\to\infty} g(x(t))$ exists (assuming the existence of a bounded solution). Also, $b(0+) = -\infty$, $b(t) \notin L_1(0, \infty)$ is not excluded in our result. Note that this answers a problem posed by Nohel [6, Section 6].

As to $f(t)$, we observe that Theorem 1 only requires (1.6) to hold. In [5], $|f'(t)| \leq K$ is assumed (in addition to (1.6)). In [2], either $|f(t)| \leq K$ or $|f'(t)| \leq K$ is assumed (depending upon the hypothesis on $b(t)$), again together with (1.6).

The proofs of the existence of $\lim_{t\to\infty} x(t)$ have essentially rested upon the Lyapunov function

$$E(t) = G(x(t)) - \frac{1}{2}b(t)\left[\int_0^t g(x(\tau))\,d\tau\right]^2 + \frac{1}{2}\int_0^t b'(t-\tau)\left[\int_\tau^t g(x(s))\,ds\right]^2 d\tau,$$

introduced in [3]. Namely, if $x(t)$ is a solution of (1.1), $f(t) \equiv 0$, and (1.3) holds, then $E'(t) \leq 0$. In the proof of Theorem 1 we show, however, that the equation (1.1) may be written in a form that immediately brings out the importance of (1.3) to the existence of $\lim_{t\to\infty} g(x(t))$. Consequently, we prove Theorem 1 without recourse to Lyapunov techniques.

In Theorem 2, we weaken the assumptions

$$xg(x) \geq 0 \ (|x| < \infty), \qquad G(x) \to \infty \ (|x| \to \infty), \qquad |g(x)| \leq K[1 + G(x)] \ (|x| < \infty),$$

made in [2] and [5] to obtain boundedness of solutions. In particular, we do not exclude $\liminf G(x) = -\infty$ $(|x| \to \infty)$.

By imposing the additional conditions

$$\int_0^\infty [B(\tau) - B(\infty)]d\tau < \infty, \qquad B(\infty) = \lim_{t\to\infty} B(t) > -\infty, \tag{1.10}$$

where $B(t) = \int_0^t b(\tau)d\tau$, we may extend our method to equations with infinite lag,

$$x'(t) = \int_{-\infty}^t b(t-\tau)g(x(\tau))d\tau + f(t) \qquad (0 \le t < \infty),$$

with initial function $\phi(t)$ $(-\infty < t \le 0)$. In fact, if $g(\phi(t))$ is bounded and (1.3) and (1.10) hold, then

$$\int_{-\infty}^0 b(t-\tau)g(\phi(\tau))d\tau \in C[0,\infty) \cap L_1[0,\infty).$$

This improves upon a result by Hale [1, Section 5.1], since we do not require that $b^{(k)}(0)$ is finite $(k = 0, 1, 2)$. Also, in [1] it is assumed that $G(x)$ is bounded from below and $f(t) \equiv 0$.

THEOREM 2. *Let* (1.2), (1.3), (1.5), *and* (1.6) *hold. Also, let*

$$\limsup G(x) = \infty \quad (|x| \to \infty), \qquad \textit{where } G(x) = \int_0^x g(u)du, \tag{1.11}$$

$$|g(x)| \le \begin{cases} K[1 + \max_{0\le y\le x} G(y)] & (x \ge 0), \\ K[1 + \max_{x\le y\le 0} G(y)] & (x \le 0), \end{cases} \tag{1.12}$$

for some constant K. *Then there exists a solution* $x(t)$ *of* (1.1) *on* $0 \le t < \infty$. *Moreover, under this hypothesis each solution of* (1.1) *on* $0 \le t < \infty$ *satisfies the condition* $\sup_{0\le t<\infty} |x(t)| < \infty$.

