THE CAUCHY PROBLEM WITH INCOMPLETE INITIAL DATA IN BANACH SPACES

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1. INTRODUCTION

Throughout this paper, E denotes a complex Banach space, and A represents a closed operator whose domain D(A) is dense in E and whose range is in E. We assume that the resolvent set $\rho(A)$ is not empty, in other words, that for some complex number λ the operator $R(\lambda; A) = (\lambda I - A)^{-1}$ is everywhere defined and bounded.

We study the problem of existence and uniqueness of solutions of the nth-order operational differential equation

$$u^{(n)}(t) = Au(t) \qquad (t \geq 0) \tag{1.1}$$

that satisfy an estimate of the form

$$|u(t)| = O(e^{\omega t}) \qquad \text{as } t \to +\infty \tag{1.2}$$

and the initial conditions

$$u^{(k)}(0) = u_k \in E \qquad (k \in \alpha). \tag{1.3}$$

Here, ω denotes a real number, n is an integer $(n > 1)$, and α is a predetermined subset of the set $\{0, 1, \cdots, n-1\}$. We also study the dependence of the solutions on the incomplete set of initial data (1.3). (By a solution of (1.1) we mean an E-valued function u that has n continuous derivatives and satisfies (1.1) for $t \geq 0$.) This is a generalization of the usual Cauchy problem, where growth conditions of the type (1.2) are absent but where α in (1.3) consists of all the integers $0, 1, \cdots, n-1$, in other words, where each of the values $u^{(k)}(0)$ $(k = 0, 1, \cdots, n-1)$ is preassigned. In order to delineate clearly the results in the present paper, we sketch briefly the available results in the usual case. We say that the problem

$$u^{(n)}(t) = Au(t) \qquad (t \geq 0), \tag{1.4}$$

$$u^{(k)}(0) = u_k \qquad (0 \leq k \leq n-1) \tag{1.5}$$

is *well posed* if solutions of (1.4), (1.5) exist (their initial data $u_0, u_1, \cdots, u_{n-1}$ arbitrarily chosen in a given dense subspace of E) and depend continuously on $u_0, u_1, \cdots, u_{n-1}$. For $n = 1$, the problem (1.4), (1.5) is well posed if and only if A generates a strongly continuous semigroup (see [12, especially Chapter I, Section 2, Theorem 2.8] and [7, Part I, Theorem 4.1]). Generators of strongly continuous semigroups are in turn characterized by the theorem of E. Hille and K. Yosida (see

Received December 3, 1970.
This research was supported in part by the National Science Foundation, under contract GP-9658.

Michigan Math. J. 18 (1971).

[12, Chapter I, Theorem 2.10] or [6, Chapter VIII, Theorem 13]). In the case $n = 2$, A must be the generator of a cosine function (or abstract cosine function; see [7, Part II, Theorem 5.9]). Operators of this type are characterized by a result somewhat similar to the theorem of Hille and Yosida ([16, Theorem 4.6], [5, Teorema 1]; see also [7, Part II, Theorem 3.1] for a different proof). Finally, when $n \geq 3$, A must be bounded (see [7, Part II, Theorem 3.1], where this condition is obtained as a particular case of a result valid in certain linear topological spaces; see also [3], [4], [8] for generalizations in different directions). Some results of roughly the same type are available for the problem (1.1) - (1.2) - (1.3); in order to gain some insight into them, it will be useful to examine in detail the case where E is the complex number field and where A is the operator of multiplication by a complex number a; that is, we examine the equation

$$u^{(n)} = a u(t) . \tag{1.6}$$

If $a = 0$, the general solution of (1.6) is a polynomial of degree at most $n - 1$; consequently, the existence of solutions of (1.6) and (1.2) for arbitrary initial data (1.3) will be possible only when $\omega > 0$ (except in the case $\omega = 0$, $m = 1$, $\alpha = \{0\}$). On the other hand, solutions of (1.6) and (1.2) will be uniquely determined by their initial data only if $\omega < 0$ (or, again, in the exceptional case $\omega = 0$, $m = 1$, $\alpha = \{0\}$). This suggests that existence and (or) uniqueness of solutions of (1.1), (1.2), (1.3) will imply in certain cases that $0 \in \rho(A)$. Indeed, we can establish such a result (see the comments following inequality (2.42)).

If $a \neq 0$, the general solution of (1.6) is

$$u(t) = \sum_{j=0}^{n-1} c_j e^{\gamma^j bt} , \tag{1.7}$$

where $c_0, c_1, \cdots, c_{n-1}$ are arbitrary constants, b is any of the nth roots of a, and $\gamma = e^{2\pi i/n}$. It is then plain that solutions of (1.4) and (1.2) with m preassigned initial data exist if and only if

(1.8) *at least* m *of the numbers* $b, \gamma b, \gamma^2 b, \cdots, \gamma^{n-1} b$ *lie in the half-plane* $\Re \lambda \leq \omega$.

Clearly, this condition will never be satisfied—no matter what the value of a—for certain combinations of n, m, and ω, namely,

$$\omega \leq 0, \quad m > [(n+1)/2] , \tag{1.9}$$

$$\omega = 0, \quad n \text{ even}, \quad m > (n+2)/2 \tag{1.10}$$

(here [p] denotes the integral part of p). This result has a counterpart for the general equation (1.1) (Corollary 2.4). On the other hand, for $\omega \geq 0$ and $m > [(n+1)/2]$, condition (1.8) holds only if the number a lies in a certain bounded region of the complex plane, a result that can be extended in a suitable form to the equation (1.1) (Corollary 2.5). Finally, in the cases

$$\omega \geq 0 \quad \text{and} \quad m = [n/2], [n/2] + 1 ,$$

condition (1.6) places some restrictions on the location of a; for the general equation (1.1), these restrictions are satisfied by $\rho(A)$, the spectrum of A (Theorem 2.1).

The restrictions, somewhat strengthened and combined with bounds on the growth of the resolvent $R(\lambda; A)$, constitute the basis for *sufficient* conditions on A for the existence and uniqueness of solutions of (1.1), (1.2), and (1.3) (see the various results in Section 3).

On the other hand, a brief examination of (1.5) shows that solutions of (1.4) satisfying the growth condition (1.2) are uniquely determined by m of their initial data if and only if

(1.11) *no more than* m *of the numbers* $b, \gamma b, \cdots, \gamma^{n-1} b$ *lie in the half-plane* $\Re \lambda \leq \omega$.

Condition (1.11) cannot be satisfied, regardless of the value of a, if

$$\omega \geq 0 \quad \text{and} \quad m < [n/2] .$$

Unfortunately, as Lemma 3.2 shows, there is no clean-cut extension of this result to the equation (1.1), except for particular choices of A (for instance, when A is normal: see Proposition 3.3)).

We note that there is no special reason for the choice of the function $e^{\omega t}$ in the growth condition (1.2), and that the function could be replaced—at least in the formulation of the problem—by any positive function K(t). However, since the main results in Section 2 are based on the theory of the Laplace transform, it is not clear whether they would be valid in this degree of generality.

The Cauchy problem with incomplete initial data, as examined in this paper, is somewhat similar to the "reduced Cauchy problem of order n and defect m - n" ("problème de Cauchy réduit d'ordre n et de défaut n - m") considered by E. Hille in [10]; although the growth condition (1.2) is absent there, Théorème 7 in [10] is related to the results in Section 3 of the present paper. More closely related to our treatment is A. V. Balakrishnan's study of the generation properties of fractional powers of certain operators [2]. In fact, the case $\omega = 0$, $n = 2$, $m = 1$, $\alpha = \{0\}$ of our formulation of the Cauchy problem with incomplete data is examined in detail in [2]; there, the main result is a sufficient condition for existence and continuous dependence, of which our results in Section 3 are an outgrowth. We obtain our results by the same means, that is, by using the fractional-power theory developed in [2], or, rather, a simplified version developed in [12] for the special case where the operator A is invertible.

On the other hand, the results in Section 2 seem to have no direct ancestors in the literature. These results are generalizations of some of the theorems contained in A. Radnitz's thesis [15]. While in [15] the existence, uniqueness, and continuous dependence on incomplete initial data are assumed, we only assume existence, in the present paper. This makes the results more general, although there is naturally some loss of detail.

The equation (1.1) has been studied in [1], although in a different vein. The results and the methods in [13] also have some relation with ours; the problem studied in [13] is that of "solving backwards" an abstract parabolic equation.

The Cauchy problem for certain partial differential equations has been extensively studied in formulations that are very close to that in the present paper. We do not attempt here to give a complete account of these studies; a fairly complete bibliography (up to 1965) can be found in [16].

We give a brief account of the type of problems that are considered in [16]. The equations studied there are of the form

$$u^{(n)}(t) = \sum_{k=0}^{\infty} P_k u^{(k)}(t) \qquad (t \geq 0), \tag{1.12}$$

where $P_0, P_1, \cdots, P_{n-1}$ are partial differential operators with constant coefficients that act in a space Q of distributions (or generalized functions). The formulation of the Cauchy problem in [16] is essentially the same as our formulation, although condition (1.2) is replaced by

$$|u(t)| = O(t^p) \qquad \text{as } t \to \infty$$

for some integer p that may depend on u. We note that our results cannot be applied to the problems in [16]; in fact, the equation (1.12) is of a form more general than (1.1), and, moreover, the spaces Q are not in general Banach spaces.

In Section 4, we use our results to answer some questions about partial differential (and more general) equations. The first of these questions can be roughly formulated as follows: How many of the partial derivatives u, u_t, u_{tt} of a bounded solution of the partial differential equation

$$u_{ttt} = u_{xx} - u$$

(the solutions are defined in $(-\infty < x < \infty,\ 0 \leq t < \infty)$) can be arbitrarily preassigned for $t = 0$? (See Section 4 for a precise formulation.) We ask then the same question about the integro-differential equation

$$u_{ttt} = u_{xx} - u + Bu,$$

where B denotes a suitable integral operator. Finally, we illustrate by means of a third example the limitations of our results.

The authors are glad to acknowledge their indebtedness to the referee for bringing [16] to their attention, also for several substantial improvements in the presentation of this paper.

2. NECESSARY CONDITIONS FOR EXISTENCE

As in Section 1, n and m denote positive integers $(1 \leq m \leq n)$, and ω denotes a real number.

By $\Lambda(n, m, \omega)$ we denote the set of all complex numbers λ such that *exactly* m of the nth roots of λ lie in the half-plane $\Re\mu \leq \omega$; we shall also write

$$\Lambda^*(n, m, \omega) = \bigcup_{k=m}^{n} \Lambda(n, k, \omega), \qquad \Lambda_*(n, m, \omega) = \mathscr{C}\Lambda^*(n, m, \omega),$$

where $\mathscr{C}$ indicates the complement. Clearly, $\Lambda^*(n, m, \omega)$ (respectively, $\Lambda_*(n, m, \omega)$) consists of all $\lambda \in C$ such that at least m (fewer than m) of the nth roots of λ lie in $\Re\mu \leq \omega$. Some properties of these sets will be discussed later in this section. A more complete description of the sets $\Lambda(n, m, \omega)$ can be found in [15], together with illustrations for some particular values of the parameters.

2.1 THEOREM. *Let* α *be a subset containing* m *elements of the set* $\{0, 1, \cdots, n-1\}$, k_0 *a fixed element of* α, p *a positive integer, and* ω *a real number. Let* A *be a closed operator defined in a dense subset of* E, *and suppose that for some* $\varepsilon > 0$

$$(-1)^n \rho(A) \cap \Lambda^*(n, n-m+1, -\omega-\varepsilon)^\circ \neq \emptyset ,$$

where $(-1)^n \rho(A) = \{\lambda; (-1)^n \lambda \in \rho(A)\}$. *Assume that for every* $u \in D(A^p)$ *there exists a solution* $u(\cdot)$ *of*

$$u^{(n)}(t) = Au(t) \tag{2.1}$$

in $t \geq 0$ *such that*

$$u^{(k)}(0) = 0 \quad \text{for } k \in \alpha \text{ and } k \neq k_0, \qquad u^{(k_0)}(0) = u, \tag{2.2}$$

$$|u(t)| \leq Ke^{\omega t} \qquad (t \geq 0), \tag{2.3}$$

where K (*not* ω) *may depend on* $u(\cdot)$. *Then*

$$\Lambda^*(n, n-m+1, -\omega-\varepsilon) \subseteq (-1)^n \rho(A) \tag{2.4}$$

for each $\varepsilon > 0$. *Moreover,*

$$|R(\lambda; A)u| \leq K\varepsilon^{-1} |\lambda|^{-(n-1-k_0)/n} \tag{2.5}$$

for $\lambda \in \Lambda^*(n, n-m+1, -\omega-\varepsilon)$ *and* $u \in D(A^p)$ (K *may depend on* u, *but for each* u *it is independent of* ε *and* λ).

Proof. Suppose $\lambda \in \Lambda^*(n, n-m+1, -\omega-\varepsilon)$. Then there are (at least) $n-m+1$ nth roots of λ in the half-plane $\Re\mu \leq -\omega-\varepsilon$. Clearly, we can write them in the form

$$\mu, \gamma\mu, \gamma^2\mu, \cdots, \gamma^{n-m}\mu , \tag{2.6}$$

where μ is a particular root of λ and $\gamma = e^{2\pi i/n}$. Now let

$$\eta(t, \lambda) = \sum_{j=0}^{n-m} c_j(\lambda) e^{(\gamma^j \mu)t} , \tag{2.7}$$

where the coefficients $c_j(\lambda)$ are chosen in such a way that

$$D_t^k \eta(0, \lambda) = 0 \quad \text{if } n-1-k \in \beta, \qquad D_t^{n-1-k_0} \eta(0, \lambda) = (-1)^{n-k_0+1} \tag{2.8}$$

(here D_t^p indicates differentiation of order p with respect to t, and the set β consists of all integers in $\{0, 1, \cdots, n-1\}$ not in α). To see that this choice is possible, observe that (2.8) will hold if the coefficients c_j satisfy the system of $n-m+1$ linear equations

(2.9) $\mu^k \sum_{j=0}^{n-m} c_j \gamma^{jk} = 0$ if $n - 1 - k \in \beta$, $\quad \mu^{n-1-k_0} \sum_{j=0}^{n-m} c_j \gamma^{j(n-1-k_0)} = (-1)^{n-k_0+1}$

The determinant of this system is

$$\Delta(\lambda) = \mu^a \det \{\gamma^{jk}\}, \tag{2.10}$$

where $j = 0, 1, \cdots, n - m$, where $n - 1 - k \in \beta \cup \{k_0\}$, and where a denotes the sum of the values k. But the determinant in the right-hand side of (2.10) is a Vandermonde determinant, hence not zero. Accordingly,

$$|\Delta(\lambda)| = K_\alpha |\mu|^a \qquad (K_\alpha > 0).$$

We have the relation

$$c_j(\lambda) = \Delta_j(\lambda)/\Delta(\lambda),$$

where $\Delta_j(\lambda)$ is the determinant obtained from $\Delta(\lambda)$ by replacing its jth column with the right-hand side of (2.9). Clearly,

$$|\Delta_j(\lambda)| = K_{\alpha,k_0} |\mu|^b,$$

where $b = \sum k$ $(n - 1 - k \in \beta)$. Consequently,

$$|c_j(\lambda)| = K_j |\mu|^{-(n-1-k_0)} = K_j |\lambda|^{-(n-1-k_0)/n}, \tag{2.11}$$

where K_j is a constant depending on α, k_0, and j.

We observe that if *more* than $n - m + 1$ of the nth roots of λ lie in the half-plane $\Re\lambda \leq \omega$, then the choice of the $m - n + 1$ roots (2.6) is not unique, since we need no longer use consecutive roots; this shows that λ does not necessarily give a complete determination of $\eta(\,\cdot\,, \lambda)$. To remove this ambiguity, we shall always suppose that a definite choice of the roots (2.6) has been made.

Suppose now that $u \in D(A^p)$, that $u(\,\cdot\,)$ is one of the solutions associated with u in the statement of Theorem 2.1, and that λ is a fixed element of $\rho(A)$. For each λ in the open region

$$\Lambda^*(n, n - m + 1, -\omega - 0) = \bigcup_{\varepsilon > 0} \Lambda^*(n, n - m + 1, -\omega - \varepsilon),$$

define

$$f(\lambda; u) = \int_0^\infty \eta(t, \lambda) R(\lambda_0; A)^p u(t)\, dt = R(\lambda_0; A)^p \int_0^\infty \eta(t, \lambda) u(t)\, dt. \tag{2.12}$$

Clearly, the integral exists; moreover, by virtue of (2.11), there exists a constant K such that

$$|(\lambda_0 I - A)^p f(\lambda, u)| \leq K\varepsilon^{-1} |\lambda|^{-(n-1-k_0)/n}, \tag{2.13}$$

for $\lambda \in \Lambda^*(n, n - m + 1, -\omega - \varepsilon)$. Observe next that

(2.14) $$|R(\lambda_0; A)^p u^{(n)}(t)| = |R(\lambda; A)^p A u(t)| \leq |R(\lambda_0; A)^p A| \, |u(t)|.$$

Thus, in view of (2.3), $|R(\lambda_0; A)^p u^{(n)}(t)| = O(e^{\omega t})$ as $t \to \infty$. Assume for the moment that $\omega \geq 0$ (this restriction will be removed later). Then, as $t \to \infty$,

$$|R(\lambda_0; A)^p u^{(k)}(t)| = \begin{cases} O(e^{\omega t}) & \text{if } \omega > 0, \\ O(t^{n-k}) & \text{if } \omega = 0, \end{cases}$$

for $0 \leq k \leq n$. We integrate the first integral in the right-hand side of (2.12) by parts n times, and we observe that

$$D_t^n \eta(t, \lambda) = \lambda \eta(t, \lambda),$$

and that, if $\varepsilon > 0$ and $\lambda \in \Lambda^*(n, n - m + 1, -\omega - \varepsilon)$, then

$$|D_t^k \eta(t, \lambda)| = O(e^{-(\omega+\varepsilon)t}) \quad \text{as } t \to \infty,$$

for all $k \geq 0$. It follows that

(2.15) $$\begin{aligned} \lambda f(\lambda, u) &= \sum_{k=0}^{n-1} (-1)^{k+1} D_t^{(n-1-k)} \eta(0, \lambda) R(\lambda_0; A)^p u^{(k)}(0) \\ &\quad + (-1)^n \int_0^\infty \eta(t, \lambda) R(\lambda_0; A)^p u^{(n)}(t)\, dt \\ &= (-1)^n R(\lambda_0; A)^p u + (-1)^n A f(\lambda, u), \end{aligned}$$

in other words, that

(2.16) $$((-1)^n \lambda I - A) f(\lambda, u) = R(\lambda_0; A)^p u$$

for $u \in D(A^p)$, $\lambda \in \Lambda^*(n, n - m + 1, -\omega - 0)$. As for the case $\omega < 0$, it is clear that the integrations by parts required in (2.15) will be permissible as soon as we establish the following result.

2.2 LEMMA. *Suppose that* $\delta > 0$ *and that* $u(\cdot)$ *is a function with values in* E, *defined and* n *times continuously differentiable in* $t \geq 0$. *Assume that*

(2.17) $$u(t) \to 0 \quad \text{as } t \to \infty,$$

(2.18) $$|u^{(n)}(t)| \leq K e^{-\delta t} \quad (t \geq 0).$$

Then there exist constants $L_0, L_1, \cdots, L_{n-1}$ *such that*

(2.19) $$|u^{(k)}(t)| \leq K L_k t^{n-1-k} e^{-\delta t} \quad (t \geq 1)$$

for $k = 0, 1, \cdots, n - 1$.

Proof. We use induction on n. When $n = 1$, we can write

$$u(t) = -\int_t^\infty u'(s)\,ds \qquad (t \geq 0).$$

This implies that

$$|u(t)| \leq K \int_t^\infty e^{-\delta s}\,ds = (K/\delta)e^{-\delta t},$$

as required. Assume now the result has been established for $k = 2, \cdots, n-1$. Then

$$u(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0) + \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} u^{(n)}(s)\,ds$$

(2.20)
$$= \sum_{k=0}^{n-1} \frac{t^k}{k!} \left\{ u^{(k)}(0) + \frac{1}{(n-k-1)!} \int_0^t (-s)^{n-k-1} u^{(n)}(s)\,ds \right\}$$

$$= \sum_{k=0}^{n-1} \frac{t^k}{k!} u_k - \sum_{k=0}^{n-1} \frac{t^k}{k!(n-k-1)!} \int_t^\infty (-s)^{n-k-1} u^{(n)}(s)\,ds,$$

where

$$u_k = u^{(k)}(0) + \frac{1}{(n-k-1)!} \int_0^\infty (-s)^{n-k-1} u^{(n)}(s)\,ds.$$

Using (2.18) in the integrals in the right-hand side of (2.20) and noting that

(2.21)
$$\int_t^\infty s^k e^{-\delta s}\,ds = (-1)^k D_\delta^k(e^{-\delta t}/\delta) \leq M_k t^k e^{-\delta t} \qquad (t \geq 1)$$

for suitable constants M_k $(k = 0, 1, \cdots)$, we see that (2.17) implies

$$u_0 = u_1 = \cdots = u_{n-1} = 0.$$

Clearly, (2.20) and (2.21) yield the estimate (2.19) for $k = 0$. Differentiating (2.20), we obtain the formula

$$u'(t) = -\sum_{k=1}^{n-1} \frac{t^{k-1}}{(k-1)!(n-k-1)!} \int_t^\infty (-s)^{n-k-1} u^{(n)}(s)\,ds$$
$$+ \left\{ \sum_{k=0}^{n-1} \frac{(-1)^{n-k-1}}{k!(n-k-1)!} \right\} t^{n-1} u^{(n)}(t).$$

Accordingly, $|u'(t)| \to 0$ as $t \to \infty$. Applying now to $u'(t)$ the result for $n-1$, we obtain all the inequalities (2.19).