2. PROOF OF THEOREM 1

Conditions (1.5) and (1.7) imply that

$$\sup_{0\le t<\infty} |g(x(t))| = M < \infty. \tag{2.1}$$

Define

$$G(x) = \int_0^x g(u)\,du \qquad (|x| < \infty).$$

Multiplication of (1.1) by g(x(t)), followed by integration, gives the formula

$$\text{(2.2)}\quad G(x(t)) = G(x(0)) + \int_0^t g(x(\tau)) \int_0^\tau b(\tau - s)\,g(x(s))\,ds\,d\tau + \int_0^t f(\tau)\,g(x(\tau))\,d\tau,$$

and thus, by (1.6) and (2.1),

$$\text{(2.3)}\qquad \left| \int_0^t g(x(\tau)) \int_0^\tau b(\tau - s)\,g(x(s))\,ds\,d\tau \right| \leq K \qquad (0 \leq t < \infty)$$

for some constant K, because $\sup_{0 \leq t < \infty} |G(x(t))| < \infty$. We also have the relation

$$\text{(2.4)}\quad \begin{cases} \displaystyle\int_0^t g(x(\tau)) \int_0^\tau b(\tau - s)\,g(x(s))\,ds\,d\tau \\ \displaystyle = \frac{1}{2}\int_0^t \int_0^\tau b''(\tau - s)\left[\int_s^\tau g(x(u))\,du\right]^2 ds\,d\tau + \frac{b(t)}{2}\left[\int_0^t g(x(\tau))\,d\tau\right]^2 \\ \displaystyle - \frac{1}{2}\int_0^t b'(t - \tau)\left[\int_\tau^t g(x(s))\,ds\right]^2 d\tau - \frac{1}{2}\int_0^t b'(\tau)\left[\int_0^\tau g(x(s))\,ds\right]^2 d\tau, \end{cases}$$

where $g(x(\tau))$ may of course be replaced by an arbitrary continuous function of τ $(0 \leq \tau \leq t)$.

We can easily verify that (2.4) holds, by differentiating both sides and then performing an integration by parts. Note that the rigour necessary for the case where one or more of b(0+), b'(0+), b"(0+) are infinite is provided by Lemma 4 of [3] and Lemma 1 of [2]. By (1.3), (2.3), and (2.4), there exists a constant K such that

$$\text{(2.5)}\qquad \int_0^t \int_0^\tau b''(v)\,\phi_v^2(\tau)\,dv\,d\tau \geq -K \qquad (0 \leq t < \infty),$$

where $\phi_v(\tau) = \int_{\tau - v}^{\tau} g(x(s))\,ds$.

We need the following two lemmas.

LEMMA 1. *Let* (1.3) *and* (1.4) *hold. Then there exists an interval* $[\eta_1, \eta_2]$ $(0 < \eta_1 < \eta_2)$ *such that* $b'(t_1) - b'(t_2) > 0$ *for any* t_1 *and* t_2 *such that* $\eta_1 \leq t_1 < t_2 \leq \eta_2$.

Proof of Lemma 1. Let $b(0+) > -\infty$. By (1.3), $b'(t) \in C(0, \infty)$. Thus, if the conclusion of the lemma does not hold, then by (1.3), $b'(t) = 0$ $(0 < t < \infty)$, and $b(t) = b(0+)$ $(0 < t < \infty)$; this violates (1.4). If $b(0+) = -\infty$, the conclusion is obvious.

LEMMA 2. *Let* (1.2), (1.3), (1.5), (1.6), *and* (1.7) *hold. Then*

$$\sup_{0 \leq t < \infty} \left| \int_0^t b(t - \tau) g(x(\tau)) d\tau \right| < \infty . \tag{2.6}$$

Proof of Lemma 2. For $1 \leq t < \infty$, we have the relation

$$\left\{ \begin{aligned} \int_0^t b(t - \tau) g(x(\tau)) d\tau &= - \int_{t-1}^t b'(t - \tau) \left[\int_\tau^t g(x(s)) ds \right] d\tau \\ &- \int_0^{t-1} b'(t - \tau) \left[\int_\tau^t g(x(s)) ds \right] d\tau + b(t) \int_0^t g(x(s)) ds . \end{aligned} \right. \tag{2.7}$$

Suppose that there exists a sequence $\{t_n\}$ $(t_n \to \infty)$ such that

$$\lim_{n \to \infty} \left| b(t_n) \int_0^{t_n} g(x(\tau)) d\tau \right| = \infty . \tag{2.8}$$

Because $|b(t)| \leq |b(1)|$ $(1 \leq t < \infty)$, (2.8) implies that

$$\lim_{n \to \infty} \left| \int_0^{t_n} g(x(\tau)) d\tau \right| = \infty ,$$

and therefore

$$\lim_{n \to \infty} b(t_n) \left[\int_0^{t_n} g(x(\tau)) d\tau \right]^2 = -\infty ;$$

together with (1.3) and (2.4), this violates (2.3). Thus, observing in addition that $\lim_{t \to 0+} b(t) t = 0$, we conclude that the absolute value of the last term in (2.7) is bounded on $0 < t < \infty$. By (1.2) and (2.1), we also have the bound

$$\left| \int_{t-1}^t b'(t - \tau) \left[\int_\tau^t g(x(s)) ds \right] d\tau \right| \leq M \int_0^1 b'(\tau) \tau d\tau < \infty .$$