We resume the proof of Theorem 2.1. The underlying idea in what follows is to continue the resolvent $R((-1)^n \lambda; A)$ to all of $\Lambda^*(n, n - m + 1, -\omega - 0)$ by means of equality (2.16). There are, however, certain technical difficulties. First, the extension in (2.16) is pointwise; second, it exists only for $u \in D(A^p)$, and third, what is extended is not $R((-1)^n \lambda; A)$ but rather $R(\lambda_0; A)^p R((-1)^n \lambda; A)$. To get rid of the first difficulty, we use the following extension argument.

2.3 LEMMA. *Let* X *be a complex Banach space, let* D_1 *and* D_2 *be two domains in the complex plane* $(D_1 \subseteq D_2)$, *and let* $\lambda \to F(\lambda)$ *be a function with values in the space* $\mathscr{L}(X)$ *of bounded operators in* X, *defined and analytic in* D_1. *Assume that for every* $u \in X$ *the* X*-valued function* $\lambda \to F(\lambda)u$ *admits an analytic extension* $\lambda \to f(\lambda, u)$ *to all of* D_2. *Then* $F(\cdot)$ *admits an* $\mathscr{L}(X)$*-valued analytic extension* $G(\cdot)$ *to all of* D_2, *defined by*

$$G(\lambda)u = f(\lambda, u) . \tag{2.22}$$

Proof. Let U be a bounded domain in the complex plane, such that $U \cap D_1 \neq \emptyset$ and $\mathrm{Cl}\, U \subseteq D_2$. Clearly, it is sufficient to show that for each such U the function $F(\cdot)$ can be extended to $D_1 \cup U$ in the way described in Lemma 2.3. Let $\mathscr{A}(U, X)$ be the Banach space of all X-valued functions $f(\cdot)$ that are analytic in U and continuous in $\mathrm{Cl}\, U$, with the norm $\|f\| = \sup\{|f(\lambda)|_X ;\ \lambda \in \mathrm{Cl}\, U\}$, and let $\mathscr{M}: X \to \mathscr{A}(U, X)$ be the operator defined by

$$\mathscr{M} u = f(\lambda, u) .$$

It is easy to see that $\mathscr{M}$ is linear and closed; since it is defined everywhere in X, an application of the closed-graph theorem shows that it is *bounded*. But this plainly implies that for every $\lambda \in U$ the operator defined by (2.22) is bounded. On the other hand, it follows again from (2.22) and from Theorem 3.10.1 in [11] that $G(\cdot)$ is analytic as an $\mathscr{L}(X)$-valued function in U.

We shall now apply Lemma 2.3 in the following situation. X will be the space $D(A^p)$ endowed with the norm

$$|u|_X = |(\lambda_0 I - A)^p|$$

(verification that X is a Banach space is simple). The domains will be

$$D_1 = (-1)^n \rho(A) \cap \Lambda^*(n, n - m + 1, -\omega - 0),$$

$$D_2 = \Lambda^*(n, n - m + 1, -\omega - 0) .$$

The function to be extended is

$$F(\lambda) = R(\lambda_0; A)^p R((-1)^n \lambda; A) \qquad (\lambda \in D_1). \tag{2.23}$$

Since for $u \in X$ we have the relations

$$|F(\lambda)u|_X = |R((-1)^n\lambda; A)u|_E = |R(\lambda_0; A)^p((-1)^n\lambda; A)(\lambda_0 I - A)^p u|_E$$
$$\leq |F(\lambda)|_{\mathscr{L}(E)} |u|_X,$$

it is clear that $F(\lambda) \in \mathscr{L}(X)$ for all $\lambda \in D_1$. That $F(\cdot)$ is analytic—as an $\mathscr{L}(X)$-valued function—follows immediately from the analyticity of $R(\cdot\,;A)$ and from the fact that $R(\lambda_0\,;A)^p$ maps E continuously into X. In the same way it can be proved that $f(\cdot\,;u)$ — as an X-valued function—is analytic in D_2 for all $u \in X$; on the other hand, it is clear from (2.16) that $f(\cdot\,;u)$ is an extension of $F(\cdot)u$ to all of D_2. From Lemma 2.3 it follows that there exists a function $G(\cdot)$ with values in $\mathscr{L}(X)$, defined and analytic in

$$\Lambda^*(n,\, n - m + 1,\, -\omega - 0)\,,$$

that coincides with $R(\lambda_0\,;A)^p R((-1)^n\lambda;A)$ in D_1. For $\lambda \in D_1$ and $u \in D(A^{2p})$ we have the equality

$$(\lambda_0 I - A)^p G(\lambda)u = G(\lambda)(\lambda_0 I - A)^p u \tag{2.24}$$

(a consequence of the definition of $G(\lambda)$ in D_1). Since both sides of (2.24) are analytic in E, (2.24) must hold for all $\lambda \in D_2$. Accordingly, if $u \in D(A^{2p})$ and $\lambda \in \Lambda^*(n,\, n - m + 1,\, -\omega - 0)$, then

$$\begin{aligned} |G(\lambda)(\lambda_0 I - A)^p u|_E &= |(\lambda_0 I - A)^p G(\lambda)u|_E = |G(\lambda)u|_X \\ &\le K|u|_X = K|(\lambda_0 I - A)^p u|_E\,. \end{aligned} \tag{2.25}$$

But $(\lambda_0 I - A)^p D(A^{2p}) = D(A^p) = X$; thus the preceding inequality implies that $G(\lambda)$ is bounded in the norm of E. Accordingly, $G(\lambda)$ can be extended to all of E as a bounded operator, and we shall denote this operator by $\tilde{G}(\lambda)$. By the definition of $\tilde{G}$,

$$((-1)^n\lambda I - A)\tilde{G}(\lambda)u = R(\lambda_0\,;A)^p u \tag{2.26}$$

for $u \in X$. But, since A is closed, $\tilde{G}(\lambda)$ is bounded, and X is dense in E (all in the norm of E), we can extend (2.25) to every $u \in E$. This shows in particular that

$$\tilde{G}(\lambda)E \subseteq D(A^{p+1})\,. \tag{2.27}$$

Next, let $u \in D(A^{p+1})$. An argument similar to the one used to establish (2.24) shows that

$$((-1)^n\lambda I - A)\tilde{G}(\lambda)u = \tilde{G}(\lambda)((-1)^n\lambda I - A)u \tag{2.28}$$

for all $\lambda \in \Lambda^*(n,\, n - m + 1,\, -\omega - 0)$. Now, given a $u \in D(A)$ and a particular λ, choose a sequence $\{v_m\}$ in X such that $v_m \to ((-1)^n\lambda I - A)u$. If we then set $u_m = R((-1)^n\lambda;A)v_m$, it is clear that

$$u_m \to u \qquad \text{and} \qquad ((-1)^n\lambda I - A)u_m \to ((-1)^n\lambda I - A)u\,.$$

Making use of (2.28) for each n, and then letting $n \to \infty$, we see that this equality holds for all $u \in D(A)$. An argument along the same lines shows that (2.24) can be extended to all $u \in X$, in other words, that

$$(\lambda_0 I - A)^p\tilde{G}(\lambda)u = \tilde{G}(\lambda)(\lambda_0 I - A)^p u \qquad (u \in X)\,. \tag{2.29}$$

Finally, we combine the preceding steps. For $\lambda \in \Lambda^*(n,\, n - m + 1,\, -\omega - 0)$, define

$$R(\lambda) = (\lambda I - A)^p G(\tilde{\lambda})\,. \tag{2.30}$$

It follows from (2.27) and from the closed-graph theorem that $R(\lambda)$ is bounded; it is also clear that

$$R(\lambda)E \subseteq D(A).$$

Premultiplying (2.26) by $(\lambda_0 I - A)^p$, we obtain the equation

$$((-1)^n \lambda I - A)R(\lambda)u = u \qquad (u \in E),$$

and using (2.28), we see that

$$R(\lambda)((-1)^n \lambda I - A)u = u \qquad (u \in D(A)).$$

This evidently shows that $R(\lambda) = R((-1)^n \lambda; A)$, and this completes the proof of (2.4). As for the estimate (2.5), observe that if $u \in D(A^p)$, then, by virtue of (2.12) and (2.30),

$$R((-1)^n\lambda; A)u = (\lambda_0 I - A)^p f(\lambda, u).$$

Applying (2.13), we obtain (2.5) immediately.

2.4 COROLLARY. *Assume that either*

(a) n *is even,* $\omega < 0$, *and* $m > n/2$, *or*

(b) n *is even,* $\omega = 0$, *and* $m > (n+2)/2$, *or*

(c) n *is odd,* $\omega \leq 0$, *and* $m > (n+1)/2$.

Then there are no operators A *satisfying the assumptions in Theorem* 2.1.

Proof. (a) If $m > n/2$ and $\omega < 0$, then $n - m + 1 < (n+2)/2$, and $\Lambda^*(n, n-m+1, -\omega-\varepsilon) = C$ if ε is so small that $-\omega - \varepsilon \geq 0$. Consequently, it follows from (2.5) that if A satisfies the conditions in Theorem 2.1, then $\rho(A) = C$. On the other hand, (2.5) shows that $R(\lambda; A)u$ is *bounded* in C for $u \in D(A^p)$. By Liouville's theorem, $R(\lambda; A)u$ is then constant, a contradiction if $u \neq 0$. To prove (b), observe that if $m > (n+2)/2$, then $n - m + 1 < n/2$. It is not difficult to see that if

$$\varepsilon > 0 \quad \text{and} \quad |\lambda| \geq \varepsilon^n \left(\sin\frac{\pi}{n}\right)^{-n},$$

then $\lambda \in \Lambda^*(n, n-m+1, -\varepsilon)$. This shows that $\rho(A) \supseteq C \setminus \{0\}$ and that

$$|R(\lambda; A)u| \leq K|\lambda|^{-(n-k_0)/n} \tag{2.31}$$

when $u \in D(A^p)$. If $k_0 \neq 0$, then $(n - k_0)/n < 1$ and the (possible) singularity of $R(\cdot\,; A)u$ at $\lambda = 0$ is removable; in this case, the proof ends like that of (a). If $k_0 = 0$, $R(\cdot\,; A)u$ may have (at most) a pole of order 1 at the origin. Since it vanishes at infinity, we see that

$$R(\lambda; A)u = \lambda^{-1}u_0 \tag{2.32}$$

for some $u_0 \in D(A)$. It follows that $(\lambda I - A)R(\lambda; A)u = u_0 - \lambda^{-1}Au_0 = u$, which implies that $Au_0 = 0$ and $u = u_0$. Since A is closed and $D(A)$ is dense in E, $A = 0$. But in that case, the only bounded solutions of (2.1) are constants, which precludes the choice (2.2) unless $k_0 = 0$, $\alpha = \{0\}$, and $m = 1$; this is absurd, in view of the condition $m > n/2 + 1$. As for (c), observe that in this case

$$n - m + 1 < (n+1)/2 .$$

If $\omega < 0$ and $\varepsilon > 0$ is such that $-\omega - \varepsilon \geq 0$, then $\Lambda^*(n, n - m + 1, -\omega - \varepsilon) = C$, and the proof ends like that of part (a). If $\omega = 0$, we need only observe that exactly as in (b) there exists a constant $k > 0$ such that $\lambda \in \Lambda^*(n, n - m + 1, -\varepsilon)$ if $|\lambda| > k\varepsilon^n$. The proof ends like that of (b).

We observe that (because of the identification above of the regions $\Lambda^*(n, n - m + 1, -\omega - \varepsilon)$) the hypothesis on $\rho(A)$ in Theorem 2.1 reduces to

$$\rho(A) \neq \emptyset,$$

which was assumed to hold from the beginning.

We look next to the same range of m for positive ω.

2.5 COROLLARY. *Let* $\omega > 0$. *Assume that* A *satisfies the assumptions in Theorem* 2.1 *when either*

(a) n *is even and* $m > (n+2)/2$ *or*

(b) n *is odd and* $m > (n+1)/2$.

Then A *is bounded.*

For the proof we shall need the following result.

2.6 LEMMA. *Let* A *be a closed operator defined in a dense subset of* E *such that* (a) $\sigma(A)$ *is bounded,* (b) *for every* u *in a dense set* D *and for all large enough* $|\lambda|$,

$$|R(\lambda; A)u| \leq K|u|^m, \tag{2.33}$$

where both the constant $K > 0$ *and the integer* m *may depend on* u. *Then* A *is bounded.*

Proof. Let $a > 0$ be such that $\sigma(A) \subset \{\lambda;\ |\lambda| < a\}$. Then $R(\cdot\,; A)$ can be developed into a Laurent series about ∞,

$$R(\lambda; A) = \sum_{n=-\infty}^{\infty} \lambda^{-n} A_n , \tag{2.34}$$

that converges in $\Re\lambda > a$ (in the $\mathscr{L}(E)$-norm), the coefficients A_n being the elements of $\mathscr{L}(E)$ given by

$$A_n = \frac{1}{2\pi i} \int_{|\lambda|=a} \lambda^{n-1} R(\lambda; A)\, d\lambda \qquad (n = \cdots, -1, 0, 1, \cdots) \tag{2.35}$$

(in particular, it follows from (2.35) that $A_n E \subseteq D(A)$ for all n). Substituting (2.34) in the identity $(\lambda I - A)R(\lambda; \lambda; A) = I$ and equating coefficients in the series thus obtained, we are led to the equations

$$A_1 = AA_0 + I, \tag{2.36}$$

$$A_{n+1} = AA_n \qquad (n \neq 0) . \tag{2.37}$$

Making use of (2.37) for $n < 0$ and of the fact that $A_n u = 0$ when $-n$ is large enough and $u \in D$ (immediate consequence of (2.33)), we obtain the equation $A_{n+1} u = \cdots = A_0 u = 0$; since A_0 is a bounded operator, $A_0 = 0$. We then see from (2.36) that $A_1 = I$, $A_2 = A$, which shows that A itself belongs to $\mathscr{L}(E)$, as we claimed.

Proof of Corollary 2.5. (a) In this case, $n - m + 1 < n/2$. But $\Lambda^*(n, n - m + 1, -\omega - \varepsilon)$ is the complement of a bounded region; thus the result follows from the estimate (2.5) *via* Lemma 2.6. The proof of (b) is entirely similar.

2.6 *Remark.* We point out that Theorem 3.1 fails to yield significant information on $\rho(A)$ when m is much smaller than n (as we shall see later in Lemma 3.2, this is due to the nature of the problem and not to the method of proof). We make this precise in what follows. First, let n be even, $m < n/2$. Then, $n - m + 1 > (n+2)/2$, and clearly the set $\Lambda^*(n, n - m + 1, -\omega - \varepsilon)$ is empty if $\omega \geq 0$ for all $\varepsilon > 0$ and bounded if $\omega > 0$ (and $-\omega - \varepsilon > 0$). In the last case, we obtain the existence of $R(\lambda; A)$ in a neighborhood of the origin. On the other hand, if $m = n/2$ and $\omega > 0$, then again $\Lambda^*(n, n - m + 1, -\omega - \varepsilon) = \emptyset$, and we obtain no information. In the case where n is odd, the situation is more or less similar; for if $m < (n+1)/2$, then $n - m + 1 > (n+1)/2$, and then the set $\Lambda^*(n, n - m + 1, -\omega - \varepsilon)$ is empty for $\omega \geq 0$ and bounded for $\omega < 0$.

We end this section by showing that in certain cases that will be of interest later, the region Λ^* can be completely identified without much trouble, and the inequality (2.5) can be written in a more manageable way.

Throughout the remainder of this section, we assume that $\omega = 0$. Then it is not difficult to see that

$$\text{(2.38)} \quad \Lambda^*(n, n/2, 0 - 0) = \begin{cases} C \setminus \{\lambda; \lambda = \eta,\ \eta \geq 0\} & \text{if } n = 4k \text{ and } k \geq 1, \\ C \setminus \{\lambda; \lambda = \eta,\ \eta \leq 0\} & \text{if } n = 4k+2 \text{ and } k \geq 0. \end{cases}$$

If $n = 4k$ and $\lambda \in \Lambda^*(n, n/2, 0 - 0)$, then $\lambda \in \Lambda^*(n, n/2, -\varepsilon)$, where $\varepsilon = |\lambda|^{1/n} \sin |(\arg \lambda)/n|$ (here $\arg \lambda$ is taken in the interval $(-\pi, \pi]$). Consequently, if the conditions of Theorem 2.1 are satisfied for $m = (n+2)/2$, then $R(\lambda; A)$ exists for all $\lambda \neq \eta$ $(\eta \geq 0)$, and it satisfies the inequality

$$\text{(2.39)} \qquad |R(\lambda; A) u| \leq K \left|\sin \frac{1}{n}(\arg \lambda)\right|^{-1} |\lambda|^{-(n-k_0)/n}$$

for each $u \in D(A^p)$. When $n = 4k+2$, the results are the "mirror image" of the previous ones, and we omit the details. Finally, assume that n is odd and that $m = (n+1)/2$. Then $R(\lambda; A)$ exists in the domain

$$\text{(2.40)} \quad -\Lambda^*(n, (n+1)/2, -0 - 0) = \begin{cases} \{\lambda; \Re \lambda > 0\} & \text{if } n = 4p+1 \text{ and } p \geq 1, \\ \{\lambda; \Re \lambda < 0\} & \text{if } n = 4p+3 \text{ and } p \geq 0. \end{cases}$$

In the first case, if $\lambda \in -\Lambda^*(n, (n+1)/2, -0 - 0)$, then λ belongs to $-\Lambda^*(n, (n+1)/2, -\omega - \varepsilon)$, where $\varepsilon < |\lambda|^{1/n} \sin \frac{1}{2n}\left(\frac{\pi}{2} - |\arg \lambda|\right)$. Accordingly, if $u \in D(A^p)$, then

$$\text{(2.41)} \qquad |R(\lambda; A) u| \leq K \left(\sin \frac{1}{2n}\left(\frac{\pi}{2} - |\arg \lambda|\right)\right)^{-1} |\lambda|^{-(n-k_0)/n}.$$

Again, the case $m = (n - 1)/2$ is the reflection through the imaginary axis of the present one, and we omit it.

A typical application of inequalities (2.39) and (2.41) follows. Assume that A satisfies the conditions of Theorem 2.1, with n *even*, $m = (n+2)/2$, and $\omega = 0$ (no other restriction on α). Let Γ be a line that is contained in $\Lambda^*(n, n - m + 1, 0 - 0)$ and passes through the origin. Then it follows from (2.39) and from the uniform-boundedness theorem that

$$|R(\lambda; A) R(\lambda_0; A)^p| \leq K |\lambda|^{-\delta} \tag{2.42}$$

for $\lambda \in \Gamma$, where K is a convenient constant and $\delta = (n - k_0)/n < 1$ (λ_0 is any fixed element of $\rho(A)$). But

$$R(\lambda; A) R(\lambda_0; A)^p = (\lambda_0 - \lambda)^{-1} (R(\lambda; A) R(\lambda_0; A)^{p-1} - R(\lambda_0; A)^p);$$

thus (2.42) holds near the origin if we replace p by $p - 1$. Iterating the argument, we arrive at the estimate

$$|R(\lambda; A)| \leq K' |\lambda|^{-\delta}$$

for (say) $\lambda \in \Gamma$ and $|\lambda| \leq 1$. But by virtue of [6, Corollary VII 3.3], we must have the inequality $|R(\lambda; A)| \geq |\lambda|^{-1}$ if $0 \in \sigma(A)$; this shows that $0 \in \rho(A)$, in other words, that A has a bounded inverse. A similar result can be obtained on the basis of (2.41) in the case where n is odd and $m = (n+1)/2$ (for results on the existence of A^{-1} in other cases, see Remark 2.7).

2.7 *Remark.* Results of the type proved here—but generally stronger—are proved in [15] under stronger assumptions on the equation (2.1). Essentially, it is assumed there that conditions (I) and (II) in the next section hold. On the basis of (I) and (II), certain operator-valued solutions of (2.1) (the *propagators*) are constructed; with their help it is proved, among other things, that the subspace D in condition (I) must contain D(A), that (I) and (II) imply (I) for the adjoint equation $u^{(n)}(t) = A^* u(t)$ in the dual space E^*, and that in all cases $\rho(A)$ must contain the origin. Also, the estimate (2.5) is shown to hold uniformly on u for all $u \in E$. In the case $m = 1$, it is proved that $A = B^n$, where B is a semigroup generator. For proofs of this and other results, the reader may consult [15].

3. SUFFICIENT CONDITIONS

As before, let α denote a subset of the set $\{0, 1, \cdots, n - 1\}$, and let m be the number of elements of α. To avoid repetition, we shall say that $A \in \Phi(n, \alpha, \omega)$ provided the following two conditions are satisfied.

(I) The initial-value problem

$$u^{(n)}(t) = A u(t) \quad (t \geq 0), \tag{3.1}$$

$$|u(t)| \leq K e^{\omega t} \quad (t \geq 0), \tag{3.2}$$

$$u^{(k)}(0) = u_k \quad (k \in \alpha) \tag{3.3}$$

has a solution $u(\cdot)$, for each u_k ($k \in \alpha$) in a dense subspace D of E (K may depend on $u(\cdot)$).

(II) If $\{u_n(\cdot)\}$ is a sequence of solutions of (3.1) that satisfy (3.2), and if

$$u_n^{(k)}(0) \to 0 \qquad (k \in \alpha),$$

then

$$e^{-\omega t} u_n(t) \to 0$$

uniformly in $t \geq 0$. (Note that (II) implies uniqueness of solutions of (3.1) having the same initial data (3.3) and satisfying the growth condition (3.2); on the other hand, no uniqueness assumption is made on solutions of (3.1) that do not satisfy the growth condition.) Although it might be interesting to separate the existence and uniqueness questions, in what follows—as was done in Section 2—we combine them for reasons of brevity. Likewise, all the results in this section will refer to the case where (I) and (II) are satisfied for all α having a fixed number of elements m; to simplify the notation later on, we define

$$\Phi(n, m, \omega) = \bigcap_{\alpha} \Phi(n, \alpha, \omega),$$

where the intersection is taken over all sets α with m elements.