Consequently, by (2.7), we see that if (2.6) does not hold, then

$$\sup_{0 \leq t < \infty} \left| \int_0^{t-1} b'(t - \tau) \left[\int_\tau^t g(x(s)) ds \right] d\tau \right| = \infty . \tag{2.9}$$

However, because (1.3) implies

$$\int_1^\infty |b''(\tau)| \tau d\tau < \infty , \tag{2.10}$$

there exists a constant K such that

$$(2.11)\quad \begin{cases} \left|\dfrac{d}{dt}\left\{\displaystyle\int_0^{t-1} b'(t-\tau)\left[\int_\tau^t g(x(s))\,ds\right]d\tau\right\}\right| \\ = \left|\displaystyle\int_0^{t-1} b''(t-\tau)\left[\int_\tau^t g(x(s))\,ds\right]d\tau \right. \\ \qquad \left. + \displaystyle\int_0^{t-1} b'(t-\tau)\,g(x(t))\,d\tau + b'(1)\int_{t-1}^t g(x(\tau))\,d\tau\right| \\ \leq M\left[\displaystyle\int_1^t |b''(\tau)|\,\tau\,d\tau - b(1) + b'(1)\right] \leq K \qquad (1 \leq t < \infty). \end{cases}$$

Thus, if (2.6) does not hold, then by (1.6), (2.7), (2.9), and (2.11), we conclude, after integrating (1.1), that $\sup_{0 \leq t < \infty} |x(t)| = \infty$, which violates (1.7). The lemma is proved.

By (1.1) and (2.6),

$$|x'(t)| \leq K + |f(t)| \qquad (0 \leq t < \infty),$$

for some constant K; therefore, remembering (1.6), we see that x(t) is uniformly continuous on $0 \leq t < \infty$. Combined with (1.5) and (1.7), this implies that g(x(t)) is uniformly continuous on $0 \leq t < \infty$.

Choose any interval $[\eta_1, \eta_2]$ satisfying the conclusion of Lemma 1. Obviously, either $\lim_{t\to\infty} \phi_v(t)$ exists for all $v \in [\eta_1, \eta_2]$, or not. To begin, let

$$(2.12)\qquad \lim_{t\to\infty} \int_{t-v}^t g(x(\tau))\,d\tau \quad \text{exist if } v \in [\eta_1, \eta_2].$$

We assert that if (2.12) holds, then $\lim_{t\to\infty} g(x(t))$ exists and is 0. Differentiating, we have the formula

$$(2.13)\qquad \frac{d\phi_v(t)}{dt} = g(x(t)) - g(x(t-v)).$$

Suppose that for some v_0 ($v_0 \in [\eta_1, \eta_2]$), $\lim_{t\to\infty} [g(x(t)) - g(x(t-v_0))]$ either does not exist, or if it exists, is not equal to zero. Then there exist a sequence $\{t_n\}$ ($t_n \to \infty$) and a number $\eta > 0$ such that for example

$$(2.14)\qquad g(x(t_n)) - g(x(t_n - v_0)) \geq \eta .$$

However, (2.13) and (2.14), combined with the uniform continuity of g(x(t)), contradict (2.12). Thus

$$(2.15)\qquad \lim_{t\to\infty} [g(x(t)) - g(x(t-v))] = 0 \qquad (v \in [\eta_1, \eta_2]).$$

Assume now that $\lim_{t\to\infty} g(x(t))$ either does not exist, or if it exists, is not 0. Then there exists a $\delta_1 > 0$ such that for example $g(x(t_n)) \geq 2\delta_1$ and

$$g(x(t)) \geq \delta_1 \qquad (t_n - T_n \leq t \leq t_n), \tag{2.16}$$

for some $\{t_n\}$ and $\{T_n\}$ $(t_n \to \infty)$. Let δ_2 be such that

$$g(x(t)) \geq \delta_1 \qquad (t_n - \delta_2 \leq t \leq t_n); \tag{2.17}$$

by the uniform continuity of $g(x(t))$, such a δ_2 exists. Let $\delta_3 = \min(\delta_2, \eta_2 - \eta_1)$. Suppose that there exist a sequence $\{t_p\}$ $(t_p \to \infty)$ and a constant $\delta_4 > 0$ such that

$$g(x(t_p)) - g(x(t_p - \delta_3)) \geq \delta_4 \tag{2.18}$$

for all sufficiently large p. By (2.15) and (2.18),

$$g(x(t_p - \delta_3 - \eta_1)) - g(x(t_p - 2\delta_3 - \eta_1)) \geq \frac{\delta_4}{2}. \tag{2.19}$$