3.1 LEMMA. *Assume that* $A = B^n$, *where* (a) $0 \in \rho(B)$, (b) *for every* k $(0 \leq k \leq m - 1)$, $\gamma^k B$ $(\gamma = e^{2\pi i/n})$ *generates a strongly continuous semigroup* $T_k(\cdot)$ *such that*

$$|T_k(t)| \leq K e^{\omega t} \qquad (t \geq 0), \tag{3.4}$$

(c) *if* $m \leq k \leq n - 1$ *and* $u_k(\cdot)$ *satisfies the conditions*

$$u'(t) = \gamma^k B u(t) \qquad (t \geq 0) \tag{3.5}$$

and

$$|u_k(t)| = O(t^n e^{\omega t}) \qquad \text{as } t \to \infty, \tag{3.6}$$

then $u(t) = 0$ $(t \geq 0)$.

Then $A \in \Phi(n, m, \omega)$.

Proof. Let α be a subset of $\{0, 1, \cdots, n - 1\}$ containing m elements. If $v_0, v_1, \cdots, v_{m-1} \in D(B^n) = D(A)$, then the function

$$u(t) = \sum_{j=0}^{m-1} T_j(t) v_j \tag{3.7}$$

is a solution of (3.1) and satisfies the growth condition (3.2). It also satisfies (3.3) if

$$\sum_{j=0}^{m-1} \gamma^{jk} v_j = B^{-k} u_k \qquad (k \in \alpha). \tag{3.8}$$

Just as in the scalar case, we can easily see that the system of linear equations (3.8) has a unique solution, its determinant $\{\gamma^{jk}\}$ $(0 \leq m - 1,\ k \in \alpha)$ being nonzero; if $u_k \in D(B^k)$, then the theorem of Rouché and Frobenius implies that $v_j \in D(B^n)$, as required. Moreover, there exists a constant K (depending only on α) such that

$$|v_j| \leq K \sum_{k\in\alpha} |u_k| \qquad (0 \leq j \leq n-1). \tag{3.9}$$

Therefore it follows from (3.7) that

$$|\tilde{u}(t)| \leq K' e^{\omega t} \sum_{k\in\alpha} |u_k| , \tag{3.10}$$

where K' is a constant independent of u_k $(k \in \alpha)$.

To verify (II), let $u(\cdot)$ be any solution of (3.1) satisfying the growth condition (3.3) and such that

$$u^{(k)}(0) = 0 \qquad (k \in \alpha) . \tag{3.11}$$

Define

$$\tilde{u}(t) = B^{-n}u(t) = A^{-1}u(t) \qquad (t \geq 0).$$

Then $\tilde{u}(\cdot)$ satisfies (3.1), (3.2), and (3.11). Moreover, since $\tilde{u}^{(n)}(t) = u(t)$, we also have the estimate

$$|\tilde{u}^{(n)}(t)| = O(e^{\omega t}) \qquad \text{as } t \to \infty.$$

Applying Lemma 2.2 and the preceding comments, we see that for $0 \leq k \leq n-1$,

$$|u^{(k)}(t)| = \begin{cases} O(e^{\omega t}) & \text{if } \omega > 0, \\ O(t^{n-k}) & \text{if } \omega = 0, \\ O(t^{n-k-1} e^{\omega t}) & \text{if } \omega < 0 \end{cases}$$

as $t \to \infty$; that is, all the derivatives of u of these orders satisfy condition (3.6). Then so do the functions

$$u_k(t) = \sum_{j=0}^{n-1} \gamma^{kj} B^{-j} \tilde{u}^{(j)}(t) \qquad (0 \leq k \leq n-1). \tag{3.12}$$

But it follows from a direct computation that $u_k(\cdot)$ satisfies the condition

$$u_k'(t) = \gamma^k B u(t) \qquad (t \geq 0,\ 0 \leq k \leq n-1),$$

and it follows from (b) that

$$u_k(t) = 0 \qquad (t \geq 0,\ m \leq k \leq n-1). \tag{3.13}$$

From (a) and a well-known result in the theory of semigroups [12, Chapter 1, Theorem 2.7] it follows that

$$u_k(t) = T_k(t) u_k(0) \qquad (t \geq 0,\ 0 \leq k \leq m-1). \tag{3.14}$$

We now write (3.12) for $t = 0$; using (3.13) and (3.11), we obtain the equation

$$\sum_{j \notin \alpha} \gamma^{kj} B^{-j} \tilde{u}^{(j)}(0) = 0 \qquad (m \le k \le n-1). \tag{3.15}$$

Since $\det\{\gamma^{kj}\}$ $(j \notin \alpha,\ m \le k \le n-1)$ is not zero, (3.15) implies that $\tilde{u}^{(k)}(0) = 0$ for $0 \le k \le n-1$. By (3.12), $u_k(0) = 0$ for all k; accordingly, it follows from (3.14) that $u_k(t) = 0$ $(t \ge 0,\ 0 \le k \le n-1)$. Finally, observing that

$$\tilde{u}(t) = u_1(t) + \cdots + u_n(t),$$

we see that $\tilde{u}(\cdot)$ —and *a fortiori* $u(\cdot)$ —vanishes identically.

The uniqueness property just established implies that every solution of (3.1), (3.2), (3.3) satisfying the condition

$$u^{(k)}(0) \in D(B^{n-k}) \qquad (k \in \alpha) \tag{3.16}$$

must be given by formula (3.7), where $v_0, \cdots, v_{m-1}$ are solutions of (3.8); in view of (3.9), this implies condition (II). As for solutions $u(\cdot)$ that do not necessarily satisfy (3.16), the previous reasoning applies to $u(t) = A^{-1}\tilde{u}(t)$; thus $\tilde{u}$ must be given by (3.7). But then, as we see by premultiplying by A, the same formula holds for $u(\cdot)$. Together with (3.9), this clearly implies condition (II). The proof of Lemma 3.1 is now complete.

We first use Lemma 3.1 to investigate the possibility of extending the nonexistence results of Section 2. In Section 1, we observed that for certain choices of ω the solutions of (3.1) satisfying the conditions (3.2) and (3.3) are never unique if the number of initial conditions is too small; we now show that this result cannot be extended to the general case. In fact, we prove that the class $\Phi(n, m, \omega)$ is never empty, except in the cases ruled out by the results of Section 2.

3.2 LEMMA. *Suppose* ω *is a real number,* m *and* n *are integers such that* $1 \le m \le n$, *and* D *is a domain in the complex plane such that*

(i) $0 \notin \mathrm{Cl}\, D$,

(ii) $\gamma^k D \subseteq \{\lambda;\ \Re\lambda \le \omega\}$ *for* $0 \le k \le m-1$,

(iii) $\gamma^k D \cap \{\lambda;\ \Re\lambda > \omega\} \ne \emptyset$ *for* $m \le k \le n-1$.

Then there exists on operator $B\ (= B_{n,m})$ *in a separable Hilbert space* H *such that* $A = B^n \in \Phi(n, m, \omega)$.

Proof. Let H be the space of all analytic functions $\zeta \to u(\zeta)$ in D such that

$$\int_D |u(\zeta)|^2 \, d\sigma < \infty,$$

and let H be endowed with the scalar product

$$\langle u(\cdot), v(\cdot) \rangle = \int_D u(\zeta)\overline{v(\zeta)}\, d\sigma \tag{3.17}$$

(here $d\sigma$ denotes the area differential in $R^2 \cong C$). Then H is a (nontrivial) Hilbert space and a subspace of the Lebesgue space $L^2(D)$. (For proofs, see [14, Chapter V, Section 10]. The only nontrivial property to be established is *completeness* of H. It can be proved with the aid of *Poincaré's inequality*

$$|u(\zeta)| \leq \pi^{-1/2}\rho^{-1}|u|_H, \tag{3.18}$$

valid for all $u \in H$ and all $\zeta \in D$; here ρ denotes the distance between ζ and the boundary of D.)

Define an operator B in H by the equation

$$(Bu)(\zeta) = \zeta u(\zeta), \tag{3.19}$$

where D(B) is the set of all $u \in H$ such that the mapping $\zeta \to \zeta u(\zeta)$ also belongs to H. It is easy to see that $\rho(B)$ consists exactly of the closure of D; in particular, B^{-1} exists and is given by the equation

$$(B^{-1}u)(\zeta) = \zeta^{-1}u(\zeta) \qquad (u \in H).$$

For $0 \leq k \leq m - 1$, define T_k by the formula

$$(T_k(t)u)(\zeta) = e^{\gamma^k \zeta t}u(\zeta) \qquad (t \geq 0).$$

Simple manipulations show that $T_k(\cdot)$ is a strongly continuous semigroup whose infinitesimal generator is $\gamma^k B$; moreover,

$$|T_k(t)| \leq e^{\omega t} \qquad (t \geq 0,\ 0 \leq k \leq m - 1). \tag{3.20}$$

We have now verified conditions (a) and (b) in Lemma 3.1. As for (c), let $m \leq k \leq n - 1$, and let $u(\cdot)$ be a solution of the equation

$$u'(t) = \gamma^k B(u(t)) \qquad (t \geq 0).$$

A simple reasoning based on Poincaré's inequality (3.18) shows that u, as a function of the two variables t and ζ, must be continuously differentiable with respect to t for fixed ζ in D, and that it must satisfy the differential equation

$$\frac{\partial u(t, \zeta)}{\partial t} = \gamma^k \zeta u(t, \zeta).$$

It follows that

$$u(t, \zeta) = e^{\gamma^k \zeta t}u(0, \zeta). \tag{3.21}$$

Assume now that $u(0, \zeta)$ is not identically zero in D. Since a nonzero analytic function can vanish only at isolated points, it is clear that we can use (iii) to show the existence of three positive numbers $\delta, \varepsilon, \omega'$ $(\omega' > \omega)$ and a point ζ_0 in D such that

$$\{\lambda; \lambda = \gamma^k\zeta, |\zeta - \zeta_0| \leq \delta\} \subset \{\lambda; \Re\lambda \geq \omega'\}, \tag{3.22}$$

$$|u(0, \zeta)| \geq \varepsilon \qquad \text{for } |\zeta - \zeta_0| \leq \delta. \tag{3.23}$$

But now it follows from (3.21) that if $t > 0$, then

$$|u(t, \cdot)|_H^2 > \int_D e^{2\Re(\gamma^k\zeta)t}|u(0, \zeta)|^2 d\sigma > \pi\varepsilon^2 e^{2\omega' t}. \tag{3.24}$$

Accordingly, $u(\cdot)$ satisfies (3.6) only if $u(0) = 0$; in view of (3.21), this implies that $u(t) \equiv 0$ for all $t \geq 0$. Consequently, (c) in Lemma 3.1 holds, and the proof of Lemma 3.2 is complete.

We point out that corresponding to each triple (n, m, ω) we can find a region D satisfying the conditions in Lemma 3.2, unless $\omega \leq 0$ and either

$$\text{n is even,} \quad m > n/2 \tag{3.25}$$

or

$$\text{n is odd,} \quad m > (n+1)/2\,. \tag{3.26}$$

In fact, if $\omega > 0$, or if $\omega \leq 0$ but m and n are related by neither (3.25) nor (3.26), the region $\Lambda(n, m, \omega)$ defined at the beginning of Section 2 has nonempty interior, which can then be taken as D. Geometric considerations make it clear that D cannot exist when $\omega \leq 0$ and (3.25) or (3.26) holds; observe that this also follows from Corollary 2.4, which implies that $\Phi(n, m, \omega) = \emptyset$ for these values of m, n, ω, except for the case where $\omega = 0$, n is even, and $m = (n+2)/2$. We shall discuss this exceptional case later (Theorem 3.8). Observe also that the region D defined by (3.27) is unbounded—thus the corresponding operator B (and the operator $A = B^n$) is unbounded—except in the case where $\omega > 0$, n is even, and $m > (n+2)/2$ or n is odd and $m > (n+1)/2$. But it is a consequence of Corollary 2.5 that for these values of m, n, and ω, each $A \in \Phi(n, m, \omega)$ is bounded. This shows that the operator A provided by Lemma 3.2 in these cases is bounded, not because of the particular nature of A but because of the nature of the problem.

A final comment concerning the uniqueness of solutions of (3.1): The following proposition implies that if $\omega \geq 0$ and m is small with respect to n, then the class $\Phi(n, m, \omega)$ does not contain any normal operators in Hilbert space.

3.3 PROPOSITION. *For each integer* n $(n > 1)$, *each integer* m $(m \leq n)$, *and each real number* ω, *a normal operator* A *in a Hilbert space* H *belongs to the class* $\Phi(n, m, \omega)$ *if and only if* $0 \in \rho(A)$ *and* $\sigma(A) \subseteq \Lambda(n, m, \omega) \cup N$, *where* N *is a set of* A-*spectral measure zero.*

The proof is elementary but not trivial. It can be found in [15], together with additional information on the examples mentioned after the proof of Lemma 3.2 and on the regions $\Lambda(n, m, \omega)$. We shall only observe that since $\Lambda(n, m, \omega) = \emptyset$ if $\omega \geq 0$ and either n is even and $m < n/2$ or n is odd and $m < (n-1)/2$, Proposition 3.3 implies that no normal operator in Hilbert space belongs to $\Phi(n, m, \omega)$ for these values of n, m, ω.

We end this section with some results giving sufficient conditions for an operator A to belong to $\Phi(n, m, \omega)$ for various values of n, m, and ω. To avoid 2-dimensional geometric complications, we shall consider only the case $\omega = 0$; extension to different values of ω is not unduly difficult.

As before, A is a closed operator defined in a dense subspace of E. We shall say that A belongs to $\Xi(\phi)$ $(0 < \phi \leq \pi)$ if $0 \in \rho(A)$,

$$\{\lambda;\ |\arg \lambda| \geq \phi\} \subseteq \rho(A)\,, \tag{3.27}$$

and

$$|R(\lambda; A)| \leq K\,|\lambda|^{-1} \tag{3.28}$$

for some constant $K > 0$ and $|\arg \lambda| \geq \phi$ (here, $\arg \lambda$ is taken in the interval $(-\pi, \pi]$).

3.4 THEOREM. (a) *If* $n = 4p + 2$ $(p = 0, 1, \cdots)$, *then* $\Xi(\pi) \subseteq \Phi(n, n/2, 0)$.

(b) *If* $n = 4p$ $(p = 1, 2, \cdots)$ *and* $-A \in \Xi(\pi)$, *then* $A \in \Phi(n, n/2, 0)$.

(c) *If* $n = 4p + 3$ $(p = 0, 1, \cdots)$ *and* $A \in \Xi(\pi/2)$, *then* $A \in \Phi(n, (n+1)/2, 0)$ *and* $-A \in \Phi(n, (n-1)/2, 0)$.

(d) *If* $n = 4p + 1$ $(p = 1, 2, \cdots)$ *and* $A \in \Xi(\pi/2)$, *then* $A \in \Phi(n, (n-1)/2, 0)$ *and* $-A \in \Phi(n, (n+1)/2, 0)$.

We shall prove Theorem 3.4 by showing that the conditions of Lemma 3.1 are satisfied for some convenient nth-root of A. We need several auxiliary results.

3.5 LEMMA. *If* $0 < \phi < \pi$, $A \in \Xi(\phi)$, K *denotes the constant in* (3.28), *and* ϕ' *is a real number such that*

$$\phi - \arcsin(1/K) < \phi' < \phi,$$

then $A \in \Xi(\phi')$.

Proof. If $\lambda_0 \in \rho(A)$, then $R(\lambda; A)$ exists for $|\lambda - \lambda_0| < 1/|R(\lambda_0; A)|$, and it is given by the series

$$R(\lambda; A) = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0; A)^{n+1}; \tag{3.29}$$

therefore we have the estimate

$$|R(\lambda; A)| \le \frac{|R(\lambda_0; A)|}{1 - |\lambda - \lambda_0|\,|R(\lambda_0; A)|}. \tag{3.30}$$

Assume now that $|\arg \lambda| \ge \phi'$, and let

$$\lambda_0 = \lambda \pm i\lambda \tan(\phi - \phi') \tag{3.31}$$

(the + sign is used when $\Re\lambda \ge 0$, the - sign when $\Re\lambda < 0$). Then it is easy to see that

$$|\arg \lambda_0| \ge \phi \quad \text{and} \quad |\lambda - \lambda_0| = |\lambda_0| \sin(\phi - \phi') < \frac{|\lambda_0|}{K} \le \frac{1}{|R(\lambda_0; A)|}. \tag{3.32}$$

Consequently, formula (3.29) is applicable and shows that $\lambda \in \rho(A)$; since $|\lambda_0| = |\lambda|/\cos(\phi - \phi')$, the estimate (3.30) now yields the estimate

$$|R(\lambda; A)| \le \frac{K}{|\lambda_0|(1 - K^{-1}\sin(\phi - \phi'))} = \frac{K\cos(\phi - \phi')}{|\lambda|(1 - K^{-1}\sin(\phi - \phi'))},$$

which shows that $A \in \Xi(\phi')$. This ends the proof of Lemma 3.6.

Suppose now that $\phi > 0$ and $A \in \Xi(\phi)$. From Lemma 3.5 and the fact that $0 \in \rho(A)$, we deduce that there exists a number $\varepsilon > 0$ such that the contour

$$\Gamma = \{\lambda;\ |\arg(\lambda - \varepsilon)| = \phi\}$$

lies entirely in $\rho(A)$ and the inequality

$$|R(\lambda; A)| \le K|\lambda|^{-1} \tag{3.33}$$

holds on Γ and in its interior (the region to the left of Γ). Given η $(0 < \eta < 1)$, we now follow [12, Chapter I, Section 5] and define

$$A^{-\eta} = \frac{1}{2\pi i} \int_{\Gamma} \lambda^{-\eta} R(\lambda; A)\, d\lambda, \tag{3.34}$$

where Γ is oriented clockwise with respect to its interior, and where $\lambda^{-\eta}$ denotes a fixed branch of the multi-valued function $\lambda \to \lambda^{-\eta}$, which is analytic in (say) the complex plane minus the negative real axis. A few simple manipulations with (3.34)—familiar from the functional calculus of bounded operators—show that

$$A^{-n} = (A^{-1})^n \qquad (n = 1, 2, \cdots), \tag{3.35}$$

$$A^{-\eta_1} A^{-\eta_2} = A^{-(\eta_1 + \eta_2)} \qquad (\eta_1, \eta_2 \geq 0) \tag{3.36}$$

(see [12, Chapter I, Section 5] for details on these and other computations related to (3.34)). Observe that if $A^{-\eta} u = 0$ for some $u \in E$ and some $\eta > 0$, it follows from (3.36) and (3.35) that $(A^{-1})^n u = 0$ for any integer n $(n \geq \eta)$; this is possible only if $u = 0$. On the other hand, it also follows from (3.36) that $A^{-\eta} E \supseteq A^{-n} E$ (η and n as before). Therefore the range of $A^{-\eta}$ is dense for all η. Making use of this and of the previously established one-to-one character of $A^{-\eta}$, we can define, for $\eta > 0$,

$$A^{\eta} = (A^{-\eta})^{-1}, \tag{3.37}$$

which will thus be a closed operator defined in a dense subset of E. It is a consequence of (3.36) that for each integer $n \geq 1$, $(A^{-1/n})^n = A^{-1}$ for $n = 1, 2, \cdots$; taking inverses, we see that

$$(A^{1/n})^n = A,$$

and this justifies the notation $A^{1/n}$. A fundamental property of A^{η} for $0 < \eta < 1$ is that $R(\lambda; A^{\eta})$ exists for $|\arg \lambda| \geq \eta\phi$ (at least), and that it is there given by the formula

$$R(\lambda; A^{\eta}) = \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{\lambda - \mu^{\eta}} R(\mu; A)\, d\mu \tag{3.38}$$

(see [12, Theorem 5.3] for a proof). By means of convenient deformations of the contour of integration in (3.38), it can also be proved that

$$|R(\lambda; A^{\eta})| \leq K |\lambda|^{-1}$$

in $|\arg \lambda| \geq \eta\phi$. Putting together all the previous remarks, we can state the following proposition.

3.6 LEMMA. *If* $0 < \phi < \pi$ *and* $A \in \Xi(\phi)$, *then* $A^{\eta} \in \Xi(\eta\phi)$ *for* $0 < \eta < 1$.

We need one more result (for a proof, see [12, Theorem 3.8] or [11, Theorem 12.8.1]).

3.7 LEMMA. *Assume that* $R(\lambda; A)$ *exists for* $\Re\lambda > \omega$ *and that for some constant* $K > 0$

$$|R(\lambda; A)| \leq K/|\lambda - \omega| \qquad (\Re\lambda \geq \omega). \tag{3.39}$$

Then A *generates a strongly continuous semigroup* $T(\cdot)$ (*which can be extended analytically to a sector containing the half-line* $t > 0$) *such that*

$$|T(t)| \leq K e^{\omega' t} \qquad (t \geq 0)$$

for all $\omega' > \omega$; *here* K *denotes a constant depending on* ω'.