But also, by (2.15),

$$g(x(t_p - \delta_3)) - g(x(t_p - \delta_3 - \eta_1)) \geq -\varepsilon_p, \tag{2.20}$$

where $\lim_{p\to\infty} \varepsilon_p = 0$, and by (2.19) and (2.20),

$$g(x(t_p - \delta_3)) - g(x(t_p - 2\delta_3 - \eta_1)) \geq \frac{\delta_4}{4} \tag{2.21}$$

if p is sufficiently large; by (2.15), this is impossible.

Thus there exists no sequence $\{t_p\}$ $(t_p \to \infty)$ such that (2.18) holds, no matter how small $\delta_4 > 0$ is taken. This, combined with the definition of δ_3, implies that $T_n \to \infty$. In particular, $T_n >> \eta_2$. But then (1.3), (2.16), and the definition of $[\eta_1, \eta_2]$ obviously imply that

$$\lim_{t\to\infty} \int_0^t \int_0^\tau b''(v)\, \phi_v^2(\tau)\, dv\, d\tau = -\infty, \tag{2.22}$$

which, by (2.5), is impossible. Thus $\lim_{t\to\infty} g(x(t)) = 0$, if (2.12) holds.

Suppose finally that for some v_1 $(\eta_1 \leq v_1 \leq \eta_2)$ $\lim_{t\to\infty} \phi_{v_1}(t)$ does not exist. By rather obvious arguments using the uniform continunity of $g(x(t))$, we again arrive at (2.22). This proves (1.8).

Using (1.8), we show next that

$$\lim_{t\to\infty} \int_0^t b(t - \tau)\, g(x(\tau))\, d\tau = 0. \tag{2.23}$$

As earlier, we write (for $0 < t < \infty$)

$$\int_0^t b(t-\tau)\, g(x(\tau))\, d\tau = b(t)\int_0^t g(x(\tau))\, d\tau - \int_0^t b'(t-\tau)\left[\int_\tau^t g(x(s))\, ds\right] d\tau. \tag{2.24}$$

Also,

$$\frac{d}{dt}\left\{ b(t) \int_0^t g(x(\tau))\,d\tau \right\} = b'(t) \int_0^t g(x(\tau))\,d\tau + b(t)\,g(x(t)). \tag{2.25}$$

Differentiating the first term on the right side of (2.25) and estimating, we obtain by (2.1) the bound

$$\left| \frac{d}{dt}\left\{ b'(t) \int_0^t g(x(\tau))\,d\tau \right\} \right| \le M\,|b''(t)t| + |b'(t)\,g(x(t))| ; \tag{2.26}$$

by (1.3) and (1.8) the second term on the right side of (2.26) tends to 0 as $t \to \infty$. Suppose now that there exist $\{t_n\}$ $(t_n \to \infty)$ and $\eta > 0$ such that

$$\left| b'(t_n) \int_0^{t_n} g(x(\tau))\,d\tau \right| \ge \eta . \tag{2.27}$$

By (2.10), (2.26), and (2.27), there exists a sequence $\{T_n\}$ $(T_n \to \infty)$ such that

$$\left| b'(t) \int_0^t g(x(\tau))\,d\tau \right| \ge \frac{\eta}{2} \qquad (t_n - T_n \le t \le t_n). \tag{2.28}$$

But then, by (1.3), (1.8), (2.25), and (2.28),

$$\lim_{n \to \infty} \left| b(t_n) \int_0^{t_n} g(x(\tau))\,d\tau - b(t_n - T_n) \int_0^{t_n - T_n} g(x(\tau))\,d\tau \right| = \infty ,$$

which is impossible because the absolute value of the last term in (2.7) is bounded, as stated in the proof of Lemma 2. Thus

$$\lim_{t \to \infty} b'(t) \int_0^t g(x(\tau))\,d\tau = 0, \tag{2.29}$$

and by (1.8) and (2.25),

$$\lim_{t \to \infty} \frac{d}{dt}\left\{ b(t) \int_0^t g(x(\tau))\,d\tau \right\} = 0. \tag{2.30}$$