Proof of Theorem 3.4 (a). Assume that $A \in \Xi(\pi)$. Then, by Lemma 3.5, $A \in \Xi(\phi)$ for some $\phi < \pi$. It follows from Lemma 3.6 that $A^{1/n} \in \Xi(\phi/n)$, and as a consequence of the definition of the classes Ξ, if we set

$$B = \gamma^{(n+2)/4} A^{1/n} \qquad (\gamma = e^{2\pi i/n}),$$

then

$$-B, -\gamma B, \cdots, -\gamma^{(n-2)/2} B \in \Xi(\pi/2). \tag{3.40}$$

It is a consequence of Lemma 3.5 that if C is an operator in $\Xi(\pi/2)$, then $-C$ satisfies (3.39) for some $\omega < 0$. It follows from Lemma 3.7 that the operators $B, \gamma B, \gamma^2 B, \cdots, \gamma^{(n-2)/2} B$ in (3.40) generate uniformly bounded semigroups $T_0, \cdots, T_{(n-2)/2}$. We have now verified assumptions (a) and (b) in Lemma 3.1. As for (c), let $n/2 \leq k \leq n - 1$, and let $u_k(\cdot)$ be a solution of the equation

$$u_k'(t) = \gamma^k B u(t) \qquad (t \geq 0) \tag{3.41}$$

such that

$$|u_k(t)| = O(t^n) \qquad \text{as } t \to \infty. \tag{3.42}$$

Let $\hat{u}_k$ be the Laplace transform of u_k, that is, let

$$\hat{u}_k(\lambda) = \int_0^\infty e^{-\lambda t} u_k(t)\, dt. \tag{3.43}$$

Clearly, $\hat{u}_k(\lambda)$ exists and is analytic for $\Re\lambda > 0$; moreover, for each $\varepsilon > 0$, (3.42) and a simple estimation of (3.43) show that there exists a constant K_ε such that

$$|u_k(\lambda)| \leq K_\varepsilon (\Re\lambda)^{n-1} \qquad (\Re\lambda \geq \varepsilon). \tag{3.44}$$

If we integrate (3.43) by parts and use (3.41), we see that $\hat{u}_k(\lambda) \in D(A)$ and

$$(\lambda I - \gamma^k B)\hat{u}_k(\lambda) = u_k(0) \qquad (\Re\lambda > 0). \tag{3.45}$$

Returning to the definition (3.40) of B and using the fact that $A^{1/n} \in \Xi(\phi/n)$, we see that if $n/2 \leq k \leq n - 1$, then

$$\sigma(\gamma^k B) \subseteq \{\lambda; \Re\lambda \geq \varepsilon\} \tag{3.46}$$

for some $\varepsilon > 0$, and that

$$|R(\lambda; \gamma^k B)| \leq K/(1 + |\lambda|) \tag{3.47}$$

for some $K > 0$ in $\Re\lambda < \varepsilon$. From (3.45) and from the fact that the strip $0 < \Re\lambda < \varepsilon$ is contained in $\rho(\gamma^k B)$, we now deduce that $\hat{u}_k(\lambda) = R(\lambda; \gamma^k B) u_k(0)$ in the strip; accordingly, $\hat{u}_k(\cdot)$ is an analytic continuation of $R(\lambda; \gamma^k B) u_k(0)$ to the entire plane.

By virtue of (3.44) and (3.47), this continuation increases (at most) like a polynomial as $|\lambda| \to \infty$. By Liouville's theorem, it must *be* a polynomial, with coefficients in E which, again in view of (3.47) must vanish identically. By the well-known uniqueness property of Laplace transforms, $u_k(t) = 0$ for all $t \geq 0$. This ends the proof of (c) and thus of part (a) of Theorem 3.4.

The proofs of parts (b), (c), and (d) are entirely similar, and therefore we shall only sketch them. In part (b), one takes

$$B = \gamma^{(n+2)/4}(-A)^{1/n} = \gamma^{n/4}(e^{i\pi/n}(-A)^{1/n}).$$

In case (c), we take

$$B = \begin{cases} \gamma^{(n+1)/4}A^{1/n} & \text{if } m = (n+1)/2\,, \\ -\gamma^{(n+1)/4}(-A)^{1/n} & \text{if } m = (n-1)/2\,. \end{cases}$$

Finally, in case (d), we use the operator

$$B = \begin{cases} \gamma^{(n+3)/4}A^{1/n} & \text{if } m = (n-1)/2\,, \\ -\gamma^{(n+3)/4}(-A)^{1/n} & \text{if } m = (n+1)/2\,. \end{cases}$$

This ends the proof of Theorem 3.4.

We close this section with a result that settles the case where E is a Hilbert space, n is even, and $m = (n+2)/2$ (as before, we assume that $\omega = 0$).

3.8 THEOREM. *Suppose* $E = H$ *is a Hilbert space,* n *is an even integer, and* $m = (n+2)/2$. *Then* $A \in \Phi(n, m, 0)$ *if and only if*

$$A = Q^{-1}(-S)^{n/2}Q\,, \tag{3.48}$$

where S *is a self-adjoint operator such that* $S \geq \varepsilon I$ *for some* $\varepsilon > 0$, *and where* Q *is a bounded, invertible, self-adjoint operator.*

Proof. If A is given by (3.48) and $u(\cdot)$ is a solution of the equation

$$u^{(n)}(t) = Au(t)\,, \tag{3.49}$$

then $\tilde{u}(t) = Qu(t)$ is a solution of the equation

$$\tilde{u}^{(n)}(t) = (-S)^{n/2}\tilde{u}(t)\,; \tag{3.50}$$

conversely, the map $u(\cdot) = Q^{-1}\tilde{u}(\cdot)$ transforms solutions of (3.50) into solutions of (3.49). On this basis, it is not difficult to prove that $A \in \Phi(n, m, 0)$ if and only if $(-S)^{n/2} \in \Phi(n, m, 0)$, and this last inclusion is assured by Proposition 3.3 whenever S satisfies the conditions in Theorem 3.8.

Assume now that $A \in \Phi(n, m, 0)$. For any $u \in D$ and any $t \geq 0$, define C by the formula

$$C(t)u = u(t)\,,$$

where $u(\cdot)$ is the bounded solution of (3.49) that satisfies the conditions

(3.51) $$u(0) = u,$$

(3.52) $$u'(0) = u''(0) = \cdots = u^{(n-1)}(0) = 0.$$

Using conditions (I) and (II) in the definition of the classes Φ, we see that $C(t)$ is well-defined and bounded in D for all $t \geq 0$, and that it can thus be extended to a bounded operator in all of E (which will be denoted by the same symbol). It is also a consequence of (II) and of the denseness of D that the function $t \to C(t)$ is strongly continuous and bounded for $t \geq 0$. If $u(\cdot)$ is any solution of (3.49) that satisfies (3.52), then

(3.53) $$u(t) = C(t)u(0).$$

This follows from the definition of $C(\cdot)$ when $u \in D$, and from a simple approximation argument in the general case. Observe also that

(3.54) $$C(0) = I.$$

We now extend $C(\cdot)$ to the entire real axis by setting $C(t) = C(-t)$ for $t < 0$. It follows from the form of the initial conditions (3.52) that for each $u \in D$ the function $t \to C(t)u$ is a solution of (3.49) in $(-\infty, \infty)$. Now, given $u \in D$ and a fixed real number t, let $u(s) = C(s+t)u + C(s-t)u$. Then $u(\cdot)$ is a solution of (3.49); by virtue of the formula (3.53), we must have the relations

$$C(s+t)u + C(s-t)u = u(s) = C(s)u(0) = 2C(s)C(t)u;$$

using once again the denseness of D, we obtain the relation

(3.55) $$C(s+t) + C(s-t) = 2C(s)C(t).$$

Combining the preceding equality with (3.54) we see that, in the nomenclature of [16], $C(t)$ is a *cosine function.* We have already observed that

(3.56) $$|C(t)| \leq K$$

for $t \geq 0$ and therefore for all t. It follows from [9, Theorem 4.1] that

(3.57) $$C(t) = Q^{-1}\cos(tS)Q \qquad (-\infty < t < \infty),$$

where Q is self-adjoint, bounded, and invertible, and where S is self-adjoint and nonnegative.

Observe next that as a consequence of the comments preceding and following inequality (2.42), we can assume that $0 \in \rho(A)$. Now let $u \in D$ and $v(t) = A^{-1}C(t)u$. Since $v(\cdot)$ is a solution of (3.49) satisfying the initial conditions (3.52) and $v(0) = A^{-1}u$, we see from (3.53) that $A^{-1}C(t)u = C(t)A^{-1}u$; since D is dense in E, we must have the relation

$$A^{-1}C(t) = C(t)A^{-1}.$$

Again, suppose that $u \in D$. Then

(3.58) $$C^{(n)}(t)A^{-1}u = A^{-1}C^{(n)}(t)u = C(t)u.$$

Integrating, we see that

$$C(t)A^{-1}u = \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}C(s)u\,ds + \sum_{j=0}^{n-1}\frac{t^j}{j!}C^{(j)}(0)A^{-1}u. \tag{3.59}$$

Let $\phi_0, \phi_1, \cdots, \phi_{n-1}$ be continuous functions with support in $0 \le t \le 1$, say, and such that

$$\int t^j \phi_k(t)\,dt = j!\,\delta_{jk} \qquad (0 \le j \le n-1,\ 0 \le k \le n-1).$$

Multiplying both sides of (3.58) by $\phi_k(t)$ and integrating the resulting equality, we immediately see that $C^{(k)}(0)A^{-1}$ is a bounded operator for $0 \le k \le n-1$; consequently, (3.58) must hold for all $u \in E$. Differentiating n times, we see that $C(\cdot)u$ is a solution of (3.49) for all $u \in D(A)$. In particular,

$$C^{(n)}(0)u = Au \qquad (u \in D(A)). \tag{3.60}$$

Going back to (3.57) we see (by a simple application of the functional calculus for self-adjoint operators) that $C(\cdot)u$ is n times continuously differentiable if and only if

$$Qu \in D((-S)^{n/2}) \qquad \text{and} \qquad C^{(n)}(0)u = Q^{-1}(-S)^{n/2}Qu.$$

This, combined with (3.60), shows that

$$A \subseteq Q^{-1}(-S)^{n/2}Q. \tag{3.61}$$

To show that the inclusion in (3.61) can be strengthened to equality, we need only observe that, because the set $\sigma(Q^{-1}(-S)^{n/2}Q) = \sigma((-S)^{n/2})$ is contained in $(-1)^{n/2}\lambda \ge 0$, $\sigma(A) \cap \sigma(Q^{-1}(-S)^{n/2}Q) \ne \emptyset$. If λ_0 belongs to the intersection, then

$$\begin{aligned}(\lambda_0 I - A)D(A) &= (\lambda_0 I - Q^{-1}(-S)^{n/2}Q)D(A) \\ &= E = (\lambda_0 I - Q^{-1}(-S)^{n/2}Q)D(Q^{-1}(-S)^{n/2}Q),\end{aligned}$$

which clearly contradicts the fact that $\lambda_0 I - Q^{-1}(-S)^{n/2}Q$ is one-to-one unless $D(A) = D(Q^{-1}(-S)^{n/2}Q)$. That $(-1)^{n/2}S \ge \varepsilon I$ follows from the invertibility of S. (Note that $(-1)^{n/2}S^{-1} = Q^{-1}A^{-1}Q$).)

3.9 *Remark.* By essentially the method used in Theorem 3.8, it is possible to show that if A satisfies conditions (I) and (II) at the beginning of the present section, then the subspace D in condition (I) contains $D(A)$. See [15] for details.

4. EXAMPLES

a) We consider the partial differential equation

$$\frac{\partial^3 u}{\partial t^3} = \frac{\partial^2 u}{\partial x^2} - u \tag{4.1}$$

in $P = \{(t, x) \in R^2;\ t \ge 0,\ -\infty < x < \infty\}$. We shall look for solutions of (4.1) (that is, ordinary real-valued functions $u(t, x)$ defined in P and having there continuous partial derivatives u_t, u_x, u_{tt}, u_{xx}, u_{ttt}, and satisfying (4.1)) such that

(4.2) $$u \text{ is bounded in } P,$$

(4.3) $$\lim |u(t, x)| = 0 \qquad (|x| \to \infty)$$

for all $t \geq 0$.

Corresponding to $N = 0, 1, \cdots$ we denote by $C_R^{(N)}$ the space of all real-valued, N times continuously differentiable functions $u(\cdot)$ defined in $-\infty < x < \infty$ such that $u^{(n)}(x) \to 0$ as $|x| \to \infty$ when $0 \leq n \leq N$. We seek an answer to the following question:

What is the number m of boundary values of partial derivatives of u,

(4.4) $$u(0, x), \quad \frac{\partial u}{\partial t}(0, x), \quad \frac{\partial^2 u}{\partial t^2}(0, x),$$

that can be arbitrarily assigned (for example, in $C_R^{(N)}$ for large enough N) and for which (4.1) has a unique solution satisfying (4.2) and (4.3) and possessing these boundary values?

Observe first that the problem does not change if we admit *complex-valued* solutions and initial data. In fact, existence for real initial data implies existence in the complex case: to see this, separate the initial data into real and imaginary parts, solve (4.1) for each, and recombine. The same holds for uniqueness; if u is a complex solution of (4.1), (4.2), and (4.3) having m of its initial data zero, then so are $\Re u$ and $\Im u$. On the other hand, assume that solutions exist in the complex case; then, if u is a solution with real boundary values, then $\Re u$ is a solution with the same boundary values, and it is clear that uniqueness in the complex case implies uniqueness in the real case.

Let then E be the space of all continuous, complex-valued functions $u(\cdot)$ that are defined for all real x and tend to zero when $|x| \to \infty$, endowed with the norm

$$|u|_E = \sup_{-\infty < x < \infty} |u(x)|.$$

Clearly, E is a Banach space. Define a linear operator A in E by the formula

$$(Au)(x) = u''(x) - u(x),$$

where D(A) consists of all the functions in E having two continuous derivatives that again belong to E. Clearly, D(A) is dense in E. It is (essentially) proved in [6, Chapter VIII] that every complex number λ not in the interval $(-\infty, -1]$ belongs to $\rho(A)$ and that

(4.5) $$(R(\lambda; A)u)(x) = \frac{1}{2(\lambda + 1)^{1/2}} \int_{-\infty}^{\infty} e^{-(\lambda+1)^{1/2}|\eta|} u(x + \eta)\, d\eta .$$

A moment's reflection shows that there is a correspondence

$$u(t, \cdot) \longleftrightarrow \tilde{u}(t)(\cdot)$$

between solutions of (4.1), (4.2), (4.3) and E-valued solutions (in the sense of the previous sections) of the operational equation

$$\tilde{u}'''(t) = A\tilde{u}(t)$$

that are bounded in $t \geq 0$. Therefore we can apply all the results previously derived (observe also that $D(A^p) = C_C^{(2p)}$, where C_C is defined similarly to C_R, but with reference to *complex-valued* functions).

From the possible choices $m = 1, 2, 3$, the case $m = 3$ is immediately ruled out by Corollary 2.4, even without the uniqueness condition; more generally, Corollary 2.5 shows that $m = 3$ is inadmissible even if the boundedness condition (4.2) is relaxed to

$$|u(x, t)| \leq K e^{\omega t} \qquad ((x, t) \in P) \tag{4.6}$$

for some real number ω. Consider now the case $m = 2 = (3 + 1)/2$. It follows from Theorem (2.1)—again, we do not use the uniqueness assumption—that if λ is any complex number such that at least two of its cube roots lie in the half-plane $\Re\mu \leq -\varepsilon < 0$, then $-\lambda \in \rho(A)$. But this means that each negative real number belongs to $\rho(A)$, and this is absurd. If (4.2) is replaced by (4.6), we conclude that every negative number with sufficiently large absolute value belongs to $\rho(A)$, which is again absurd. The only case left is

$$m = 1 = (3 - 1)/2 .$$

It is easy to see that (4.4) implies

$$|R(\lambda; A)| = |\lambda + 1|^{-1/2}\, \Re((\lambda + 1)^{-1/2}), \tag{4.7}$$

and that this last inequality implies in turn that $-A \in \Xi(\pi/2)$. It follows from Theorem 3.4 (c) that $A \in \Phi(3, 1, 0)$, and this shows that our question has an affirmative answer in the present case. Observe also that by using Theorem 3.4 in full force, we may take the initial data in $D(A)$—instead of taking them in an unspecified $C^{(N)}$—and that we have continuous dependence of the solutions on any one of their boundary values. In other words, there exists a constant $K > 0$ such that each solution $u(\cdot, \cdot)$ of (4.1), (4.2), and (4.3) satisfies the condition

$$|u(t, x)| \leq K \sup_{-\infty < x < \infty} |\phi(x)| \qquad ((t, x) \in P),$$

where ϕ is any one of the boundary values (4.4).

b) To emphasize that our methods apply not only to partial differential equations, we consider now the integro-differential equation

$$\frac{\partial^3 u}{\partial t^3}(t, x) = \frac{\partial^2 u}{\partial x^2}(t, x) - u(t, x) + a \int_{-\infty}^{\infty} e^{-b|x-\eta|}\, u(t, \eta)\, d\eta , \tag{4.8}$$

where a and b are constants ($b > 0$). About the equation (4.8) we ask the same questions that we asked about the equation (4.1). In other words, we want again to determine the number m of boundary values of partial derivatives (4.4) that can be arbitrarily assigned and for which (4.8) has a solution in P that satisfies (4.2) and (4.3) and has the preassigned boundary values.

We can reduce the equation (4.8) to an operational differential equation in the space E by a reasoning similar to that of Example a). This time, we obtain the operational equation

$$u'''(t) = (A + B)u(t).$$

The operator A is defined as in Example a), whereas the bounded operator B is defined by the formula

$$(Bu)(x) = a \int_{-\infty}^{\infty} e^{-b|x-\eta|} u(\eta)\, d\eta \qquad (u \in E).$$

It is easy to see that $|B| = 2|a|/b$.

Assume that

$$|B| = 2|a|/b < 1, \tag{4.9}$$

and let r be a positive real number such that $|B| < 1 - r$. It is easy to see that the minimum of

$$|R(\lambda; A)|^{-1} = |\lambda + 1|^{1/2}\, \Re((\lambda + 1)^{1/2})$$

in the half-plane $\Re\lambda \geq -r$ equals $1 - r$. Then, by a well-known perturbation result [10], the resolvent $R(\lambda; A + B)$ exists and is given there by the formula

$$R(\lambda; A + B) = \sum_{n=0}^{\infty} R(\lambda; A)(BR(\lambda; A))^n \qquad (\Re\lambda \geq -r), \tag{4.10}$$

where the series in the right-hand side of (4.10) converges in the topology of $\mathscr{L}(E)$. From (4.10) we obtain the estimate

$$|R(\lambda; A + B)| \leq \left(1 - \frac{|B|}{1 - r}\right)^{-1} |R(\lambda; A)| \qquad (\Re\lambda \geq -r).$$

Using this estimate, we obtain for the equation (4.8) results analogous to those established in Example a) for the equation (4.1). We leave the details to the reader.

We make some further comments concerning equation (4.8). The results we have obtained are based only on the fact that $|B| < 1$; clearly, they will be valid for any bounded operator B of norm less than 1. For the case $|B| \geq 1$, we cannot decide whether $A + B \in \Phi(3, 1, 0)$, at least not without a more careful examination of B. On the other hand, we can deduce by essentially the same methods that the cases $m = 2$ and $m = 3$ are inadmissible for equation (4.8). The details are again omitted.

c) The last example illustrates some of the limitations of our results. More precisely, it shows that there is a wide gap between the necessary conditions of Section 2 and the sufficient conditions of Section 3.

Instead of (4.1) or (4.8), consider now the equation

$$\frac{\partial^3 u}{\partial t^3} = i\left(\frac{\partial^2 u}{\partial x^2} - u\right) \tag{4.11}$$

or the equivalent operational equation in E,

$$u'''(t) = iAu(t), \tag{4.12}$$

where A is defined as in Example a). We can still deduce that the case $m = 3$ is inadmissible; in fact, this result does not depend at all on the operator in the right-hand side of (4.12). But our methods do not allow us to decide whether the cases $m = 1$ and $m = 2$ are admissible. In fact, a few simple computations show that iA satisfies the necessary conditions of Section 2 both for $m = 1$ and $m = 2$. On the other hand, iA does not satisfy the sufficient conditions in Section 3, either for $n = 1$ or for $n = 2$.

REFERENCES

1. A. V. Balakrishnan, *Abstract Cauchy problems of the elliptic type.* Bull. Amer. Math. Soc. 64 (1958), 290-291.

2. ———, *Fractional powers of closed operators and the semigroups generated by them.* Pacific J. Math. 10 (1960), 419-437.

3. J. Chazarain, *Problèmes de Cauchy au sens des distributions vectorielles et applications.* C. R. Acad. Sci. Paris Sér. A-B 266 (1968), A10-A13.

4. ———, *Problèmes de Cauchy abstraits et applications.* Séminaire Lions-Schwartz, exposé du 26 Janvier 1968.

5. G. Da Prato and E. Giusti, *Una caratterizzazione dei generatori di funzioni coseno astratte.* Boll. Un. Mat. Ital. (3) 22 (1967), 357-362.

6. N. Dunford and J. T. Schwartz, *Linear operators. I. General theory.* Interscience, N. Y., 1958.

7. H. O. Fattorini, *Ordinary differential equations in linear topological spaces,* I and II. J. Differential Equations 5 (1969), 72-105, and 6 (1969), 50-70.

8. ———, *Sur quelques équations différentielles pour les distributions vectorielles.* C. R. Acad. Sci. Paris Sér. A-B 268 (1969), A707-A709.

9. ———, *Uniformly bounded cosine functions in Hilbert space.* J. Math. Mech. 20 (1970), 411-425.

10. E. Hille, *Une généralization du problème de Cauchy.* Ann. Inst. Fourier Grenoble 4 (1952), 31-48.

11. E. Hille and R. S. Phillips, *Functional analysis and semi-groups.* Amer. Math. Soc. Colloquium Publications, vol. 31, rev. ed. Amer. Math. Soc., Providence, R. I., 1957.

12. S. G. Kreĭn, *Linear differential equations in Banach space* (Russian). Izdat. "Nauka," Moscow, 1967.

13. S. G. Kreĭn and O. I. Prozorovskaya, *Analytic semi-groups and incorrect problems for evolutionary equations* (Russian). Dokl. Akad. Nauk SSSR 133 (1960), 277-280.

14. Z. Nehari, *Conformal mapping.* McGraw Hill, New York, 1952.

15. A. Radnitz, *The incomplete Cauchy problem in Banach spaces.* Dissertation, University of California (Los Angeles), 1970.

16. G. E. Shilov, *Well-posed boundary-value problems in a half-space for linear partial differential equations with constant coefficients* (Russian). Uspehi Mat. Nauk 19 (1964), no. 3 (117), 3-52. English translation: Russian Math. Surveys 19 no. 3 (1964), 1-51.

17. M. Sova, *Cosine operator functions.* Rozprawy Mat. 49 (1966), 47 pp.

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