Consider now the second term on the right side of (2.24). For each $\varepsilon > 0$ and each finite T, condition (1.8) together with the fact that $\int_0^T b'(\tau)\,\tau\,d\tau < \infty$, implies that the inequality

$$\left| \int_0^t b'(t-\tau)\left[\int_\tau^t g(x(s))\,ds \right] d\tau \right| \le \varepsilon + \left| \int_0^{t-T} b'(t-\tau)\left[\int_\tau^t g(x(s))\,ds \right] d\tau \right| \tag{2.31}$$

holds for all sufficiently large t. Also, choosing T sufficiently large but fixed, we deduce from (1.3), (1.8), (2.1), and (2.10) that

$$(2.32)\qquad \begin{cases} \left|\dfrac{d}{dt}\left\{\displaystyle\int_0^{t-T} b'(t-\tau)\left[\int_\tau^t g(x(s))\,ds\right]d\tau\right\}\right| \\ = \left|\displaystyle\int_0^{t-T} b''(t-\tau)\left[\int_\tau^t g(x(s))\,ds\right]d\tau\right. \\ \quad \left. + \displaystyle\int_0^{t-T} b'(t-\tau)\,g(x(t))\,d\tau + b'(T)\int_{t-T}^t g(x(\tau))\,d\tau\right| \le \varepsilon\,, \end{cases}$$

if t is sufficiently large. Combining (2.24), (2.30), (2.31), (2.32), we see that if (2.23) does not hold, then, for example,

$$(2.33)\qquad \int_0^t b(t-\tau)\,g(x(\tau))\,d\tau \ge \eta > 0 \qquad (t_n - T_n \le t \le t_n),$$

for some $\{t_n\}$ and $\{T_n\}$ ($\lim_{n\to\infty} t_n = \lim_{n\to\infty} T_n = \infty$). Integrating (1.1) over $[t_n - T_n, t_n]$ and invoking (1.6), we then obtain a contradiction to (1.7). The validity of (2.23) now follows. Conditions (1.1), (1.9), and (2.23) together imply $\lim_{t\to\infty} x'(t) = 0$.

This completes the proof.

3. PROOF OF THEOREM 2

Let x(t) be a solution of (1.1) on some t-interval ($t \ge 0$). Then, by (1.3), (1.6), (1.12), (2.2), and (2.4), there exists a constant K_1 such that

$$(3.1)\qquad G(x(t)) \le G(x(0)) + \int_0^t f(\tau)\,g(x(\tau))\,d\tau \le K_1 + K\int_0^t |f(\tau)|\,G(\tau)\,d\tau\,,$$

where the nonnegative, nondecreasing function G(t) is defined as

$$G(t) = \max\,(0, \max_{0\le\tau\le t} G(x(\tau)))\,.$$

In (3.1) we have also used the obvious inequalities

$$\max_{x(0)\le y\le x(\tau)} G(y) \le \max_{0\le s\le\tau} G(x(s)) \qquad (x(\tau) \ge x(0)),$$

$$\max_{x(\tau)\le y\le x(0)} G(y) \le \max_{0\le s\le\tau} G(x(s)) \qquad (x(\tau) \le x(0))\,.$$

The inequalities in (3.1) are obviously valid if t is such that G(x(t)) = G(t). Therefore, observing in addition that the last integrand in (3.1) is nonnegative, we conclude that

(3.2) $$G(t) \leq K_1 + K \int_0^t |f(\tau)|\, G(\tau)\, d\tau \qquad (t \geq 0).$$

Applying the Gronwall inequality to (3.2), we see, by (1.6), that

(3.3) $$G(x(t)) \leq K_2 \qquad (t \geq 0)$$

for some constant K_2. By (1.11) and (3.3),

(3.4) $$|x(t)| \leq K_3 \qquad (t \geq 0)$$

for some constant K_3.

The bound obtained in (3.4) is seen to be an *a priori* bound. Thus any local solution (by the present hypothesis and a result in [6], such a solution exists) can be continued to $0 \leq t < \infty$.

This completes the proof.

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